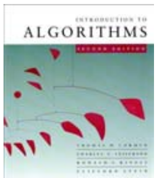
CS 5633 -- Spring 2006



Red-black trees

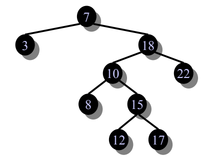
Carola Wenk

Slides courtesy of Charles Leiserson with small changes by Carola Wenk

Search Trees

• A binary search tree is a binary tree. Each node stores a key. The tree fulfills the **binary search tree property**:

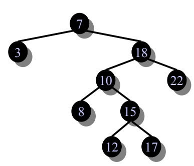
For every node x holds:
- $left(x) \leq x$, if x's left child $left(x)$ exists
- $x \leq right(x)$, if x's right child $right(x)$ exists



Search Trees

Different variants of search trees:

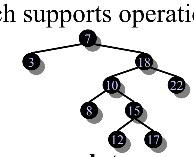
- Balanced search trees (guarantee height of $log\ n$ for n elements)
- k-ary search trees (such as B-trees, 2-3-4-trees)
- Search trees that store the keys only in the leaves, and store additional split-values in the internal nodes



ADT Dictionary / Dynamic Set

Abstract data type (ADT) Dictionary
(also called Dynamic Set):

A data structure which supports operations
- Insert
- Delete
- Find



Using **balanced binary search trees** we can implement a dictionary data structure such that each operation takes $O(\log n)$ time.

Balanced search trees

Balanced search tree: A search-tree data structure for which a height of $O(\log n)$ is guaranteed when implementing a dynamic set of n items.

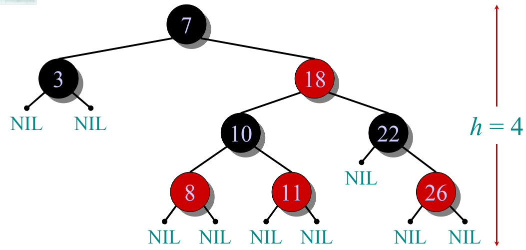
Examples:
- AVL trees
- 2-3 trees
- 2-3-4 trees
- B-trees
- Red-black trees

Red-black trees

This data structure requires an extra one-bit color field in each node.

Red-black properties:
1. Every node is either red or black.
2. The root is black.
3. The leaves (NIL's) are black.
4. If a node is red, then both its children are black.
5. All simple paths from any node x, excluding x, to a descendant leaf have the same number of black nodes = black-height(x).

Example of a red-black tree



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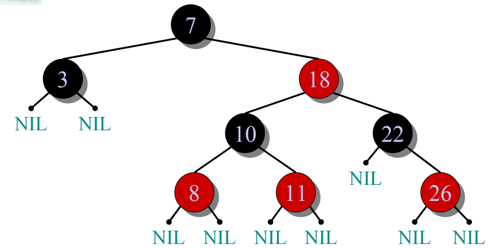

1. Every node is either red or black.

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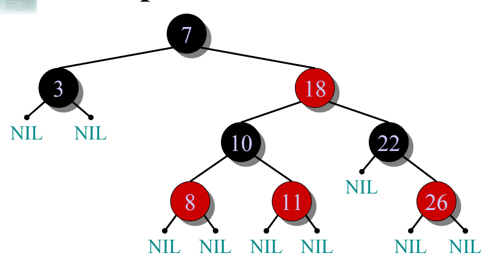
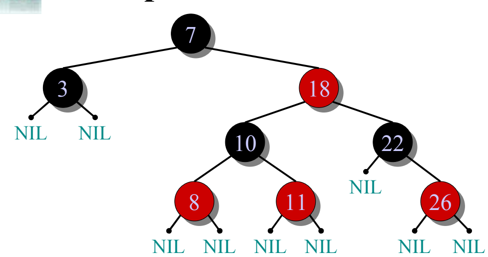


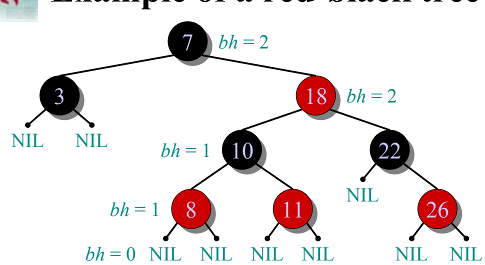
2., 3. The root and leaves (NIL's) are black.

Example of a red-black tree



4. If a node is red, then both its children are black.

Example of a red-black tree



5. All simple paths from any node x, excluding x, to a descendant leaf have the same number of black nodes = *black-height*(x).

Height of a red-black tree

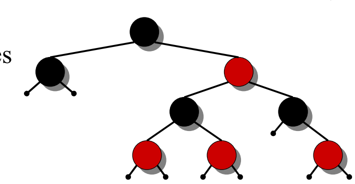
Theorem. A red-black tree with n keys has height

$$h \le 2 \log(n + 1).$$

Proof. (The book uses induction. Read carefully.)

INTUITION:

- Merge red nodes into their black parents.



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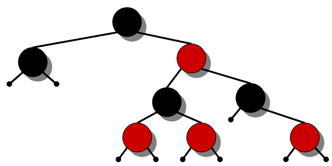

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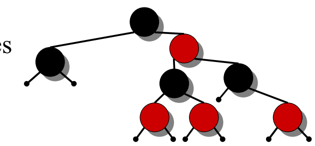

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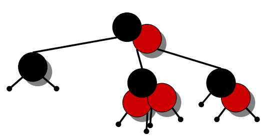

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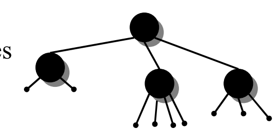

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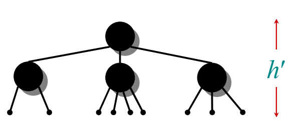
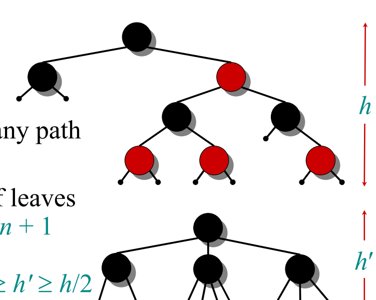


- This process produces a tree in which each node has 2, 3, or 4 children.
- The 2-3-4 tree has uniform depth h' of leaves.

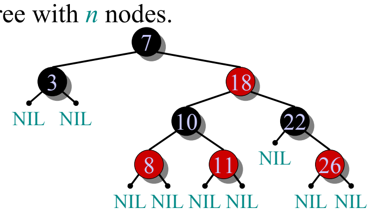
Proof (continued)

- We have $h' \geq h/2$, since at most half the leaves on any path are red.
- The number of leaves in each tree is $n+1$
  $\Rightarrow n+1 \geq 2^{h'}$
  $\Rightarrow \log(n+1) \geq h' \geq h/2$
  $\Rightarrow h \leq 2\log(n+1)$. □



Query operations

Corollary. The queries SEARCH, MIN, MAX, SUCCESSOR, and PREDECESSOR all run in $O(\log n)$ time on a red-black tree with n nodes.

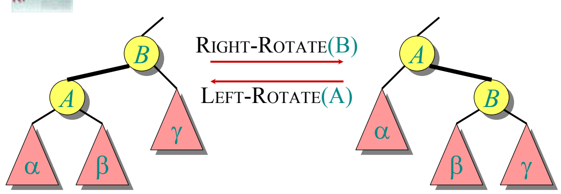


Modifying operations

The operations INSERT and DELETE cause modifications to the red-black tree:

1. the operation itself,
2. color changes,
3. restructuring the links of the tree via ***"rotations"***.

Rotations



- Rotations maintain the inorder ordering of keys:
  $a \in \alpha, b \in \beta, c \in \gamma \Rightarrow a \le A \le b \le B \le c$.
- Rotations maintain the binary search tree property
- A rotation can be performed in $O(1)$ time.

Red-black trees

This data structure requires an extra one-bit color field in each node.

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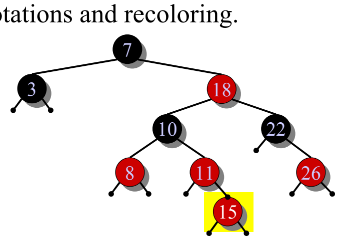
Insertion into a red-black tree

IDEA: Insert x in tree. Color x red. Only red-black property 4 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

Example:

- Insert x =15.



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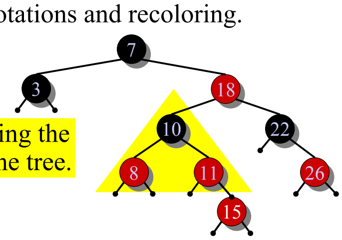
Example:

- Insert x =15.
- Recolor, moving the violation up the tree.



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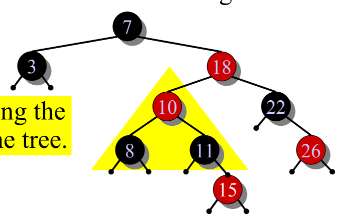

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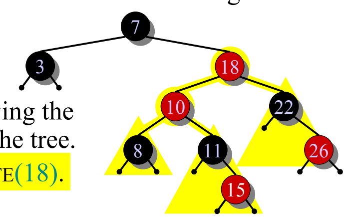
- RIGHT-ROTATE(18).



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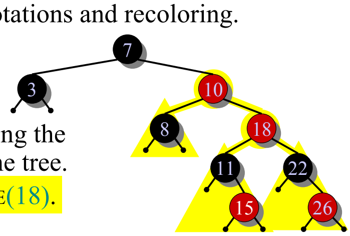

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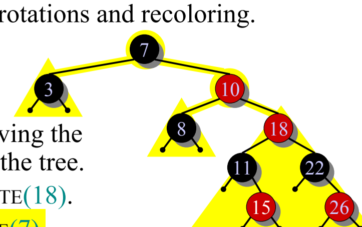
- RIGHT-ROTATE(18).
- LEFT-ROTATE(7)



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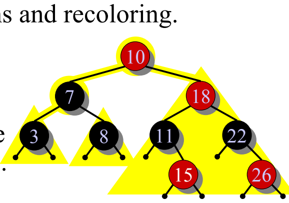

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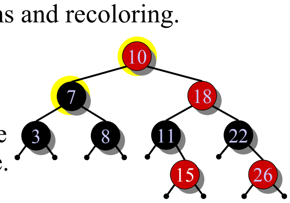
Example:
- Insert x =15.
- Recolor, moving the violation up the tree.
- RIGHT-ROTATE(18).
- LEFT-ROTATE(7) and recolor.



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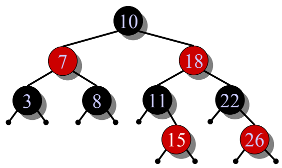


Pseudocode

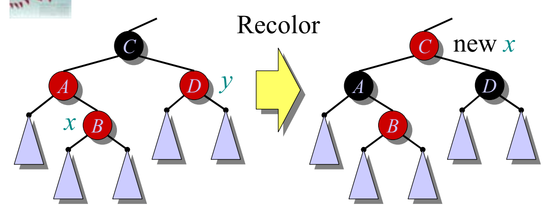
```
RB-INSERT(T, x)
    TREE-INSERT(T, x)
    color[x] ← RED      ▷ only RB property 4 can be violated
    while x ≠ root[T] and color[p[x]] = RED
       do if p[x] = left[p[p[x]]
            then y ← right[p[p[x]]      ▷ y = aunt/uncle of x
                 if color[y] = RED
                  then ⟨Case 1⟩
                  else  if x = right[p[x]]
                          then ⟨Case 2⟩    ▷ Case 2 falls into Case 3
                        ⟨Case 3⟩
            else ⟨"then" clause with "left" and "right" swapped⟩
    color[root[T]] ← BLACK
```


Graphical notation

Let △ denote a subtree with a black root.

All △'s have the same black-height.

Case 1



(Or, A's children are swapped.)

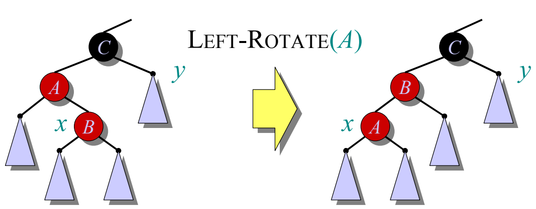
$p[x] = left[p[p[x]]$

$y = right[p[p[x]]$

$color[y]$ = RED

Push C's black onto A and D, and recurse, since C's parent may be red.

Case 2



$p[x] = left[p[p[x]]$

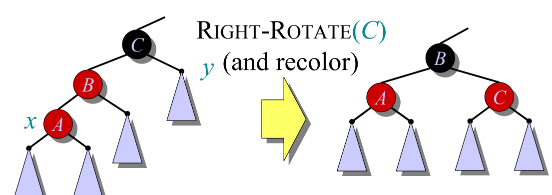
$y = right[p[p[x]]$

$color[y]$ = BLACK

$x = right[p[x]]$

Transform to Case 3.

Case 3



$p[x] = left[p[p[x]]$

$y = right[p[p[x]]$

$color[y]$ = BLACK

$x = left[p[x]]$

Done! No more violations of RB property 4 are possible.

Analysis

- Go up the tree performing Case 1, which only recolors nodes.
- If Case 2 or Case 3 occurs, perform 1 or 2 rotations, and terminate.

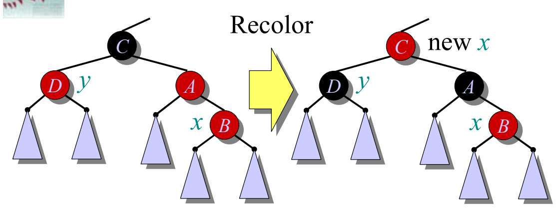
Running time: $O(\log n)$ with $O(1)$ rotations.

RB-DELETE — same asymptotic running time and number of rotations as RB-INSERT (see textbook).

Pseudocode (part II)

else ⟨**"then"** clause with "*left*" and "*right*" swapped⟩
▹ $p[x] = right[p[p[x]]]$
then $y \leftarrow left[p[p[x]]]$ ▹ y = aunt/uncle of x
if $color[y]$ = RED
then ⟨**Case 1'**⟩
else if $x = left[p[x]]$
then ⟨**Case 2'**⟩ ▹ Case 2' falls into Case 3'
⟨**Case 3'**⟩
$color[root[T]] \leftarrow$ BLACK

Case 1'



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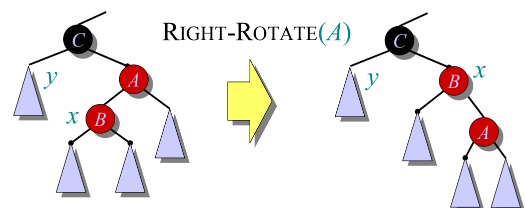
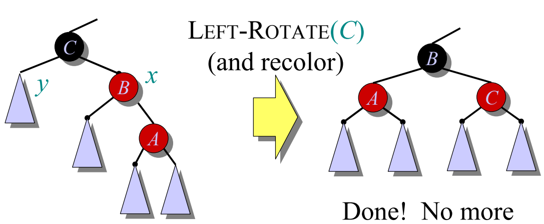
Case 2'



$p[x] = right[p[p[x]]]$
$y = left[p[p[x]]]$
$color[y]$ = BLACK
$x = left[p[x]]$

Transform to Case 3'.

Case 3'



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