Midterm Review

CMPS/MATH 2170: Discrete Mathematics

Overview

- Midterm
  - closed book, closed notes, one page cheat sheet (single-sided) allowed
  - Time & Place: **Thursday, Oct 18, 5:00 pm- 6:15 pm,** Gibson Hall 126
- Office hours in the week of Oct 15
  - Lecturer: MTW 11-12 pm, Stanly Thomas 307B
  - TA: Tue 3:30-5:30 pm, Stanly Thomas 309

Topics

Propositional logic: 1.1-1.3

Predicate logic: 1.4-1.5

Intro to Proofs: 1.6-1.8

Sets and Set Operations: 2.1-2.2

Functions: 2.3

Cardinality of Sets: 2.5

Mathematical Induction: 5.1

Propositional Logic (1.1-1.3)

- A *proposition* is a declarative sentence that is either true or false, but not both
- *Compound propositions* can be formed from simple propositions using connectives (logical operators)
- Logical operators: $\neg$, $\wedge$, $\vee$, $\oplus$, $\rightarrow$, $\leftrightarrow$
- Translation: from English to logic, and logic to English
- Logical equivalences: $A \equiv B$ ($A \leftrightarrow B$ is a tautology)
  - Proving logical equivalences using truth tables
  - Proving logical equivalences using known logical equivalences
- ~~Representing Truth Tables: Disjunctive Normal Form (DNF)~~

Key Logical Equivalences

- Identity laws: $p \wedge \mathbf{T} \equiv p$ $\quad p \vee \mathbf{F} \equiv p$
- Domination laws: $p \vee \mathbf{T} \equiv \mathbf{T}$ $\quad p \wedge \mathbf{F} \equiv \mathbf{F}$
- Idempotent laws: $p \vee p \equiv p$ $\quad p \wedge p \equiv p$
- Double negation law: $\neg(\neg p) \equiv p$
- Negation laws: $p \vee \neg p \equiv \mathbf{T}$ $\quad p \wedge \neg p \equiv \mathbf{F}$

➢ p and q can be substituted by any propositional forms.

Key Logical Equivalences

- Commutative laws: $p \vee q \equiv q \vee p$ $\quad p \wedge q \equiv q \wedge p$
- Associative laws: $(p \vee q) \vee r \equiv p \vee (q \vee r)$ $\quad (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive Laws: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- De Morgan's laws: $\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\quad \neg(p \vee q) \equiv \neg p \wedge \neg q$
- Absorption laws: $p \vee (p \wedge q) \equiv p$ $\quad p \wedge (p \vee q) \equiv p$

Key Logical Equivalences

- Implication law: $p \to q \equiv \neg p \lor q$

- Contrapositive law: $p \to q \equiv \neg q \to \neg p$

- Logical equivalences involving biconditional statements

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$

$$p \leftrightarrow q \equiv \neg q \leftrightarrow \neg p$$

Predicates and Quantifiers (1.4-1.5)

- Statements involving subjects, predicates, and quantifiers
- Quantifiers: $\forall x\ P(x), \exists x\ P(x)$
- Nested Quantifiers
- Negating quantifiers using De Morgan's laws:

$$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x),\ \neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$$

- Translations of statements involving quantifiers
  - E.g., "Every real number has an inverse"

Rules of Inference (1.6)

- An argument: a sequence of propositions that end with a conclusion
- A valid argument: it is impossible for all the premises to be true and the conclusion to be false
- Rules of Interference: templates of valid arguments
  - Know how to use rules of inference to establish formal proofs

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore\ q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore\ \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore\ p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore\ q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism

p $\therefore\ p \lor q$	$p \rightarrow (p \lor q)$	Addition
$p \land q$ $\therefore\ p$	$(p \land q) \rightarrow p$	Simplification
p q $\therefore\ p \land q$	$((p) \land (q)) \rightarrow (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore\ q \lor r$	$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$	Resolution

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Using Rules of Inference to Build Arguments

Ex. 3: Suppose all these statements are known:

premises

"It is not sunny this afternoon and it is colder than yesterday" — "It is not sunny this afternoon" ($\neg p$), "it is colder than yesterday" (q): $\neg p \wedge q$

"We will go swimming only if it is sunny this afternoon" — "We will go swimming" (r), "it is sunny this afternoon" (p): $r \rightarrow p$

"If we do not go swimming, then we will take a canoe trip" — "we do not go swimming" ($\neg r$), "we will take a canoe trip" (s): $\neg r \rightarrow s$

"If we take a canoe trip, then we will be home by sunset" — "we take a canoe trip" (s), "we will be home by sunset" (t): $s \rightarrow t$

conclusion

Show that "We will be home by sunset" t

Intro to Proofs (1.7-1.8)

- Direct Proofs: want to show $p \to q$
- Proof by Contraposition: want to prove $p \to q$, actually prove $\neg q \to \neg p$
- Proof by Contradiction: want to prove p, actually prove $\neg p \to \mathbf{F}$
- Proof by Cases
- Prove a collection of statements are equivalent
- Existence and Uniqueness Proofs
- Know basic facts about integers, rational, and irrational numbers

Set Theory (2.1-2.2)

- A set is an unordered collection of objects (duplicates not allowed)
  - $A = \{1, 3, 5, 7, 9\} = \{x \in \mathbb{Z}^+ | x \text{ is odd and x} < 10\}$
  - Often used sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$
- Set relations: element of, subset of, equality
  - To prove $A \subseteq B$, show that for any a, if $a \in A$ then $a \in B$
  - To prove $A = B$, show that $A \subseteq B$ and $B \subseteq A$
- Power sets
- Cartesian products of sets
- Set operations: $A \cup B$, $A \cap B$, $A \backslash B$, $\bar{A}$

Set Identities

TABLE 1 Set Identities.

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Functions (2.3)

- Definition of a function: domain, codomain, range, image, preimage
- Injection, Surjection, Bijection - you should be able to prove or disprove a function is any of these, and give examples
  - Pay attention to the domain and the codomain of a function
- Inverse Functions
- Composition of Functions
- Floor and Ceiling Functions

Cardinality (2.5)

- Finite set: $|S| = n$ if S contains n distinct elements

  $|\mathcal{P}(A)| = 2^{|A|}$

  $|A \times B| = |A||B|$

  $|A \cup B| = |A| + |B| - |A \cap B|$

- $|A| = |B|$ if there is a bijection between A and B
- $|A| \leq |B|$ if there is an injection from A to B
- A set S is countably infinite if $|S| = |\mathbb{Z}^+|$: $O^+,\ \mathbb{Z},\ \mathbb{Q}^+$
- A set is countable if it is finite or countably infinite
- Uncountable sets: $\mathbb{R}$, (0,1)

Cardinality

- To show that a set A is countably infinite
  - Find a bijection between $\mathbb{Z}^+$ and A
  - Find a way to list the elements of A in a sequence
  - Show that A is a subset of a countable set
- To show that a set A is uncountable
  - Find an injection from an uncountable set to A
  - Show that A is a superset of an uncountable set

Mathematical Induction (5.1)

- Want to prove $\forall n \in \mathbb{Z}^+ : P(n)$
  - Base case: verify that $P(1)$ is true
  - Inductive step: show that $P(k) \rightarrow P(k+1)$ for any $k \in \mathbb{Z}^+$

- Want to prove $P(n)$ is true for $n = b, b+1, b+2, \ldots,$ where $b \in \mathbb{Z}$
  - Base case: verify that $P(b)$ is true
  - Inductive step: show that $P(k) \rightarrow P(k+1)$ for any $k = b, b+1, b+2, \ldots$