Set Theory

CMPS/MATH 2170: Discrete Mathematics

Outline

- Sets and Set Operations (2.1-2.2)
- Functions (2.3)
- Sequences and Summations (2.4)
- Cardinality of Sets (2.5)

Introduction to Sets

- A set is an unordered collection of objects, called elements or members of the set
  - Usually: duplicates are not allowed
  - $a \in A$: a is an element of the set A
  - $a \notin A$: a is not an element of the set A
- Examples

$$A = \{1, 3, 5, 7, 9\} \quad B = \{1, 2, 3, \dots, 99\}$$

Roster method

$$A = \{x | x \text{ is an odd positive integer less than } 10\}$$

$$A = \{x \in \mathbb{Z}^+ | x \text{ is odd and x} < 10\}$$

Set builder notation

the set of positive integers

Often Used Sets

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$, the set of natural numbers

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the set of integers

$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$, the set of positive integers

$\mathbb{Q} = \{p/q | p \in \mathbb{Z}, q \in \mathbb{Z}, \text{and } q \neq 0\}$, the set of rational numbers

$\mathbb{R}$, the set of real numbers

$\mathbb{R}^+$, the set of positive real numbers

$\mathbb{C}$, the set of complex numbers

Sets vs. Tuples

- A set is an unordered collection of objects
  - two sets are equal if and only if they have the same elements
  - $A = B$ iff $\forall a \colon a \in A \leftrightarrow a \in B$
  - $\{1,3,5\} = \{3,5,1\}$
- An n-tuple $(a_1, a_2, \ldots, a_n)$ is an ordered collection of elements
  - $(3,5,1)$ is a 3-tuple
  - $(3,5,1) \neq (1,3,5)$
  - $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_m)$ iff $n = m$, $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$

Subsets

- A is a subset of B if every element of A is also an element of B
  - $A \subseteq B$
  - $\forall x \in A\colon\ x \in B$
  - $\forall x\colon\ x \in A \rightarrow x \in B$

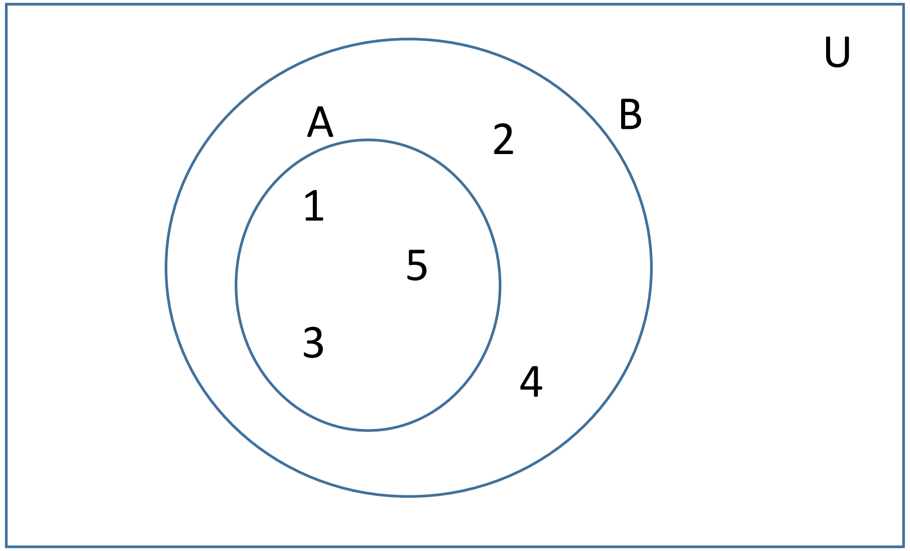
- B is a superset of A if A is a subset of B
  - $B \supseteq A$

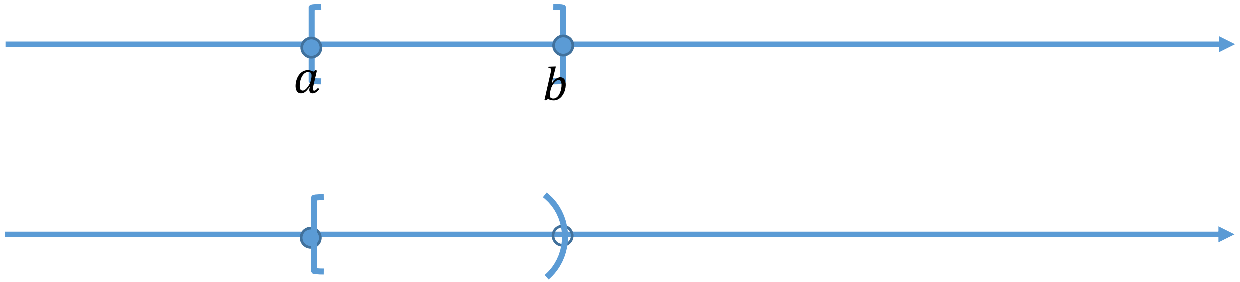
Subsets

- Ex. 1: $A = \{1, 3, 5\}, B = \{1, 2, 3, 4, 5\}$
- Ex. 2: Intervals of real numbers

  $[a, b] = \{x | a \leq x \leq b\}$
  $[a, b) = \{x | a \leq x < b\}$
  $(a, b] = \{x | a < x \leq b\}$
  $(a, b) = \{x | a < x < b\}$

Venn Diagram





Subsets

- To show that $A \subseteq B$, show that if $a \in A$ then $a \in B$
- To show that $A \nsubseteq B$, show that there is $a \in A$ such that $a \notin B$
- $S \subseteq S$ for any set S
- $\emptyset \subseteq S$ for any set S: $\emptyset$ - empty set {}
- $A = B$ iff $A \subseteq B$ and $B \subseteq A$
- A is a proper subset of B if A is a subset of B but $A \neq B$
  - $A \subset B$
  - $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

The Size of a Set

- If a set S contains n distinct elements, we say that S is a finite set and n is the cardinality of S, denoted by $|S| = n$
  - $|\emptyset| = 0$
  - $|\{1, 2, 6\}| = 3$
- A set is said to be infinite if it is not finite
  - The set of positive integers is infinite
  - How to compare the sizes of two infinite sets?

Power Sets

- The power set of a set A is the set of all subsets of A
  - $\mathcal{P}(A) = \{B | B \subseteq A\}$
- Ex: $A = \{1, 2, 3\}$
  - $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
  - $|\mathcal{P}(A)| = 8 = 2^3 = 2^{|A|}$
- Theorem: for any finite set A, $|\mathcal{P}(A)| = 2^{|A|}$
  - A proof by mathematical induction will be given in Chapter 5

Cartesian Products

- Let A and B be two sets. The Cartesian product of A and B is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

- Ex: $A = \{a, b\},\ B = \{1, 2, 3\}$

$$A \times B = \{(a, 1),\ (a, 2),\ (a, 3),\ (b, 1),\ (b, 2),\ (b, 3)\}$$

- Ex: $\mathbb{R}$ is the set of real numbers

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) | x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$ is the set of all points in the Cartesian plane

Cartesian Products

- Ex: $A = \{a, b\},\ B = \{1, 2, 3\}$

  $A \times B = \{(a, 1),\ (a, 2),\ (a, 3),\ (b, 1),\ (b, 2),\ (b, 3)\}$

- For any finite sets A and B, $|A \times B| = |A||B|$

- Cartesian product of multiple sets
  - $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1,2, \ldots, n\}$

True or False

Suppose $A = \{a, b, c\}$

- $\emptyset \subseteq A$ True
- $\{\emptyset\} \subseteq A$ False
- $\{a, c\} \in A$ False
- $\{b, c\} \in \mathcal{P}(A)$ True
- $\{a, b\} \in A \times A$ False

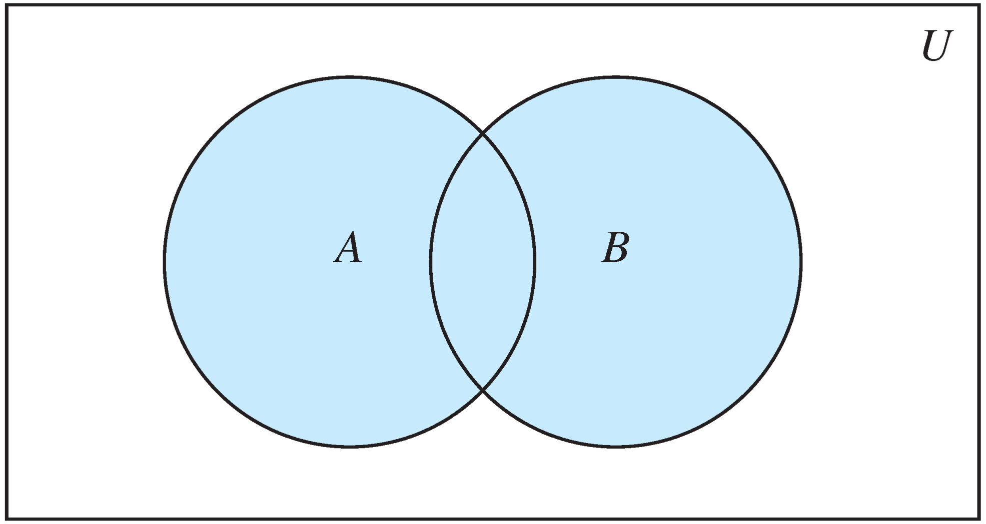
Set Operations

- Set Operations
  - Union -- Disjunction
  - Intersection -- Conjunction
  - Difference & Complement -- Negation
- Set Identities -- Logical equivalences

Set Operations

- The union of set A and set B, denoted by $A \cup B$, is the set that contains those elements that are either in A or B, or in both

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

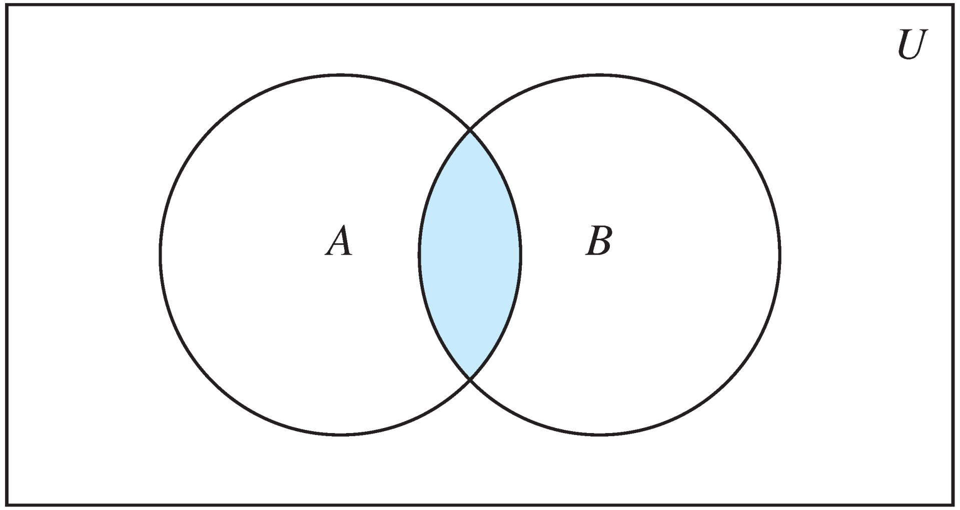


$A \cup B$ is shaded.

Set Operations

- The intersection of A and B, denoted by $A \cap B$, is the set containing those elements that are in both A and B

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

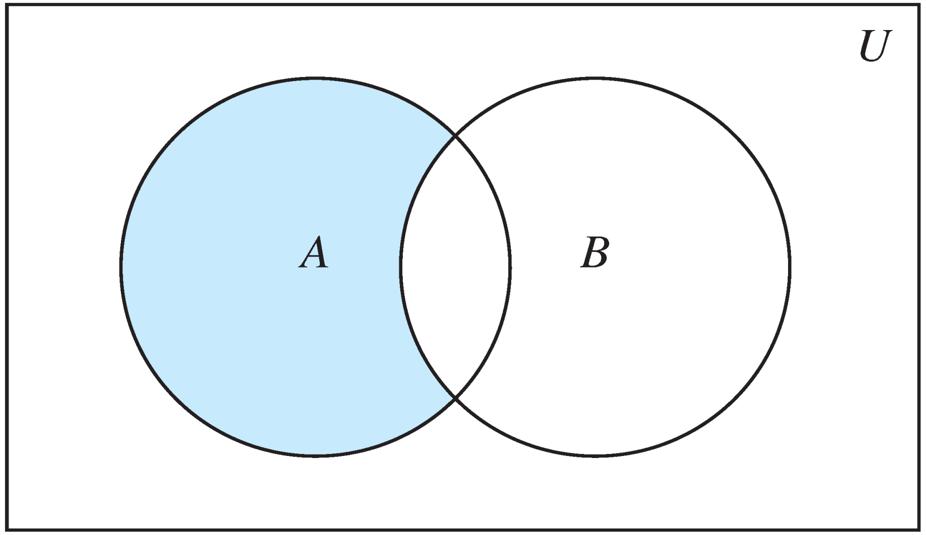


$A \cap B$ is shaded.

Set Operations

- The difference of A and B, denoted by $A \backslash B$ (or $A - B$) is the set containing those elements that are in A but not in B

$$A \backslash B = \{x \mid x \in A \land x \notin B\}$$

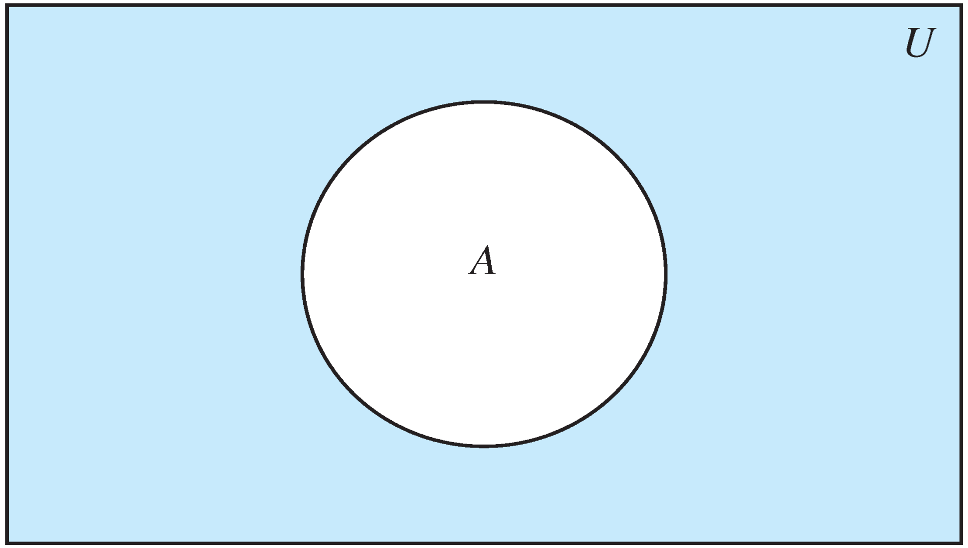


$A - B$ is shaded.

Set Operations

- The complement of a set A with respect to a universe U, denoted by $\bar{A}$, is the set containing those elements that are not in A

$$\bar{A} = \{x \in U \mid x \notin A\} = U \backslash A$$



$\bar{A}$ is shaded.

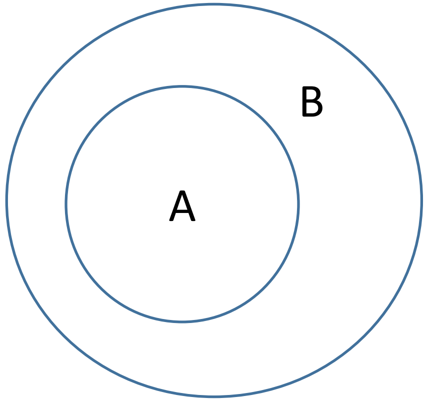
Set Operations

Ex: $A = \{-2, 3, 4\} \quad B = \{1, 3, 4, 7\}$

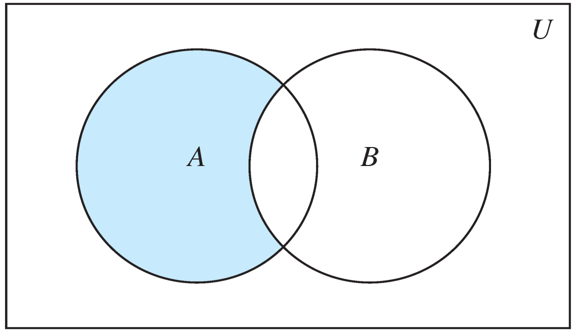
$A \cup B = \{-2, 1, 3, 4, 7\}$

$A \cap B = \{3, 4\}$

$A \setminus B = \{-2\}$



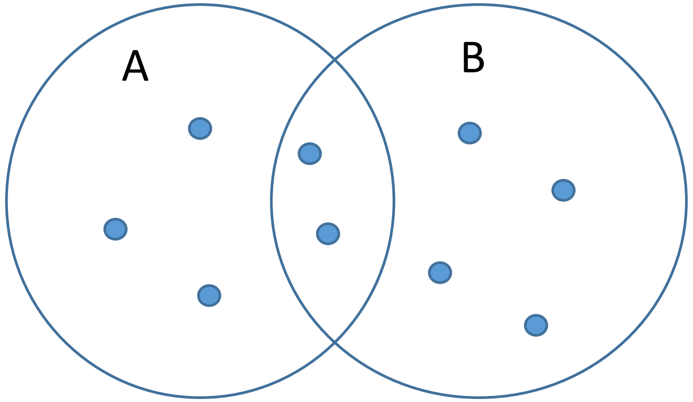
- If $A \subseteq B$, then $A \cup B = B$ and $A \cap B = A$
- $A \setminus B = A \cap \bar{B}$



Set Operations

Theorem: If A and B are two finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Corollary: If two sets A and B are finite and disjoint, $|A \cup B| = |A| + |B|$

- Two sets are called disjoint if their intersection is the empty set

Set Identities

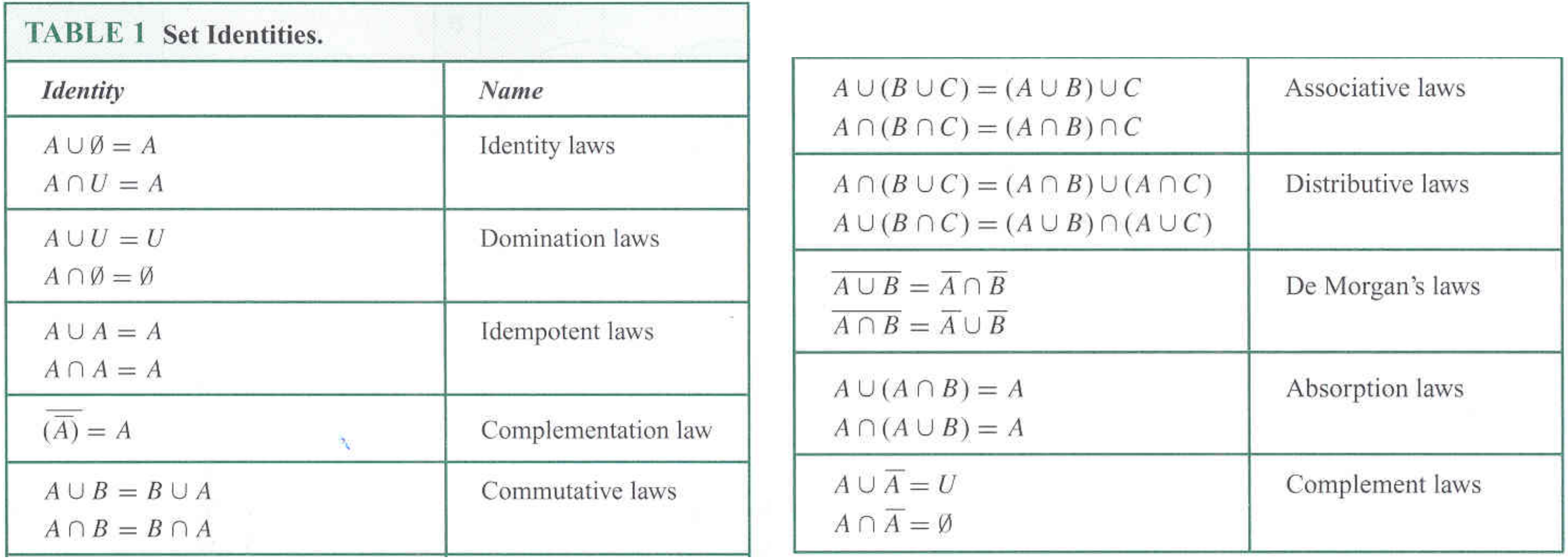


TABLE 1 Set Identities.

Identity	***Name***
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Set Identities

- De Morgan's laws for sets

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws for sets

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Generalized Union and Intersections

- $A \cup B \cup C = A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cap B \cap C = A \cap (B \cap C) = (A \cap B) \cap C$

- The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

- The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

Generalized Union and Intersections

- Ex: $B_1 = \{1\}, B_2 = \{1, 2\}, \ \dots, B_n = \{1,2,3, \dots, n\}, \dots$

$$\bigcup_{n=1}^{\infty} B_n = \{1\} \cup \{1,2\} \cup \cdots \cup \{1,2,3, \dots, n\} \cup \cdots$$

$$= \{1,2,3 \dots\} = \mathbb{Z}^+$$

$$\bigcap_{n=1}^{\infty} B_n = \{1\} \cap \{1,2\} \cap \cdots \cap \{1,2,3, \dots, n\} \cap \cdots$$

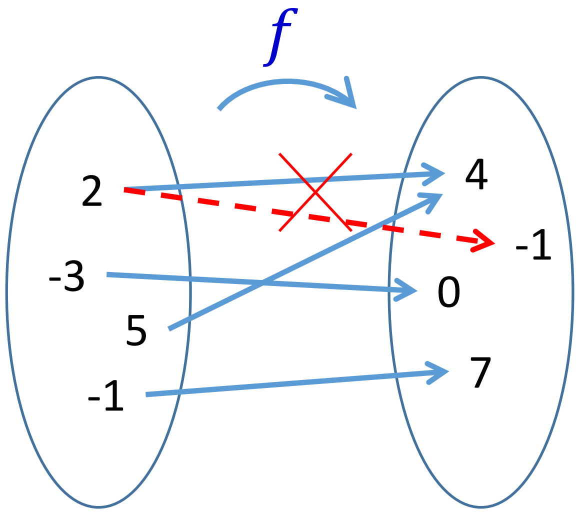
$$= \{1\}$$

Outline

- Sets and Set Operations
- Functions
- Sequences and Summations
- Cardinality of Sets

Functions

- Let X and Y be nonempty sets. A function $f\colon X \to Y$ maps every element of X to exactly one element in Y.
  - X is called the domain, Y is called the codomain
  - Write $f(x) = y$ where y is the unique element of Y assigned by f to $x \in X$
    - y is called image of x and x is the preimage of y
- Let $S \subseteq X$. Then $f(S) = \{f(s) | s \in S\}$ is the image of S
  - $f(X)$ is the range of f



$X = \{-3, -1, 2, 5\}$ $\quad Y = \{-1, 0, 4, 7\}$

$f(-3) = 0$
$f(-1) = 7$
$f(2) = 4$
$f(5) = 4$

$f(\{2, 5\}) = \{4\}$
$f(X) = \{0, 4, 7\}$

Functions

- Let $f_1, f_2 : X \to \mathbb{R}$ be two functions from X to $\mathbb{R}$

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

- Ex.1: $f, g : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^2, g(x) = x - x^2$$

$$(f + g)(x) = f(x) + g(x) = x^2 + (x - x^2) = x$$

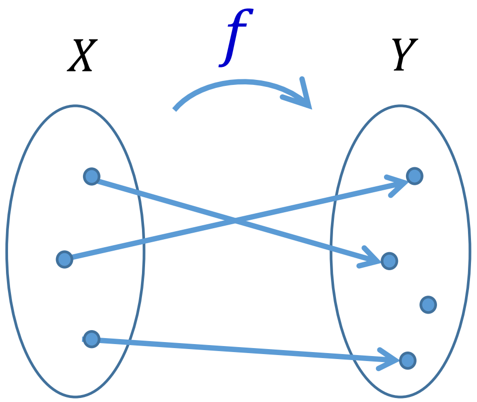
$$(fg)(x) = x^2(x - x^2) = x^3 - x^4$$

Injective and Surjective Functions

Let $f: X \rightarrow Y$ be a function

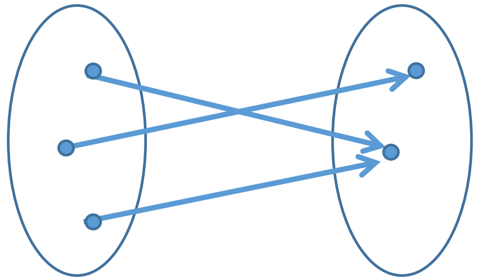
- f is said be one-to-one, or injective, if

$$\forall x_1, x_2 \in X\colon\ f(x_1) = f(x_2) \ \rightarrow\ x_1 = x_2$$

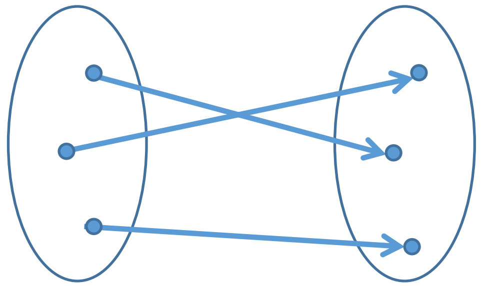


- f is said be onto, or surjective, if

$$\forall y \in Y\ \exists x \in X\colon\ f(x) = y$$



- f is said be one-to-one correspondence, or bijective, if it is both injective and surjective



Injective and Surjective Functions

Ex. 2: $f: \mathbb{R} \to \mathbb{R}$

$x \longmapsto 2x + 1$ (same as $f(x) = 2x + 1$)

bijective

Ex. 3: $g: \mathbb{R} \to \mathbb{R}$

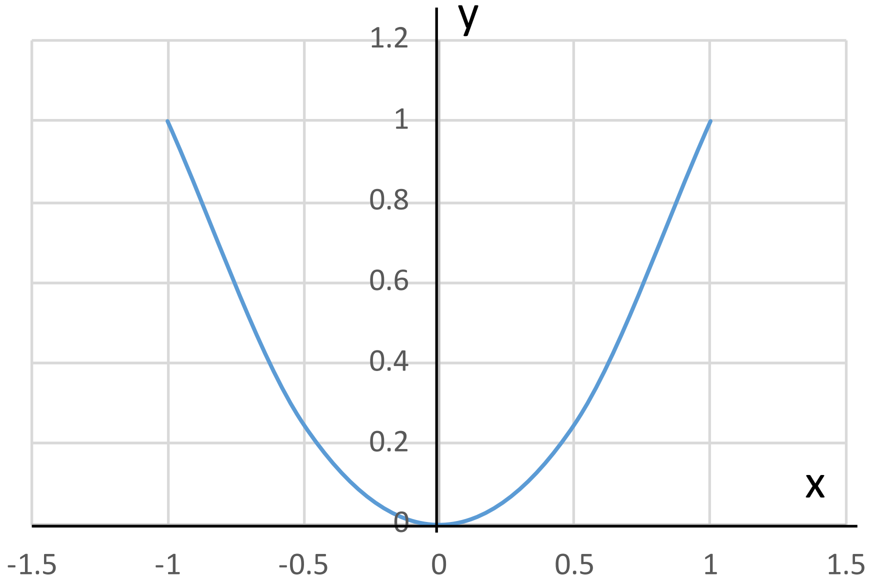
$x \longmapsto x^2$

neither injective nor surjective

Ex. 4: $h: \mathbb{R} \to \mathbb{R}_0^+$ (non-negative real numbers)

$x \longmapsto x^2$

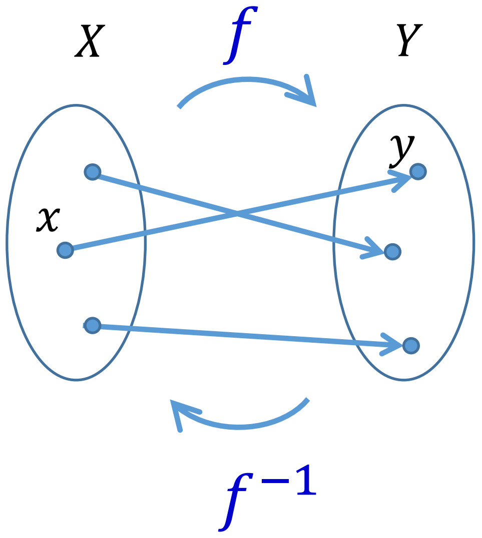
surjective but not injective



Inverse Functions

- Let $f: X \to Y$ be a bijective function. Then

  $f^{-1}: Y \to X,\ \ f^{-1}(y) = x$ such that $f(x) = y$ is the inverse of f



Ex. 5: $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x + 1$

$f^{-1}: \mathbb{R} \to \mathbb{R}$ where $f^{-1}(y) = (y - 1)/2$

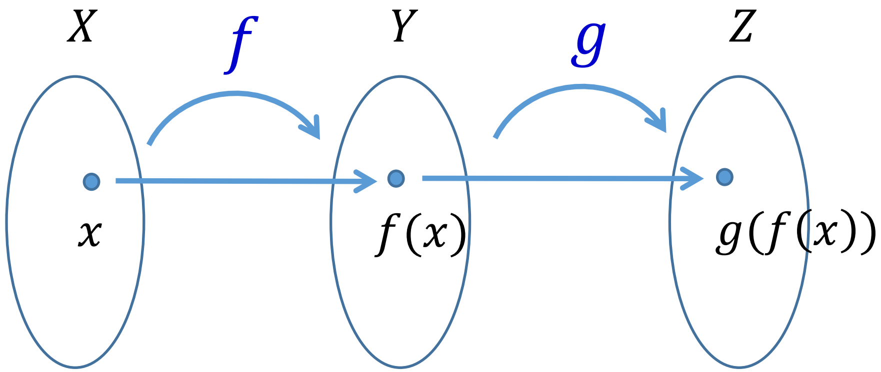
Ex. 6: $f: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ where $f(x) = x^2$

$f^{-1}: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ where $f^{-1}(y) = \sqrt{y}$

- $(f^{-1})^{-1} = f$

Compositions of Functions

- Let $f: X \to Y$ and $g: Y \to Z$. Then $g \circ f: X \to Z$ where $(g \circ f)(x) = g(f(x))$ is the composition of g and f



For $g \circ f$ to be defined, the range of f must be a subset of the domain of g

Ex. 7: $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2x, \quad g: \mathbb{R} \to \mathbb{R},\ g(x) = x + 3$

$$g \circ f:\ \mathbb{R} \to \mathbb{R},\ (g \circ f)(x) = g\big(f(x)\big) = g(2x) = 2x + 3$$

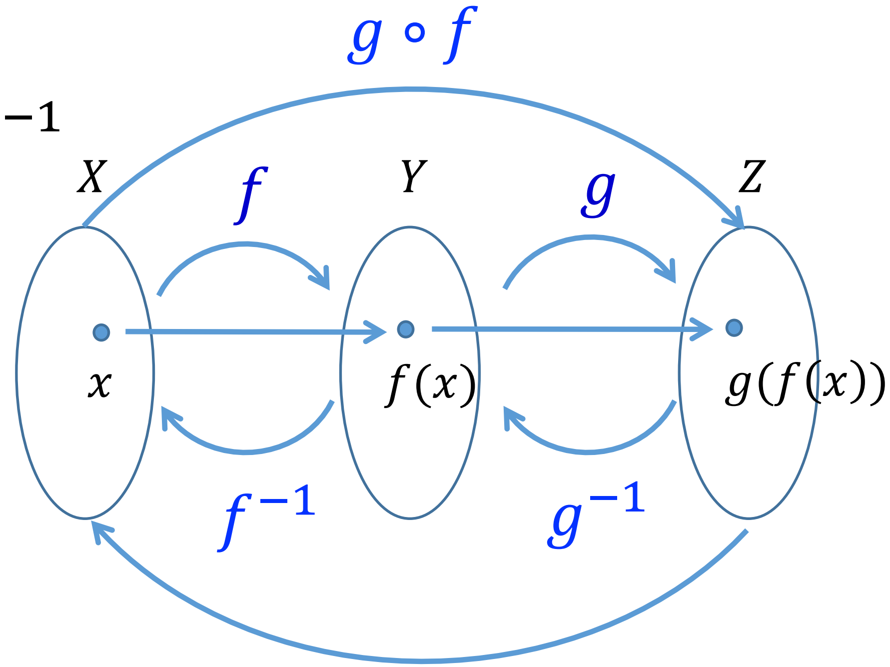
$$f \circ g:\ \mathbb{R} \to \mathbb{R},\ (f \circ g)(x) = f\big(g(x)\big) = f(x + 3) = 2(x + 3) = 2x + 6$$

Compositions of Functions

- Assume $f: X \to Y$ and $g: Y \to Z$ are bijective. Then

  (1) $g \circ f$ is bijective

  (2) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$



$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Floor and Ceiling Functions

- Floor function: $\lfloor\ \rfloor: \ \mathbb{R} \to \mathbb{Z}$

  $x \to \lfloor x \rfloor$ (the largest integer less than or equal to x)

- Ceiling function: $\lceil\ \rceil: \ \mathbb{R} \to \mathbb{Z}$

  $x \to \lceil x \rceil$ (the smallest integer greater than or equal to x)

Useful properties:

- $x - 1 < \lfloor x \rfloor \leq x \qquad x \leq \lceil x \rceil < x + 1$
- For all $x \in \mathbb{Z}$: $\left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{2} \right\rceil = x$

Outline

- Sets and Set Operations
- Functions
- Sequences and Summations (to be discussed after fall break)
- Cardinality of Sets

Cardinality of Sets

Recall: For a finite set S, $|S| = n$ if S contains n distinct elements

How to compare the sizes of two infinite sets?

Hilbert’s Grand Hotel

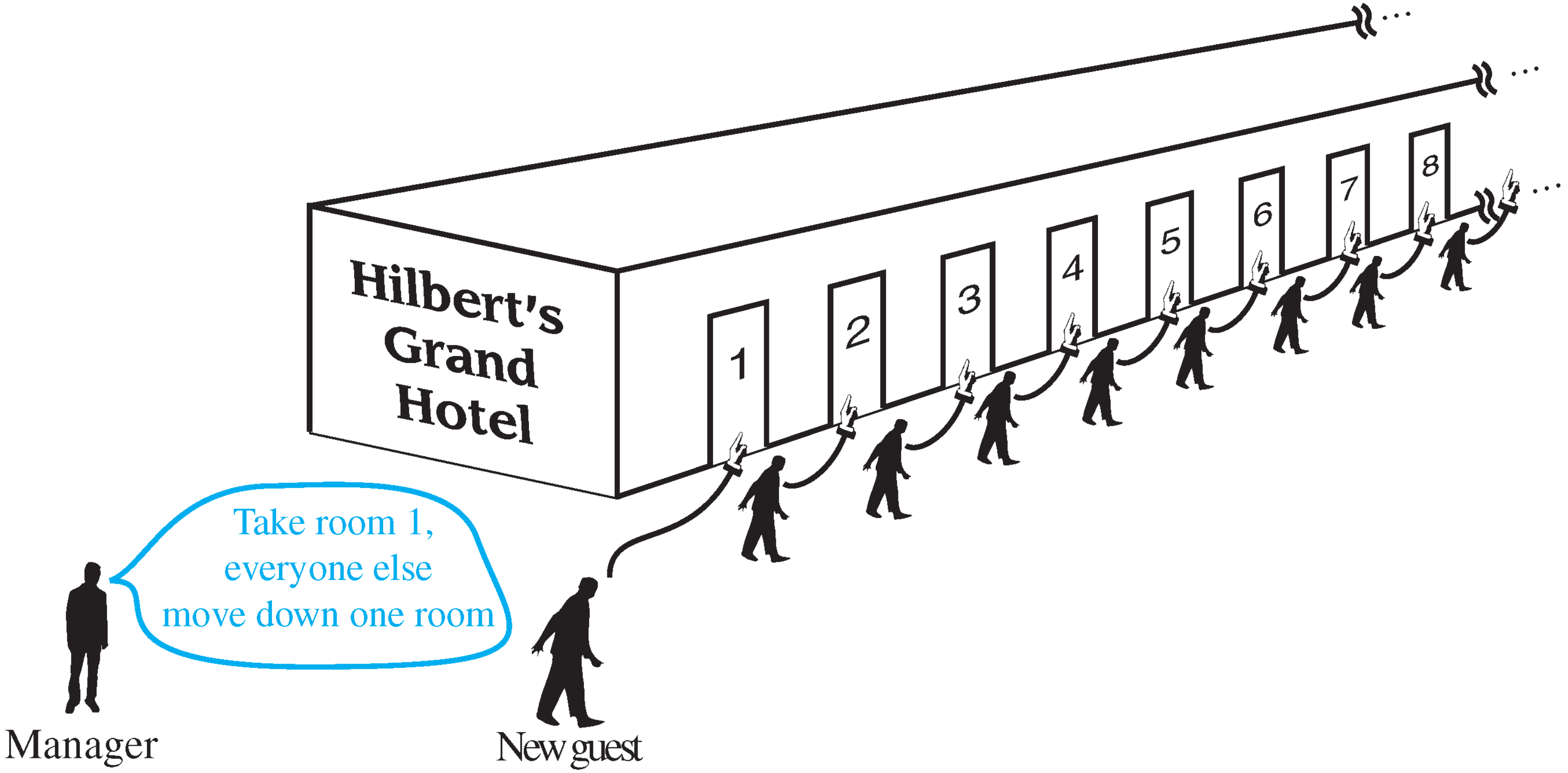
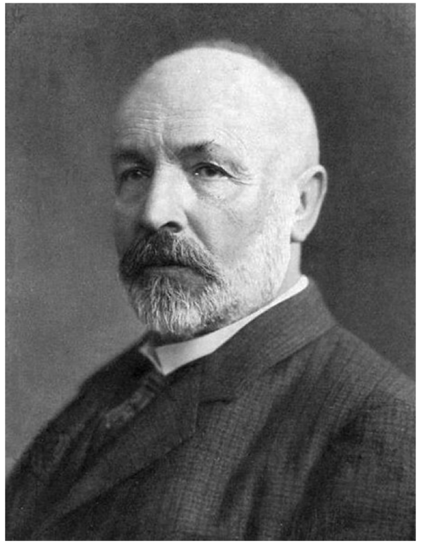


FIGURE 2 **A New Guest Arrives at Hilbert’s Grand Hotel.**

David Hilbert

Cardinality of Sets



Georg Cantor
(1845-1918)

How to compare the sizes of two infinite sets?

Definition 1: Two sets A and B have the same cardinality, denoted by $|A| = |B|$, if there is a bijection between A and B

Ex: Let S and T be finite sets with $|S| = |T|$. Find a bijection between S and T.

Cardinality of Sets

Theorem: Let O^+ be the set of odd positive integers. Show that $|\mathbb{Z}^+| = |O^+|$

Proof: $f: \mathbb{Z}^+ \rightarrow O^+ ,\ f(n) = 2n - 1$

Theorem: Show that $|\mathbb{Z}| = |\mathbb{Z}^+|$

Proof: $f: \mathbb{Z} \rightarrow \mathbb{Z}^+,\ f(n) = \begin{cases} 2n & \text{if } n > 0 \\ -2n + 1 & \text{if } n \leq 0 \end{cases}$

Countable and Uncountable Sets

Definition 3: Let S be a set.

- S is countably infinite if $|S| = |\mathbb{Z}^+| = \aleph_0$ (“aleph null”)
  - E.g., both O^+ and $\mathbb{Z}$ are countably infinite
- S is countable if S is finite or countably infinite
- If S is not countable, it is uncountable

- Definition 4: We say that $|A| \leq |B|$ if there is an injection $f: A \to B$, and $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$
  - If A is finite and B is uncountable, then $|A| < \aleph_0 < |B|$