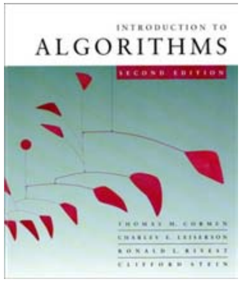
# CS 5633 -- Spring 2005




## *Quicksort*

**Carola Wenk**

Slides courtesy of Charles Leiserson with small changes by Carola Wenk



# Quicksort

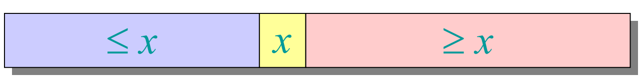
- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts “in place” (like insertion sort, but not like merge sort).
- Very practical (with tuning).



# Divide and conquer

Quicksort an $n$-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** $x$ such that elements in lower subarray $\leq x \leq$ elements in upper subarray.




2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

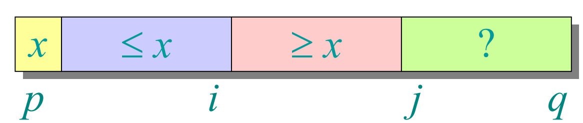
**Key:** *Linear-time partitioning subroutine.*

# Partitioning subroutine

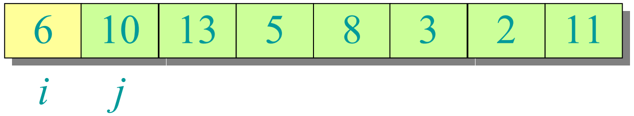
```
PARTITION(A, p, q)  ⊳ A[p . . q]
    x ← A[p]          ⊳ pivot = A[p]
    i ← p
    for j ← p + 1 to q
        do if A[j] ≤ x
              then  i ← i + 1
                    exchange A[i] ↔ A[j]
    exchange A[p] ↔ A[i]
    return i
```

Running time $= O(n)$ for $n$ elements.

***Invariant:***



# Example of partitioning



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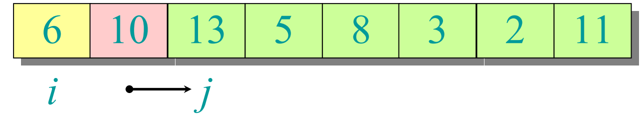




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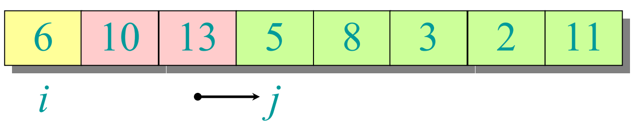


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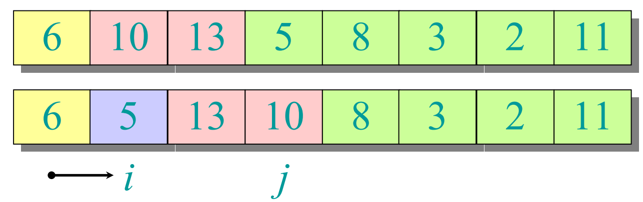

# Example of partitioning

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | 5 | 13 | 10 | 8 | 3 | 2 | 11 |
| | *i* | | | *j* | | | |



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→ *i* (under 2)  *j* (under 10)



# Example of partitioning

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*i* (under 2)  → *j* (under 11)



# Example of partitioning

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| 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |
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| 6 | 5 | 3 | 2 | 8 | 13 | 10 | 11 |

*i* (under 2)  → *j* (past the end)

# Example of partitioning

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |
| 6 | 5 | 13 | 10 | 8 | 3 | 2 | 11 |
| 6 | 5 | 3 | 10 | 8 | 13 | 2 | 11 |
| 6 | 5 | 3 | 2 | 8 | 13 | 10 | 11 |
| 2 | 5 | 3 | 6 | 8 | 13 | 10 | 11 |

*i* (under 6)

# Pseudocode for quicksort

QUICKSORT$(A, p, r)$
  **if** $p < r$
    **then** $q \leftarrow$ PARTITION$(A, p, r)$
      QUICKSORT$(A, p, q-1)$
      QUICKSORT$(A, q+1, r)$

**Initial call:** QUICKSORT$(A, 1, n)$



# Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of $n$ elements.



# Worst-case of quicksort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \end{aligned}$$

***(arithmetic series)***

# Worst-case recursion tree

$T(n) = T(0) + T(n-1) + cn$

## Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

$T(n)$



## Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

$cn$

$T(0)$ $T(n-1)$



## Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

$cn$

$T(0)$ $c(n-1)$

$T(0)$ $T(n-2)$

## Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

$cn$

$T(0)$ $c(n-1)$

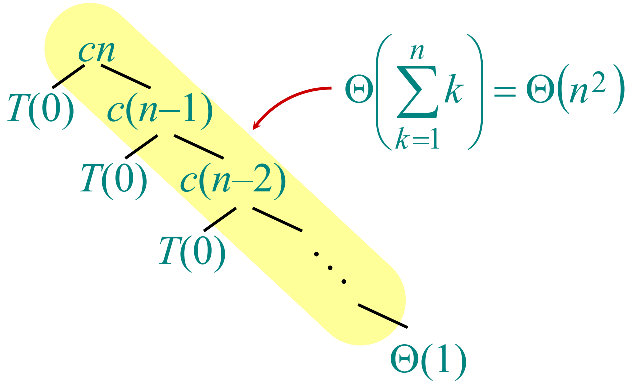
$T(0)$ $c(n-2)$

$T(0)$ $\ddots$

$\Theta(1)$

# Worst-case recursion tree

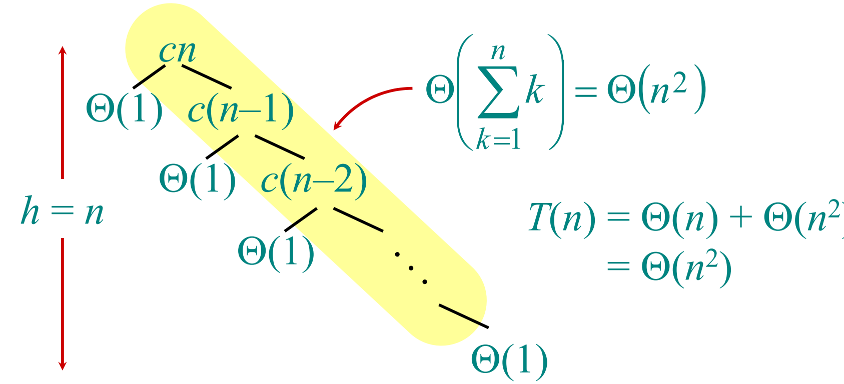
$T(n) = T(0) + T(n–1) + cn$






# Worst-case recursion tree

$T(n) = T(0) + T(n–1) + cn$






# Best-case analysis

***(For intuition only!)***

If we're lucky, PARTITION splits the array evenly:

$$T(n) = 2T(n/2) + \Theta(n)$$
$$= \Theta(n \log n) \quad \text{(same as merge sort)}$$

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

# Analysis of "almost-best" case

$T(n)$

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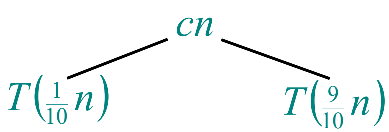
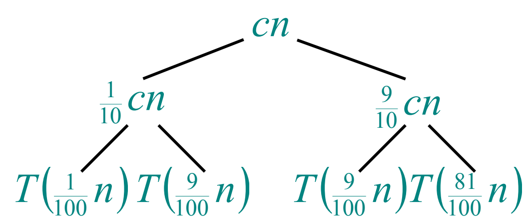


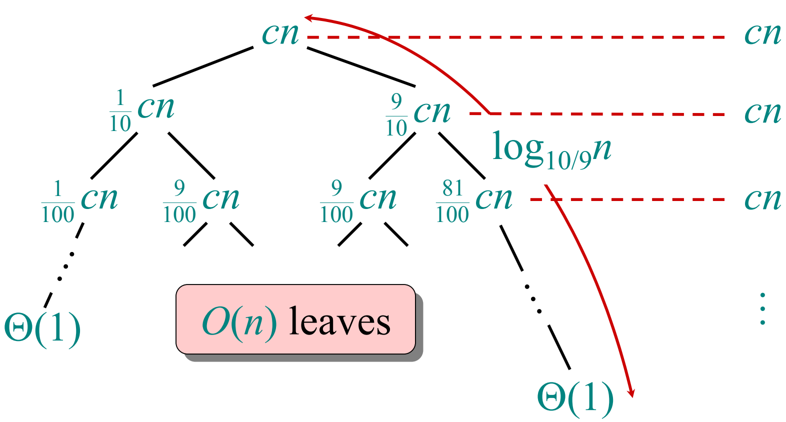



# Analysis of “almost-best” case

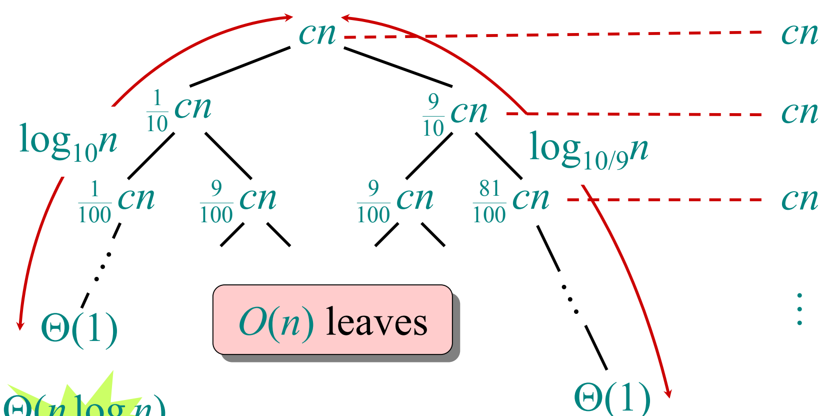





# Analysis of “almost-best” case




# Analysis of “almost-best” case




$\Theta(n \log n)$

***Lucky!***

$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$

# Randomized quicksort

**IDEA**: Partition around a ***random*** element.

- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.



# Randomized quicksort analysis

Let $T(n)$ = the random variable for the running time of randomized quicksort on an input of size $n$, assuming random numbers are independent.

For $k = 0, 1, \ldots, n–1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.



# Analysis (continued)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad\vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big).$$

# Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

Take expectations of both sides.

## Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$
$$= \sum_{k=0}^{n-1} E[X_k(T(k) + T(n-k-1) + \Theta(n))]$$

Linearity of expectation.



## Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$
$$= \sum_{k=0}^{n-1} E[X_k(T(k) + T(n-k-1) + \Theta(n))]$$
$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

Independence of $X_k$ from other random choices.



## Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$
$$= \sum_{k=0}^{n-1} E[X_k(T(k) + T(n-k-1) + \Theta(n))]$$
$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$
$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$ .

## Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$
$$= \sum_{k=0}^{n-1} E[X_k(T(k) + T(n-k-1) + \Theta(n))]$$
$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$
$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$
$$= \frac{2}{n}\sum_{k=0}^{n-1} E[T(k)] + \Theta(n)$$

Summations have identical terms.

# Hairy recurrence

$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

**Prove:** $E[T(n)] \le a n \log n$ for constant $a > 0$.

- Choose $a$ large enough so that $a n \log n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

**Use fact:** $\sum_{k=2}^{n-1} k \log k \le \frac{1}{2} n^2 \log n - \frac{1}{8} n^2$ (exercise).



# Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \log k + \Theta(n)$$

Substitute inductive hypothesis.



# Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \log k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2 \log n - \frac{1}{8}n^2\right) + \Theta(n)$$

Use fact.

# Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \log k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2 \log n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an \log n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as ***desired*** – ***residual***.

# Substitution method

$$
\begin{aligned}
E[T(n)] &\le \frac{2}{n}\sum_{k=2}^{n-1} ak\log k + \Theta(n) \\
&= \frac{2a}{n}\left(\frac{1}{2}n^2\log n - \frac{1}{8}n^2\right) + \Theta(n) \\
&= an\log n - \left(\frac{an}{4} - \Theta(n)\right) \\
&\le an\log n\ ,
\end{aligned}
$$

if $a$ is chosen large enough so that $an/4$ dominates the $\Theta(n)$.



# Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.