Counting

CMPS/MATH 2170: Discrete Mathematics

Counting problems

- How many different subsets that a finite set *A* has?
- How many different North American telephones numbers are possible?
- How many different sequences of bases in the DNA of simple organisms?
- How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Outline

- The Basics of Counting (6.1)
- The Pigeonhole Principle (6.2)
- Permutations and Combinations (6.3, 6.5)

The Product Rule

Ex. 1: Two 6-sided dices (one red and one green) are rolled. How many different outcomes are there? $6 \cdot 6 = 36$

Ex. 2: How many bit strings of length seven are there? $2^7 = 128$

Ex. 3: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$$

The product rule: suppose that a procedure can be broken down into a sequence of $T_1, T_2, \ldots, T_m$ tasks. If each task T_i can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 n_2 \ldots n_m$ ways to carry out the procedure.

The Product Rule

Ex. 4: How many functions are there from a set with m elements to a set with n elements? n^m

Ex. 5: How many injective functions are there from a set with m elements to a set with n elements?

$$0 \text{ if } m > n$$

$$n(n-1)(n-2)\cdots(n-m+1) \text{ if } m \leq n$$

The Sum Rule

Ex. 6: 30 students are taking CMPS/MATH 2170 this semester, and 35 students are taking MATH 3070 (Intro To Probability) this semester. Assume no one is taking both courses. How many students are taking 2170 or 3070 this semester?

$$30 + 35 = 65$$

Ex. 7: $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$ when $A_i \cap A_j = \emptyset$ for all i, j.

The sum rule: if a task can be done either in one of n_1 ways, in one of n_2 ways, … or in one of n_m ways, with all these ways different from each other, then there are $n_1 + n_2 + \cdots + n_m$ ways to do the task.

The Subtraction Rule

Ex. 8: 30 students are taking CMPS/MATH 2170 this semester, 35 students are taking MATH 3070 this semester, and 5 students are taking both courses. How many students are taking 2170 or 3070 this semester? $30 + 35 - 5 = 60$

Ex. 9: How many bit strings of length eight start with a 1 bit or end with the two bits 00? $2^7 + 2^6 - 2^5 = 128 + 64 - 32 = 160$

Ex. 10: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ for finite sets A_1 and A_2

The subtraction rule: if a task can be done either in n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways that are common in both sets. (Inclusion-exclusion principle)

Solving Counting problems

Find structures in a counting problem

- Break it into stages $\Rightarrow$ product rule
- Break it into disjoint subcases $\Rightarrow$ sum rule
    - If the subcases are non-disjoint $\Rightarrow$ subtraction rule

More Complex Counting Problems

Ex. 11: How many possible passwords of length 8? Assume each character in a password is an uppercase letter or a digit, and each password must contain at least one digit. $36^8 - 26^8$

Ex. 12: How many ways are there to form a three-letter word using a, b, c, d, e, f?

1. With repetition of letters allowed? $6 \cdot 6 \cdot 6 = 216$
2. Without repetition? $6 \cdot 5 \cdot 4 = 120$
3. Without repetition and containing letter e? $6 \cdot 5 \cdot 4 - 5 \cdot 4 \cdot 3 = 60$
4. With repetition and containing letter e? $6 \cdot 6 \cdot 6 - 5 \cdot 5 \cdot 5 = 91$

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- The Basics of Counting (6.1)
- The Pigeonhole Principle (6.2)
- Permutations and Combinations (6.3,6.5)

The Pigeonhole Principle

The Pigeonhole Principle: if $k + 1$ or more objects (pigeons) are placed into k boxes (pigeonholes), then there is at least one box containing two or more of the objects.

- Proof by contraposition

The Pigeonhole Principle

Ex. 1: Among any group of 367 people, there must be at least two with the same birthday.

Ex. 2: In any group of 27 English words, there must be at least two that begin with the same letter.

The Generalized Pigeonhole Principle

Ex. 3: Among 100 people there are at least 9 who were born in the same month.

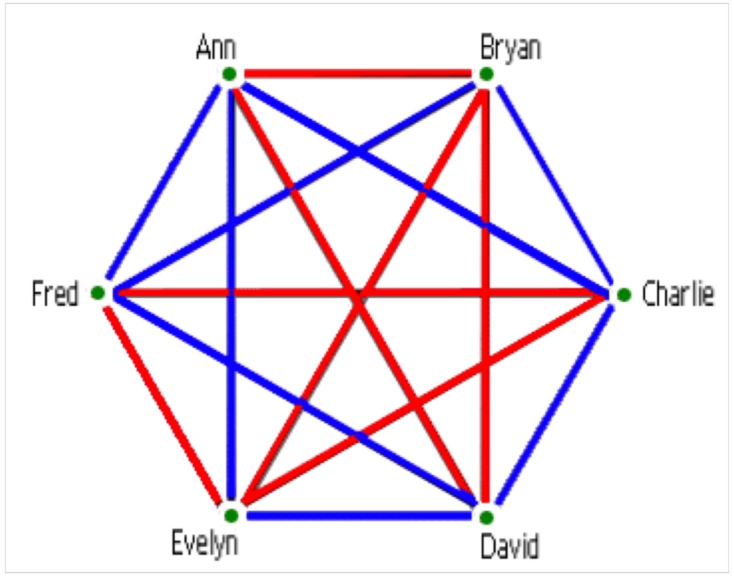
The Generalized Pigeonhole Principle: if N objects (pigeons) are placed into k boxes (pigeonholes), then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Ex. 4: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

$$\lceil N/4 \rceil = 3 \quad \Rightarrow N = 9$$

More Applications of the Pigeonhole Principle

Ex. 5: In any party of 6 people where every two people are either friends or strangers, at least three of them are mutual friends or at least three of them are mutual strangers.



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Permutations and Combinations

Ex. 1: In how many ways can we select three students from a group of four students to stand in a line for a picture? $4 \cdot 3 \cdot 2 = 24$

Ex. 2: How many different committees of three students can be formed from a group of four students? $= \#$ 3-element subsets $= \#$ 1-element subsets $= 4$

Permutations

Definition: A permutation of a set of n elements is an ordering of these elements. An r-permutation of a set of n elements is an ordering of a subset of $r \leq n$ of these elements (no repetition allowed)

Ex. 3: $S = \{a, b, c\}$

1-permutations: $a,\ b,\ c$

2-permutations: $ab,\ ba,\ ac,\ ca,\ bc,\ cb$

3−permutations: $abc,\ acb,\ bac,\ bca,\ cab,\ cba$

0−permutation : $\{\}$

Permutations

Theorem: The number of r-permutations of a set with n elements where $1 \le r \le n$ equals to $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$

- Proof by the product principle

Define $P(n, 0) = 1$: there is exactly one way to order zero elements.

Then for $0 \le r \le n$, $P(n, r) = \frac{n!}{(n-r)!}$. In particular, $P(n, n) = n!$

Ex. 4: How many permutations of the letters $ABCDEFGH$ contain the string ABC?

$P(6,6) = 6! = 720$

Combinations

How many different committees of three students can be formed from a group of four students? $= \#\ 3\text{-element subsets} = \#\ 1\text{-element subsets} = 4$

Definition: An r-combination of a set of n elements is an unordered selection of r elements from the set (a subset of r of the n elements)

- No repetition allowed

Ex. 5: $S = \{a, b, c\}$

1-combinations: $\{a\}$, $\{b\}$, $\{c\}$

2-combinations: $\{a, b\}$, $\{a, c\}, \{b, c\}$

3-combinations: $\{a, b, c\}$

0-combination: $\{\}$

Combinations

Theorem: The number of r-combinations of a set with n elements where $0 \le r \le n$ equals $C(n,r) = \frac{n!}{r!(n-r)!}$

- $C(n,r)$ is also denoted by $\binom{n}{r}$ (often read as "n choose k") and is called a binomial coefficient
- $C(n,r) = C(n, n-r)$

Ex. 6: A committee of 2 men and 2 women is to be chosen from 5 men and 7 women. How many ways are there to form a committee? $\binom{5}{2}\binom{7}{2} = 210$

Ex. 7: How many 5-card hands can be formed from a 52-card deck? $\binom{52}{5} = 2{,}598{,}960$

How many flushes (5 cards of same suit) are there? $\binom{13}{5} \times 4 = 5148$

Permutations with Repetition

Ex.8: How many strings of length r can be formed from the uppercase letters of the English alphabet? 26^r

Theorem: The number of r-permutations of a set of n elements with repetition allowed is n^r

Combinations with Repetition

- Ex.9: How many ways are there to select five bills from a cash box containing \$1 bills, \$5 bills, and \$10 bills? Assume that the order in which the bills are chosen does not matter and there are at least five bills of each type. $\binom{5+2}{2}$

- Theorem: The number of r-combinations of a set of n elements with repetition allowed is $\binom{r+n-1}{n-1} = \binom{r+n-1}{r}$

Permutations and Combinations

	Permutations	Combinations
Without repetition	$P(n,r) = \dfrac{n!}{(n-r)!}$	$C(n,r) = \binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$
With repetition	n^r	$\binom{r+n-1}{r}$