Discrete Probability

CMPS/MATH 2170: Discrete Mathematics

Applications of Probability in Computer Science

- Algorithms
- Complexity
- Machine learning
- Combinatorics
- Networking
- Cryptography
- Information theory
- …

Agenda

- Discrete Probability Law (7.1,7.2)
- Independence (7.2)
- Random Variables (7.2)
- Expected Value (7.4)

Examples

- Ex. 1: consider rolling a pair of 6-sided fair dice
  - Sample space $\Omega = \{(i, j): i, j = 1, 2, 3, 4, 5, 6\}$
  - Each outcome has the same probability of 1/36
  - What is the probability that the sum of the rolls is 6?

  Let A denote the event that the sum of the rolls is 6

  $A = \{(1,5), (5,1), (2,4), (4,2), (3,3)\}$

  $\mathrm{P}(A) = 5/36$

Examples

- Ex. 2: consider rolling a 6-sided biased (loaded) die
  - Sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$
  - Assume $P(3) = \frac{2}{7}, P(1) = P(2) = P(4) = P(5) = P(6) = \frac{1}{7}$
  - What is the probability of getting an odd number?

Let B denote the event of getting an odd number

$B = \{1, 3, 5\}$

$$P(B) = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} = \frac{4}{7}$$

Experiment and Sample Space

- Experiment: a procedure that yields one of a given set of possible outcomes
  - Ex: flip a coin, roll two dice, draw five cards from a deck, etc.
- Sample space Ω: the set of possible outcomes
  - We focus on countable sample space: Ω is finite or countably infinite
  - In many applications, Ω is uncountable (e.g., a subset of $\mathbb{R}$)
- Event: a subset of the sample space
  - Probability is assigned to events
  - For an event $A \subseteq \Omega$, its probability is denoted by $\mathrm{P}(A)$
    - Describes beliefs about likelihood of outcomes

Discrete Probability

- Discrete Probability Law
  - A function $\mathrm{P}: \mathcal{P}(\Omega) \to [0,1]$ that assigns probability to events such that:
    - $0 \leq \mathrm{P}(\{s\}) \leq 1$ for all $s \in \Omega$ (Nonnegativity)
    - $\mathrm{P}(A) = \sum_{s \in A} \mathrm{P}(\{s\})$ for all $A \subseteq \Omega$ (Additivity)
    - $\mathrm{P}(\Omega) = \sum_{s \in \Omega} \mathrm{P}(\{s\}) = 1$ (Normalization)
- Discrete uniform probability law: $|\Omega| = n, \mathrm{P}(A) = \frac{|A|}{n} \; \forall \, A \subseteq \Omega$

Properties of Probability Laws

- Consider a probability law, and let $A, B,$ and C be events
  - If $A \subseteq B$, then $\mathrm{P}(A) \leq \mathrm{P}(B)$
  - $\mathrm{P}(\overline{A}) = 1 - \mathrm{P}(A)$
  - $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
  - $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ if A and B are disjoint, i.e., $A \cap B = \emptyset$
- Ex. 3: What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

$$\frac{50}{100} + \frac{20}{100} - \frac{10}{100} = 0.6$$

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Independence

- Two events A and B are independent if and only if $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$
- Ex. 4: Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely and have probability $1/16$. Are the following pair of events independent?

(a) $A = \{\text{ 1st roll is 1 }\}$, $B = \{\text{ sum of two rolls is 5 }\}$ Yes

(b) $A = \{\text{ 1st roll is 4 }\}$, $B = \{\text{ sum of two rolls is 4 }\}$ No

Bernoulli Trials

- Bernoulli Trial: an experiment with two possible outcomes
  - E.g., flip a coin results in two possible outcomes: head (H) and tail (T)
- Independent Bernoulli Trials: a sequence of Bernoulli trails that are mutually independent

Bernoulli Trials

- Ex.5: Consider an experiment involving five independent tosses of a biased coin, in which the probability of heads is p.
  - What is the probability of the sequence *HHHTT*?
    - $A_i = \{i\text{–th toss is a head}\}$
    - $\mathrm{P}\left(A_1 \cap A_2 \cap A_3 \cap \overline{A}_4 \cap \overline{A}_5\right) = \mathrm{P}(A_1)\mathrm{P}(A_2)\mathrm{P}(A_3)\mathrm{P}\left(\overline{A}_4\right)\mathrm{P}\left(\overline{A}_5\right) = p^3(1-p)^2$
  - What is the probability that exactly three heads come up?
    - $\mathrm{P}(\text{exactly three heads come up}) = \binom{5}{3}p^3(1-p)^2$

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Random Variables

- A random variable (r.v.) is a real-valued function of the experimental outcome.
- Ex. 6: Consider an experiment involving three independent tosses of a fair coin.
  - $\Omega = \{HHH,\ HHT,\ HTH,\ THH,\ HTT,\ THT,\ TTH,\ TTT\}$
  - $X(s) =$ the number of heads that appear for outcome $s \in \Omega$. Then

    $X(HHH) = 3, X(HHT) = X(HTH) = X(THH) = 2,$

    $X(HTT) = X(THT) = X(TTH) = 1,\ X(TTT) = 0$
  - $\mathrm{P}(X = 2) = \mathrm{P}(\{s \in \Omega: X(s) = 2) = \mathrm{P}(\{HHT, HTH, THH\}) = 3/8$
  - $\mathrm{P}(X < 2) = \mathrm{P}(\{HTT, THT, TTH, TTT\}) = 4/8 = 1/2$

Random Variables

- A random variable is a real-valued function of the outcome of the experiment.
  - A random variable is called discrete if the sample space Ω is finite or countably infinite
- We can associate with each random variable certain "averages" of interest, such as the expected value and the variance.

Expected Value

- The expected value (also called the expectation or the mean) of a random variable X on the sample space Ω is equal to

$$E(X) = \sum_{s \in \Omega} X(s)\, \mathrm{P}(\{s\})$$

- Ex. 7: Consider an experiment of tossing a biased coin once where the probability of heads is p.
  - $\Omega = \{H, T\}$
  - Let X be a r.v. where $X = 1$ if "Head" and $X = 0$ if "Tail"
  - $E(X) = 1 \cdot p + 0 \cdot (1 - p) = p$

Expected Value

- Ex. 8: Consider an experiment involving three independent tosses of a biased coin in which the probability of heads is p
  - $\Omega = \{HHH,\ HHT,\ HTH,\ THH,\ HTT,\ THT,\ TTH,\ TTT\}$
  - $X(s)$ = the number of heads that appear for outcome $s \in \Omega$
  - $E(X) = 3 \cdot p^3 + 2 \cdot p^2(1-p) \cdot\ 3 + 1 \cdot\ p(1-p)^2 \cdot 3 + 0 \cdot (1-p)^3$

    $= \ 3p^3 + 6p^2(1-p) + 3p(1-p)^2$

    $= 3p$

Expected Value

- Ex. 9: Consider an experiment involving n independent tosses of a biased coin in which the probability of heads is p
  - $X(s) =$ the number of heads that appear for outcome $s \in \Omega$
  - $E(X) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$

    $= np$