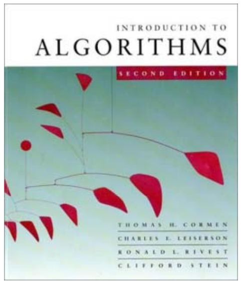



# CMPS 2200 – Fall 2012

# *More on Shortest Paths*

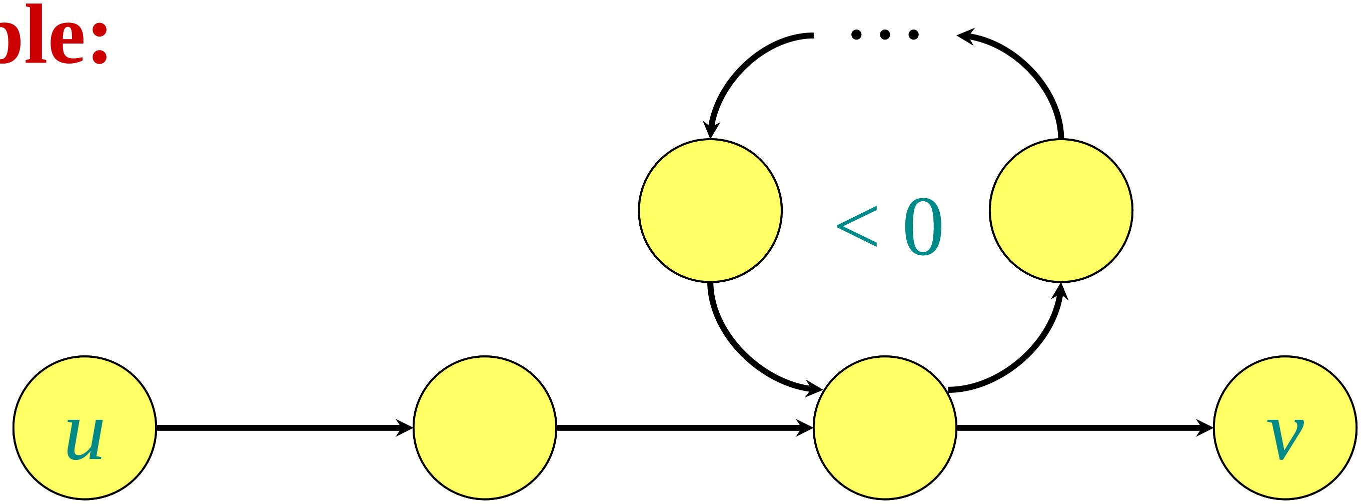
**Carola Wenk**

Slides courtesy of Charles Leiserson
with changes by Carola Wenk

# Negative-weight cycles

**Recall:** If a graph $G = (V, E)$ contains a negative-weight cycle, then some shortest paths may not exist.

**Example:**




***Bellman-Ford algorithm:*** Finds all shortest-path weights from a ***source*** $s \in V$ to all $v \in V$ or determines that a negative-weight cycle exists.

# Bellman-Ford algorithm

$d[s] \leftarrow 0$
**for** each $v \in V - \{s\}$
  **do** $d[v] \leftarrow \infty$

} initialization

**for** $i \leftarrow 1$ **to** $|V| - 1$ **do**
 **for** each edge $(u, v) \in E$ **do**
  **if** $d[v] > d[u] + w(u, v)$ **then**
   $d[v] \leftarrow d[u] + w(u, v)$
   $\pi[v] \leftarrow u$

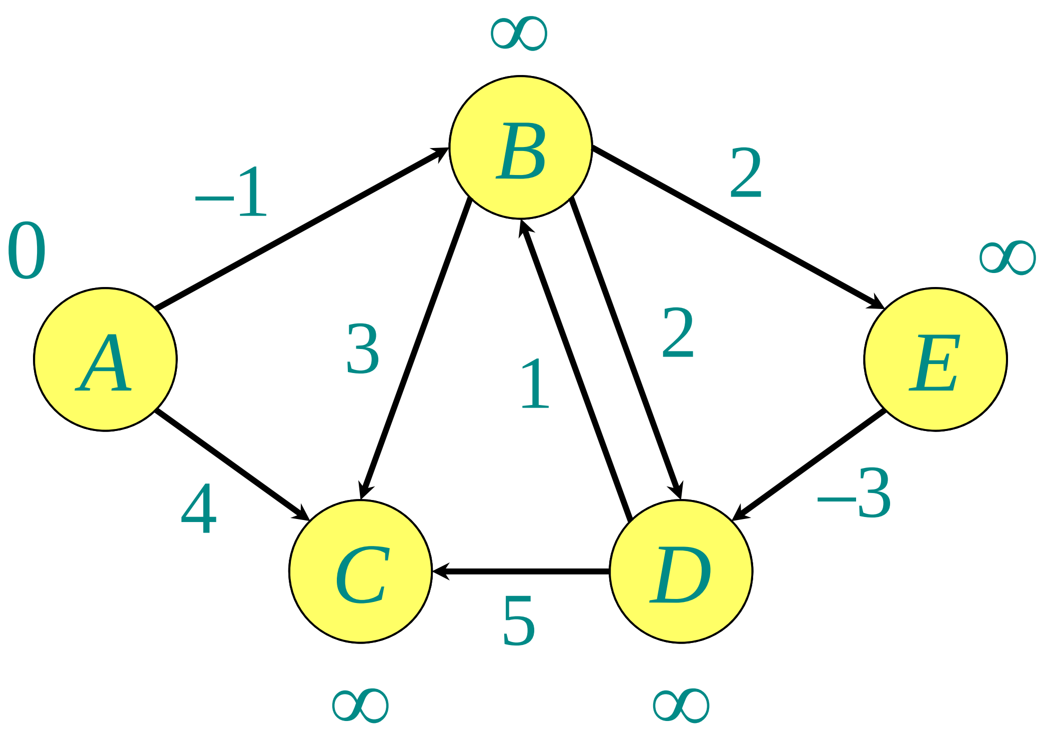
} ***relaxation step***

**for** each edge $(u, v) \in E$
  **do if** $d[v] > d[u] + w(u, v)$
   **then** report that a negative-weight cycle exists

At the end, $d[v] = \delta(s, v)$. Time $= O(|V||E|)$.

# Example of Bellman-Ford

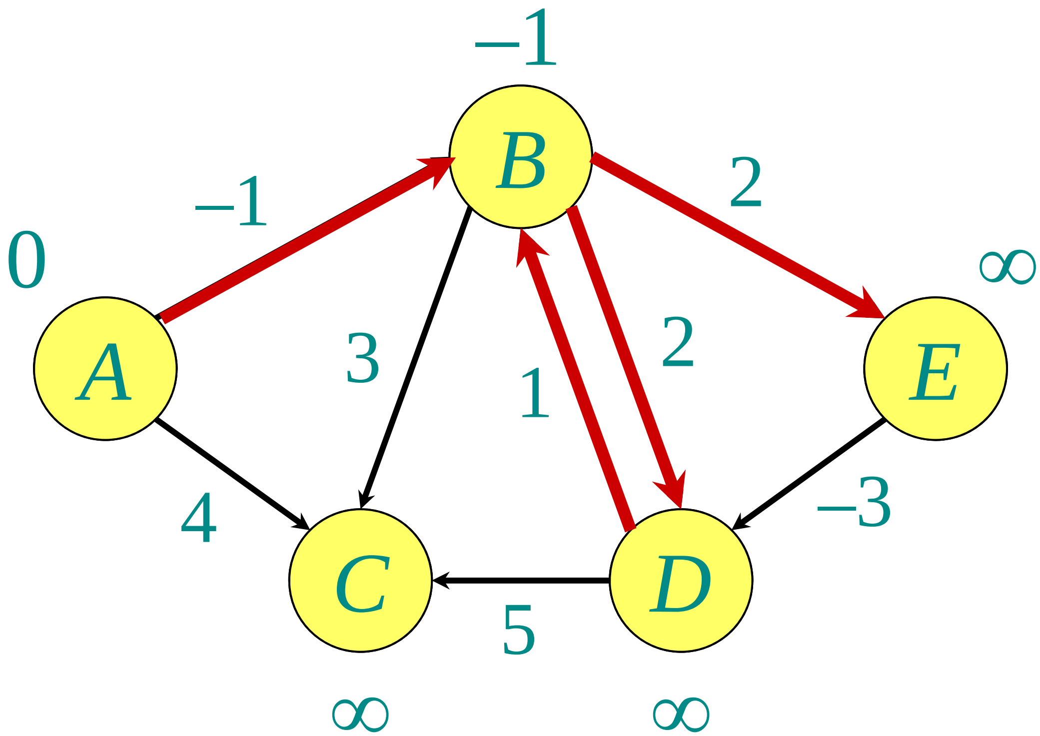
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*




| *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|
| 0 | ∞ | ∞ | ∞ | ∞ |

# Example of Bellman-Ford

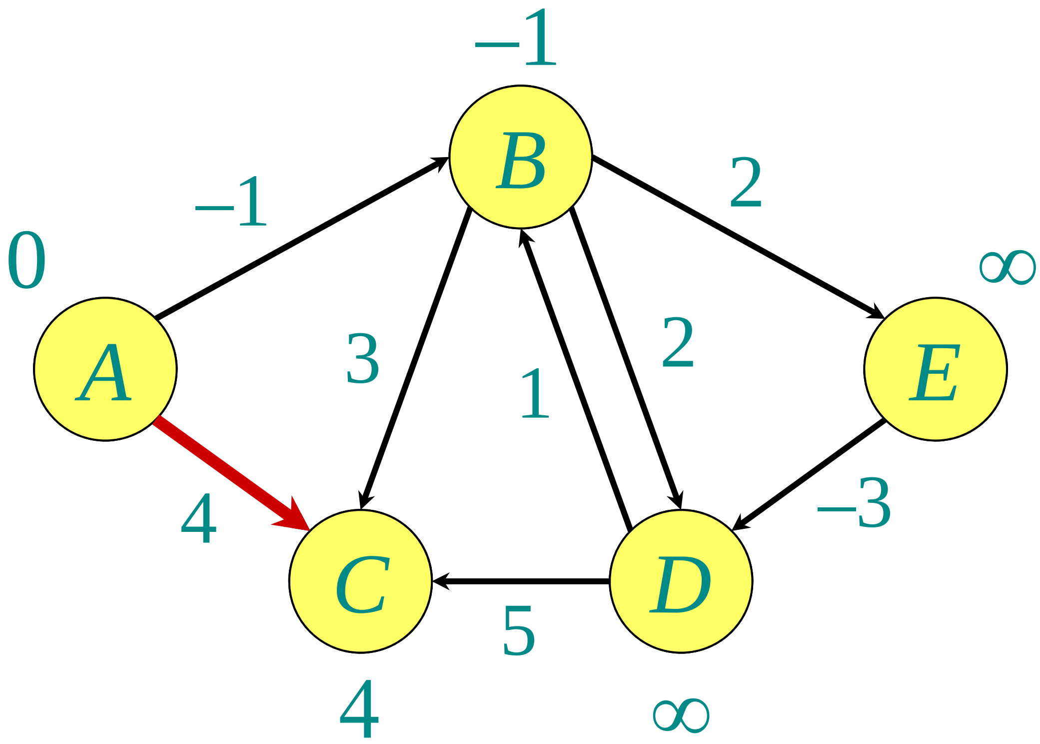
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*




| *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | –1 | ∞ | ∞ | ∞ |

# Example of Bellman-Ford

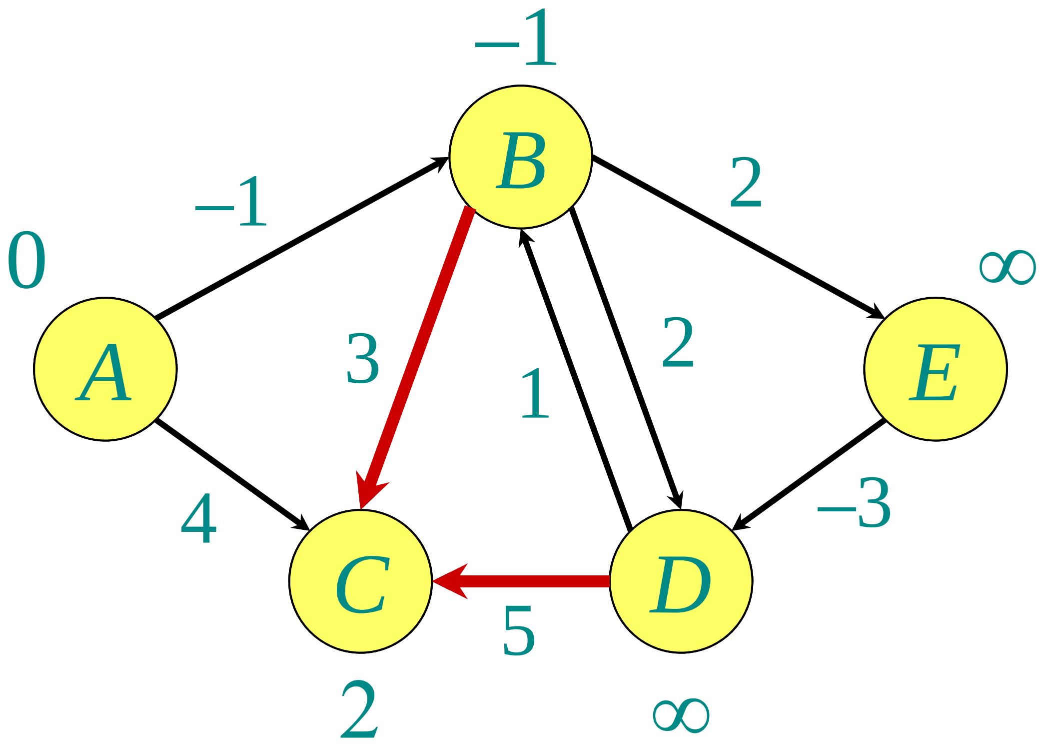
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*




| *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | –1 | ∞ | ∞ | ∞ |
| 0 | –1 | 4 | ∞ | ∞ |

# Example of Bellman-Ford

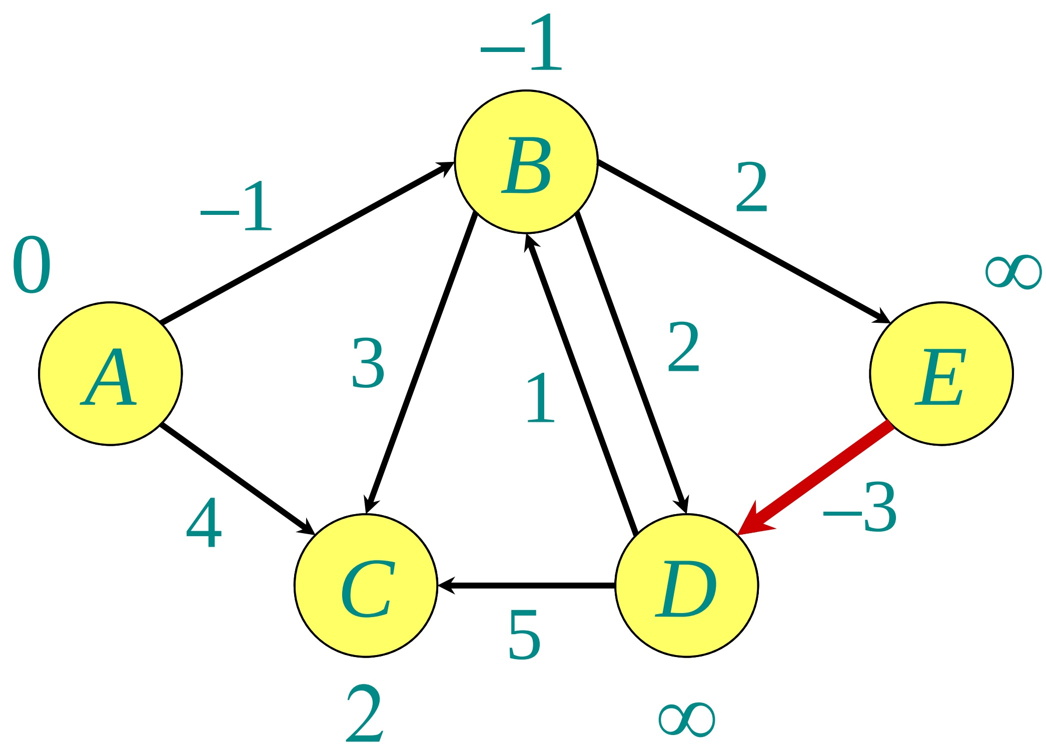
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*




| *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | –1 | ∞ | ∞ | ∞ |
| 0 | –1 | 4 | ∞ | ∞ |
| 0 | –1 | 2 | ∞ | ∞ |

# Example of Bellman-Ford

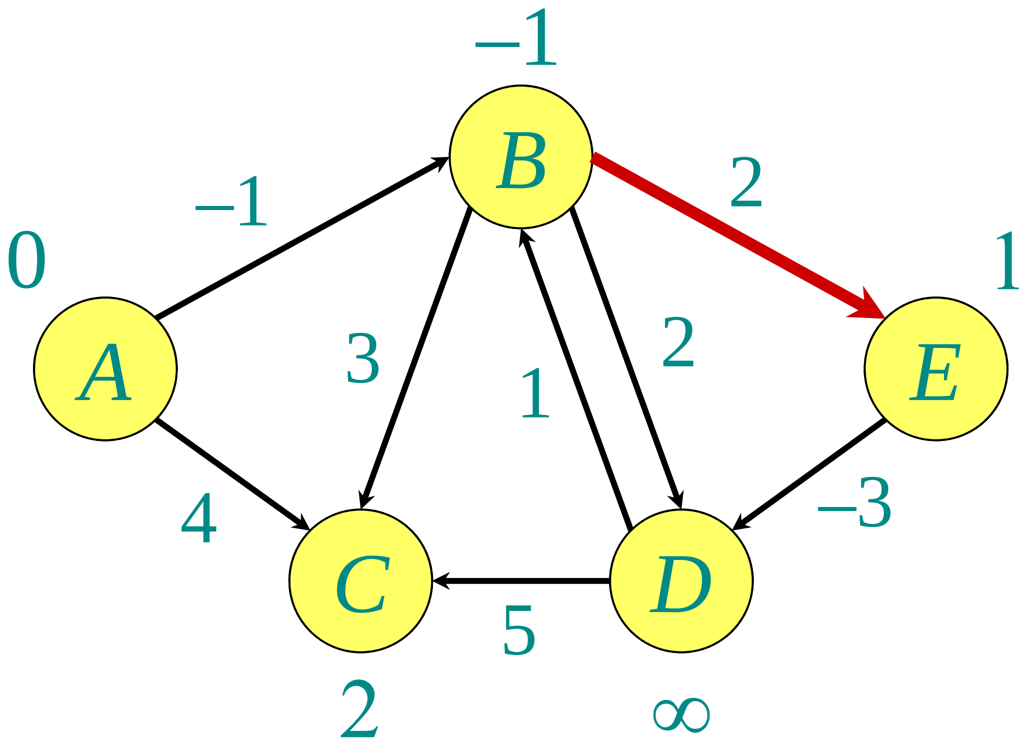
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*




| *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | –1 | ∞ | ∞ | ∞ |
| 0 | –1 | 4 | ∞ | ∞ |
| 0 | –1 | 2 | ∞ | ∞ |

# Example of Bellman-Ford

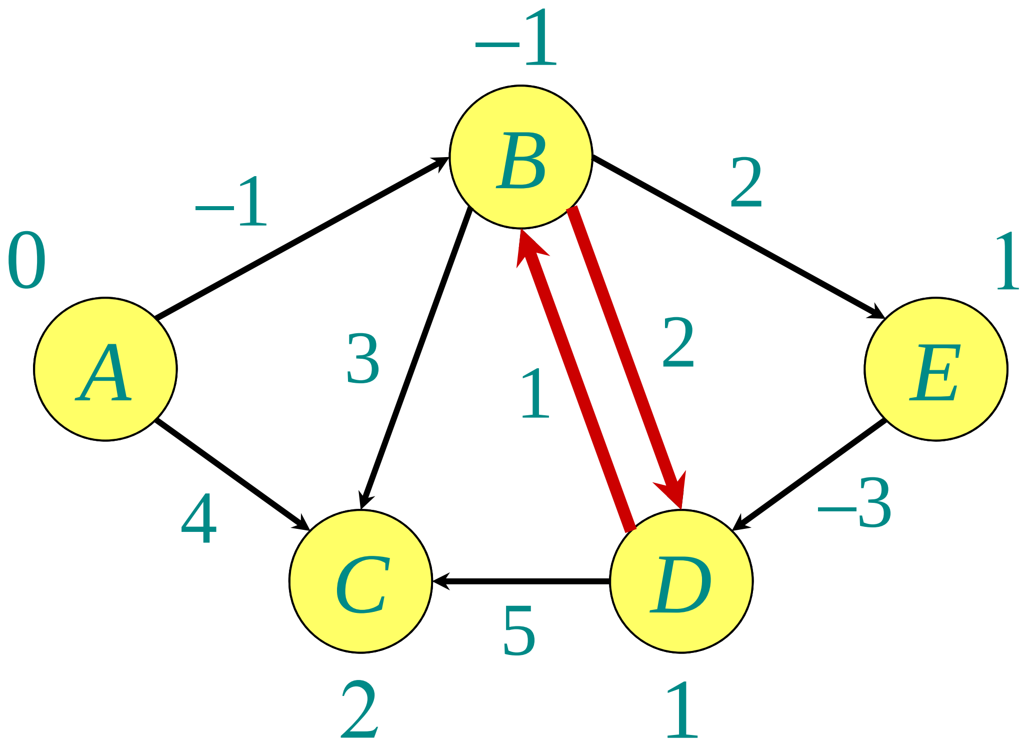
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*




| *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | –1 | ∞ | ∞ | ∞ |
| 0 | –1 | 4 | ∞ | ∞ |
| 0 | –1 | 2 | ∞ | ∞ |
| 0 | –1 | 2 | ∞ | 1 |

# Example of Bellman-Ford

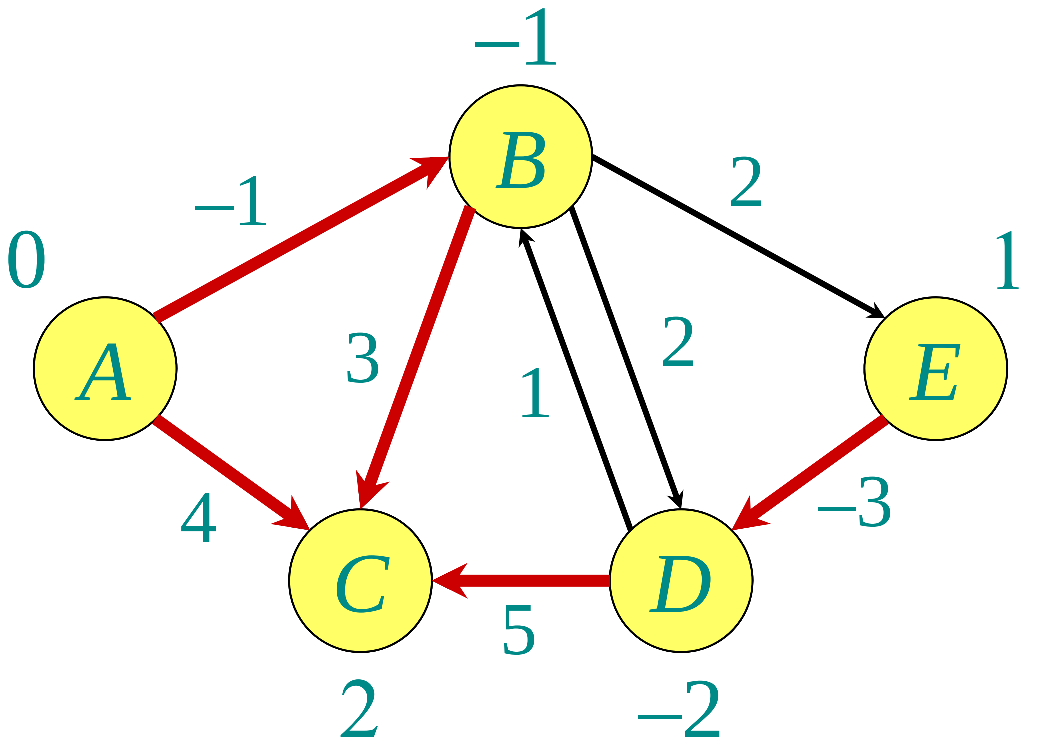
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*




| *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | –1 | ∞ | ∞ | ∞ |
| 0 | –1 | 4 | ∞ | ∞ |
| 0 | –1 | 2 | ∞ | ∞ |
| 0 | –1 | 2 | ∞ | 1 |
| 0 | –1 | 2 | 1 | 1 |

# Example of Bellman-Ford

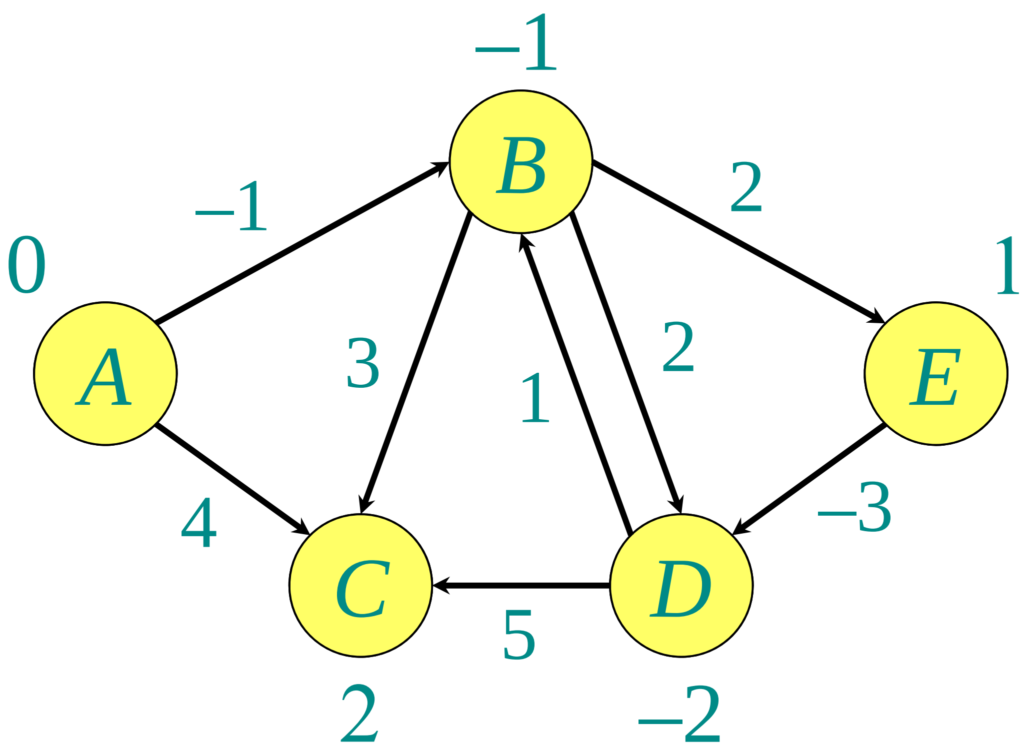
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*




| *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | –1 | ∞ | ∞ | ∞ |
| 0 | –1 | 4 | ∞ | ∞ |
| 0 | –1 | 2 | ∞ | ∞ |
| 0 | –1 | 2 | ∞ | 1 |
| 0 | –1 | 2 | 1 | 1 |
| 0 | –1 | 2 | –2 | 1 |

# Example of Bellman-Ford

Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*




| *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | –1 | ∞ | ∞ | ∞ |
| 0 | –1 | 4 | ∞ | ∞ |
| 0 | –1 | 2 | ∞ | ∞ |
| 0 | –1 | 2 | ∞ | 1 |
| 0 | –1 | 2 | 1 | 1 |
| 0 | –1 | 2 | –2 | 1 |

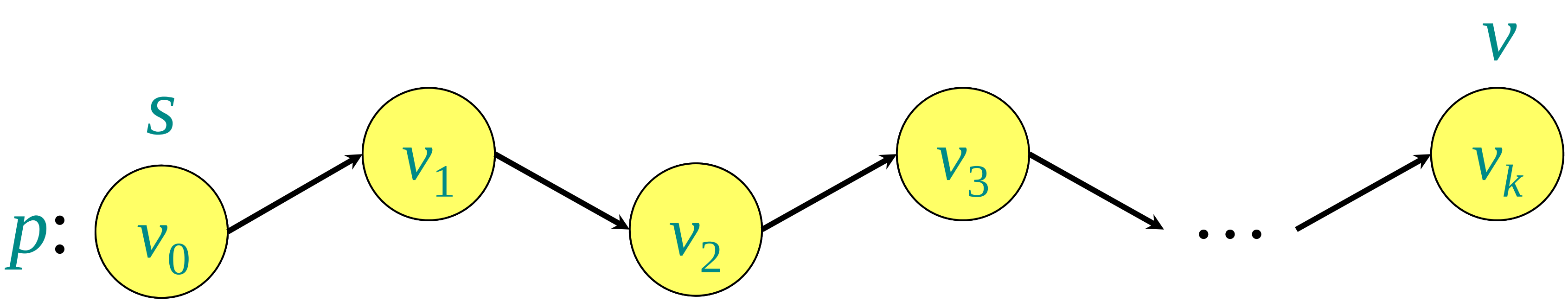
… and 2 more iterations

**Note:** Values decrease monotonically.

# Correctness

**Theorem.** If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.

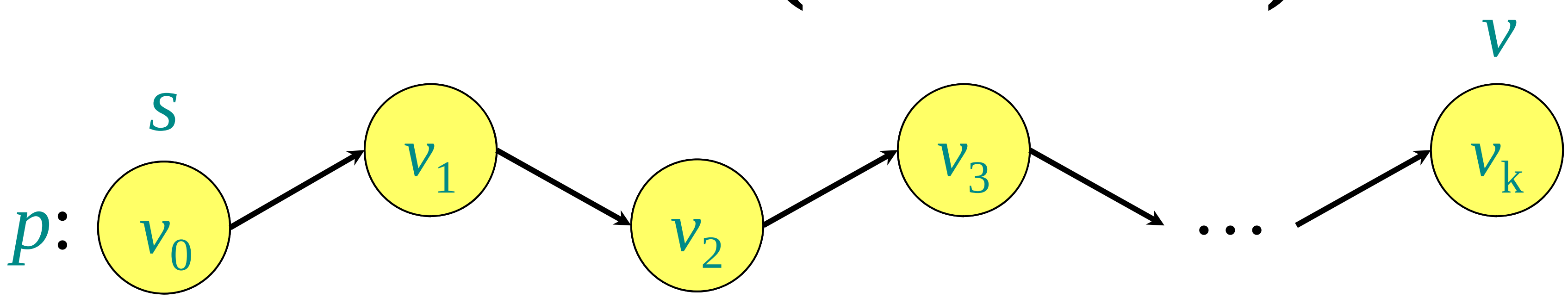
*Proof.* Let $v \in V$ be any vertex, and consider a shortest path $p$ from $s$ to $v$ with the minimum number of edges.




Since $p$ is a shortest path, we have

$$\delta(s, v_i) = \delta(s, v_{i-1}) + w(v_{i-1}, v_i) .$$

# Correctness (continued)




Initially, $d[v_0] = 0 = \delta(s, v_0)$, and $d[s]$ is unchanged by subsequent relaxations.

- After 1 pass through $E$, we have $d[v_1] = \delta(s, v_1)$.
- After 2 passes through $E$, we have $d[v_2] = \delta(s, v_2)$.

  ...
- After $k$ passes through $E$, we have $d[v_k] = \delta(s, v_k)$.

Since $G$ contains no negative-weight cycles, $p$ is simple. Longest simple path has $\leq |V| - 1$ edges. □

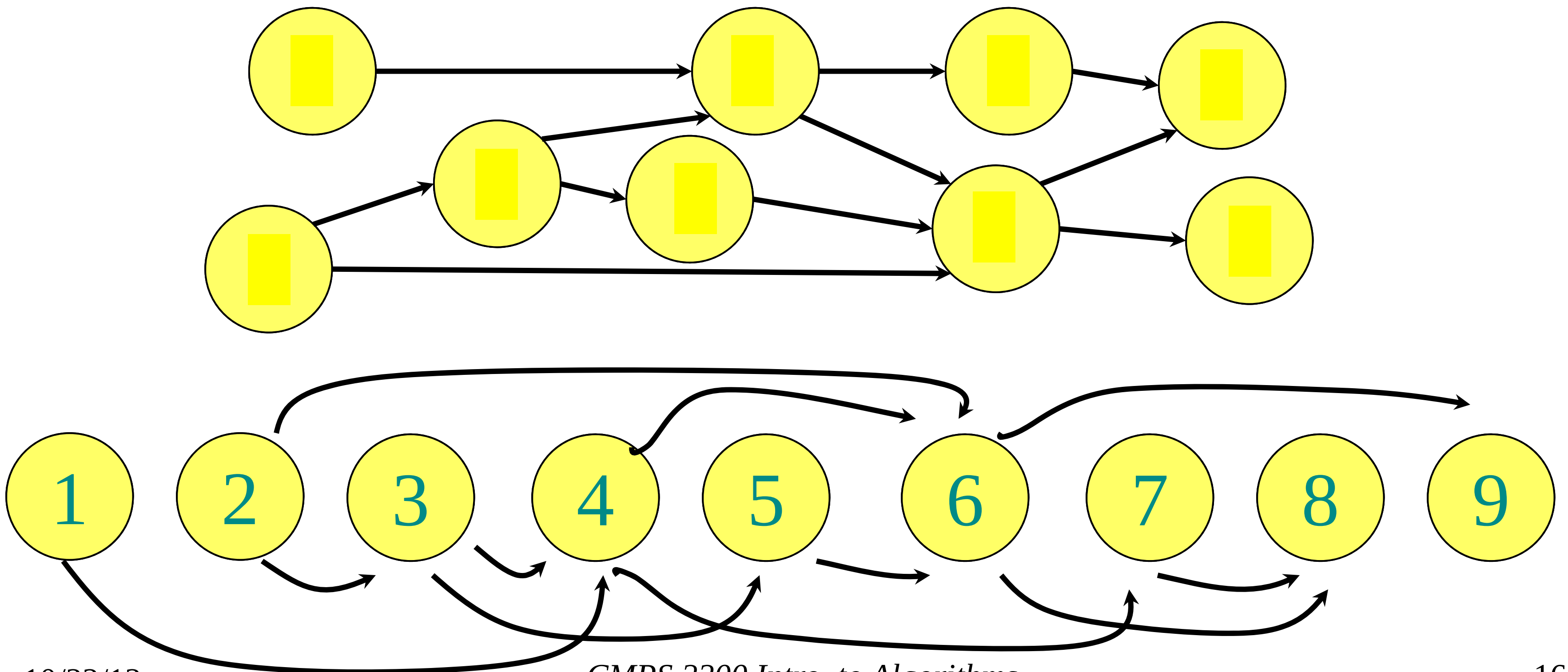
# Detection of negative-weight cycles

**Corollary.** If a value $d[v]$ fails to converge after $|V| - 1$ passes, there exists a negative-weight cycle in $G$ reachable from $s$. □

# DAG shortest paths

If the graph is a ***directed acyclic graph*** (***DAG***), we first ***topologically sort*** the vertices.

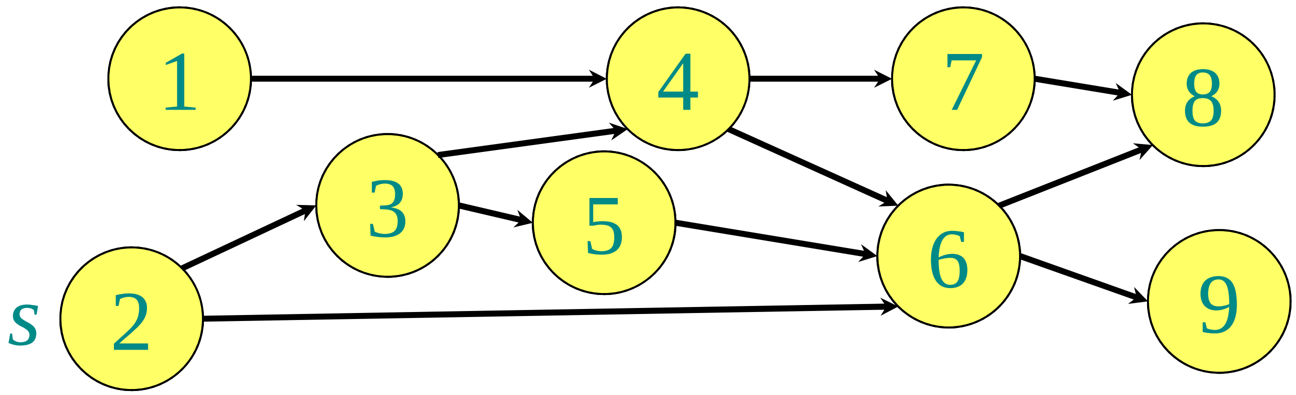
- Determine $f : V \to \{1, 2, \ldots, |V|\}$ such that $(u, v) \in E$ $\Rightarrow f(u) < f(v)$.



# DAG shortest paths

If the graph is a ***directed acyclic graph*** (***DAG***), we first ***topologically sort*** the vertices.

- Determine $f : V \to \{1, 2, \ldots, |V|\}$ such that $(u, v) \in E \Rightarrow f(u) < f(v)$.
- $O(|V| + |E|)$ time




- Walk through the vertices $u \in V$ in this order, relaxing the edges in $Adj[u]$, thereby obtaining the shortest paths from $s$ in a total of $O(|V| + |E|)$ time.

# Shortest paths

**Single-source shortest paths**

- Nonnegative edge weights
  - Dijkstra's algorithm: $O(|E| \log |V|)$
- General: Bellman-Ford: $O(|V||E|)$
- DAG: One pass of Bellman-Ford: $O(|V| + |E|)$