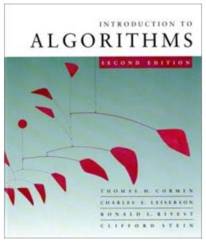

CMPS 2200 – Fall 2012

Randomized Algorithms & Quicksort II

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Slides courtesy of Charles Leiserson with additions by Carola Wenk

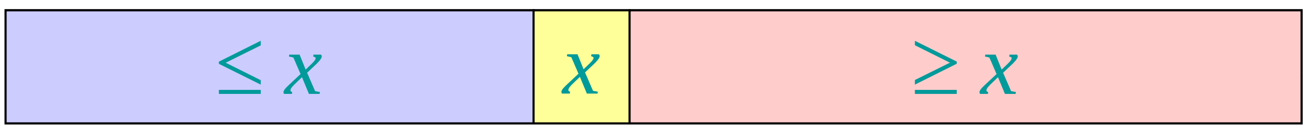
Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts "in place" (like insertion sort, but not like merge sort).
- Very practical (with tuning).
- We are going to perform an expected runtime analysis on randomized quicksort

Quicksort: Divide and conquer

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.



2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

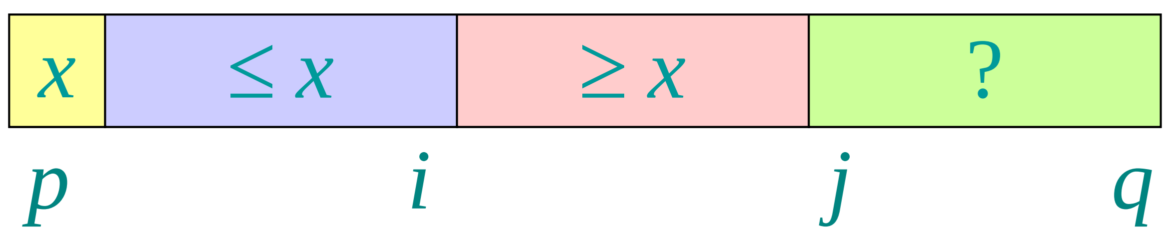
Key: *Linear-time partitioning subroutine.*

Partitioning subroutine

PARTITION(A, p, q) ⊳ $A[p \, . \, . \, q]$
    $x \leftarrow A[p]$ ⊳ pivot $= A[p]$
    $i \leftarrow p$
    for $j \leftarrow p + 1$ **to** q
        do if $A[j] \leq x$
            then $i \leftarrow i + 1$
                exchange $A[i] \leftrightarrow A[j]$
    exchange $A[p] \leftrightarrow A[i]$
    return i

Running time $= O(n)$ for n elements.

Invariant:



Example of partitioning

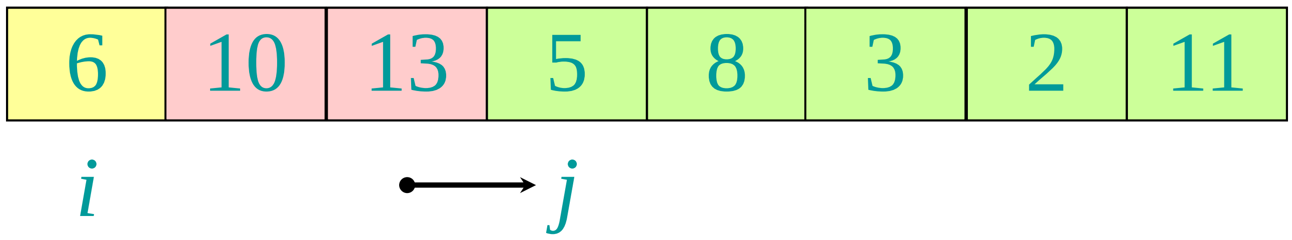
6	10	13	5	8	3	2	11
i	*j*						

Example of partitioning

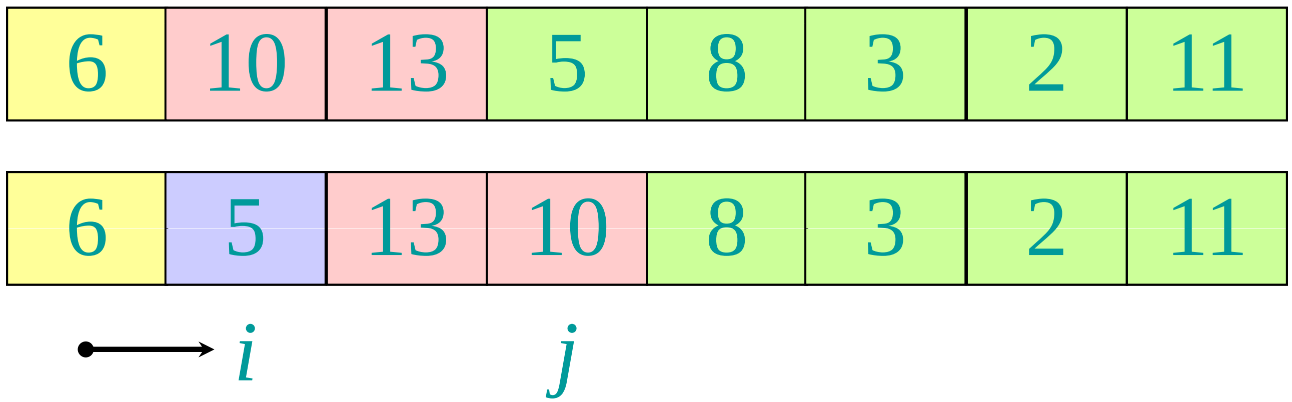
6	10	13	5	8	3	2	11

i ⟶ j

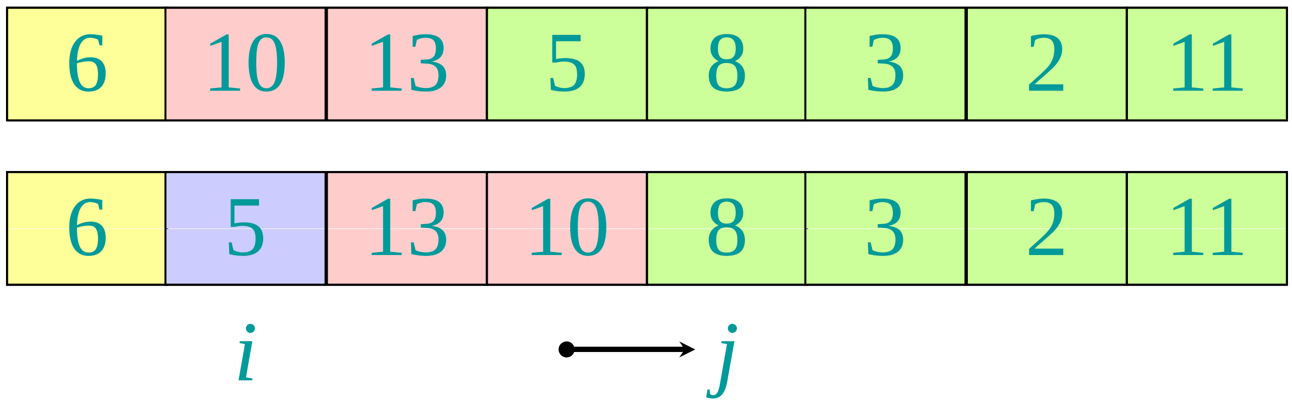
Example of partitioning



Example of partitioning



Example of partitioning



Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

i $\longrightarrow$ j

Example of partitioning

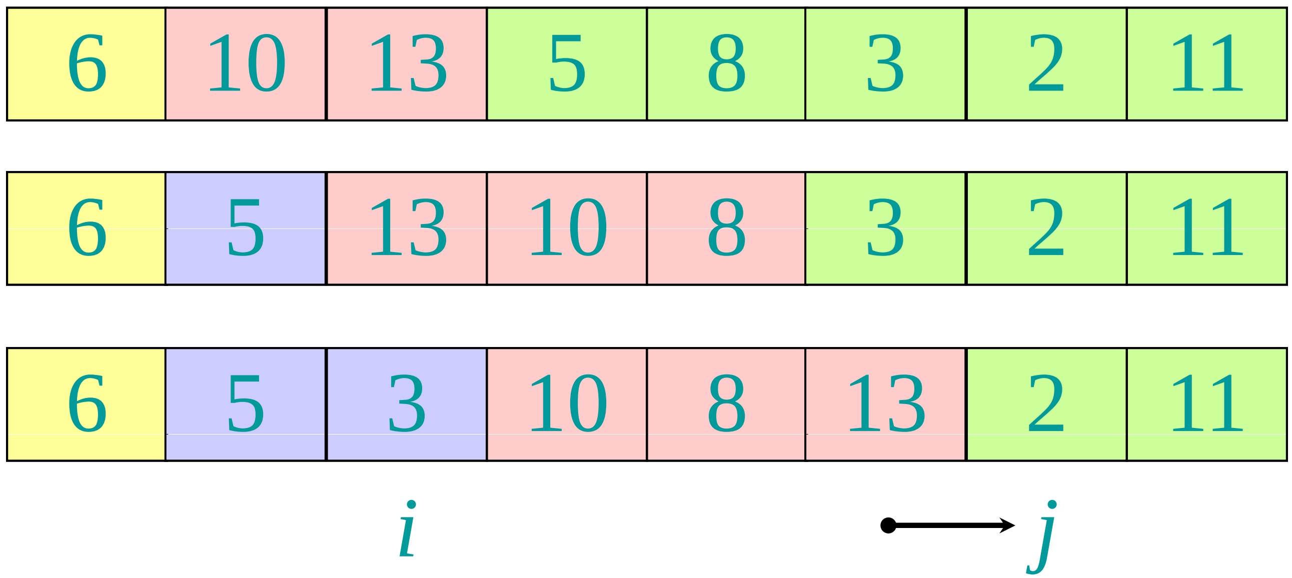
6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

$\longrightarrow i$ $\qquad\qquad j$

Example of partitioning



Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

6	5	3	2	8	13	10	11
			i			j	

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

6	5	3	2	8	13	10	11

i → *j*

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11

i *j*

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11
2	5	3	6	8	13	10	11
			i				

Pseudocode for quicksort

QUICKSORT(A, p, r)
    if $p < r$
        then $q \leftarrow$ PARTITION(A, p, r)
            QUICKSORT$(A, p, q-1)$
            QUICKSORT$(A, q+1, r)$

Initial call: QUICKSORT$(A, 1, n)$

Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.

Worst-case of quicksort

```
QUICKSORT(A, p, r)
  if p < r
    then q ← PARTITION(A, p, r)
      QUICKSORT(A, p, q–1)
      QUICKSORT(A, q+1, r)
```

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \end{aligned}$$

(arithmetic series)

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

Worst-case recursion tree

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$T(n)$

Worst-case recursion tree

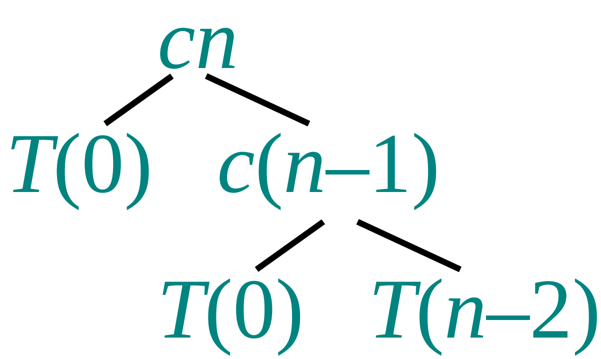
$$T(n) = T(0) + T(n–1) + cn$$

cn

$T(0)$ $T(n–1)$

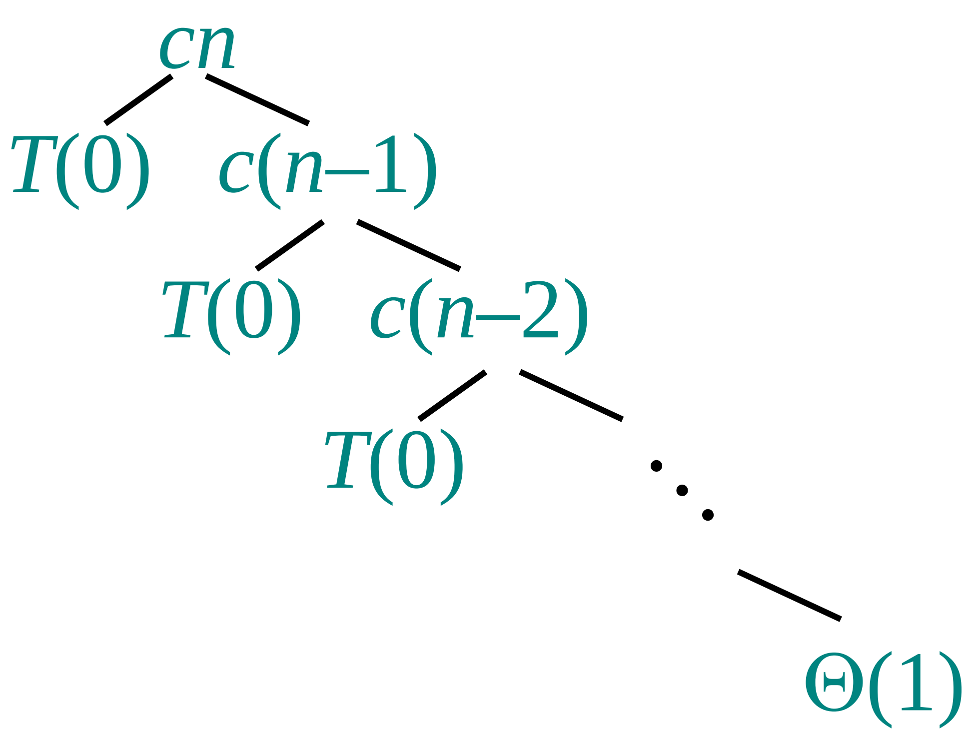
Worst-case recursion tree

$$T(n) = T(0) + T(n–1) + cn$$



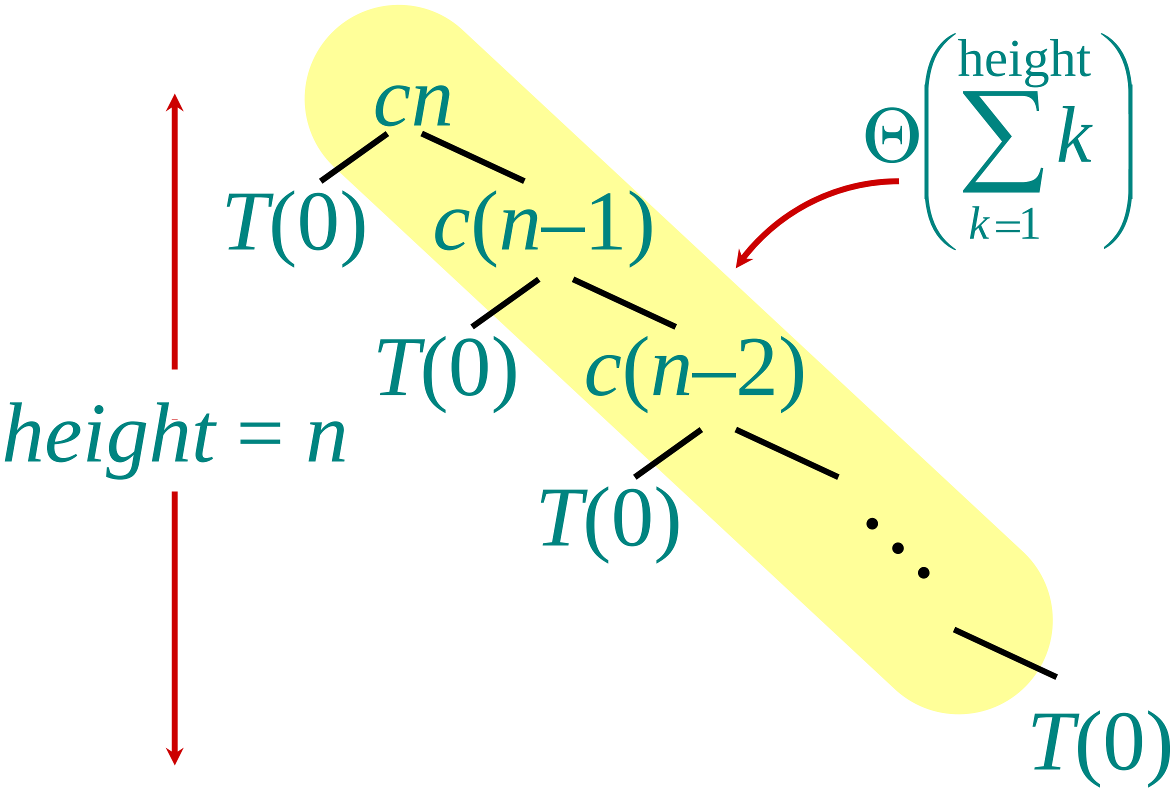
Worst-case recursion tree

$$T(n) = T(0) + T(n–1) + cn$$



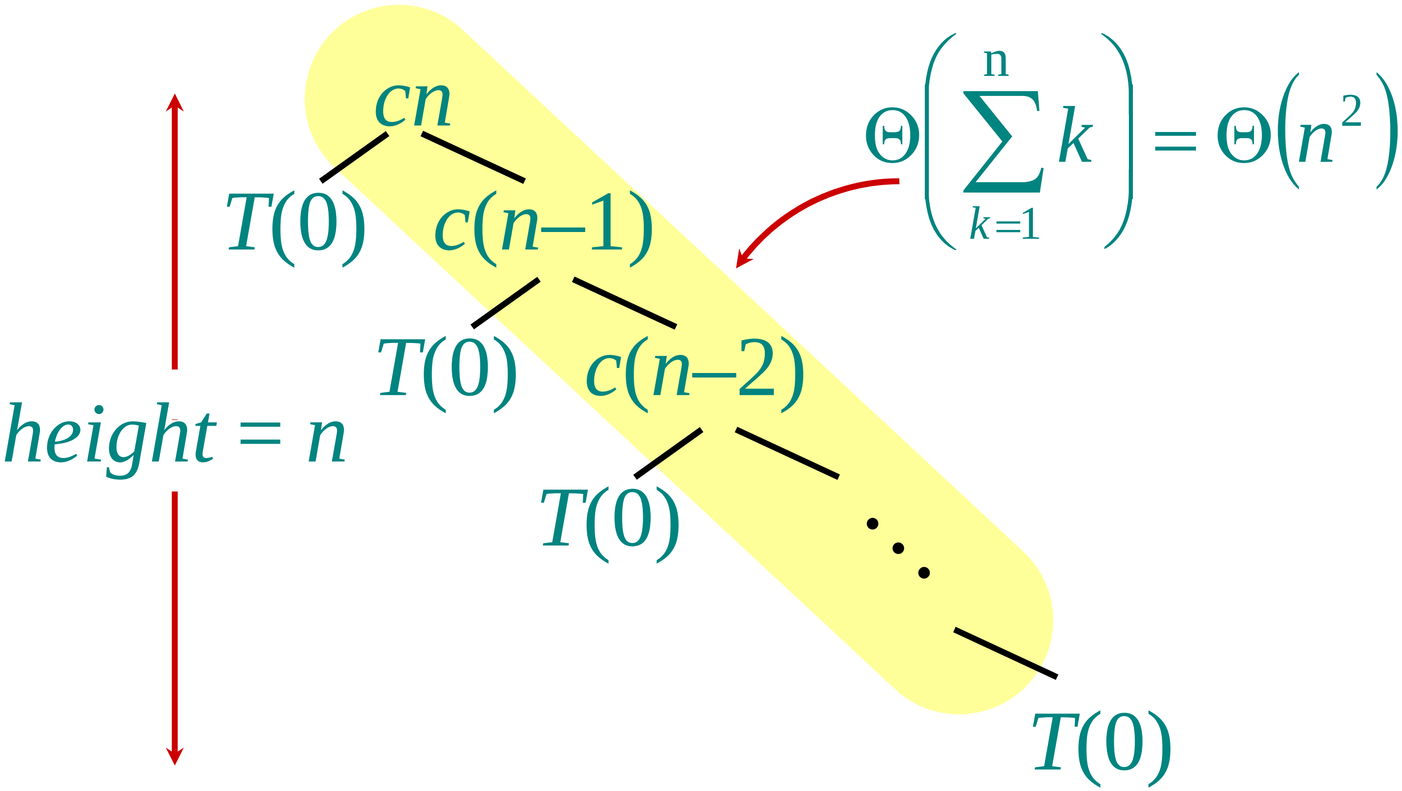
Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



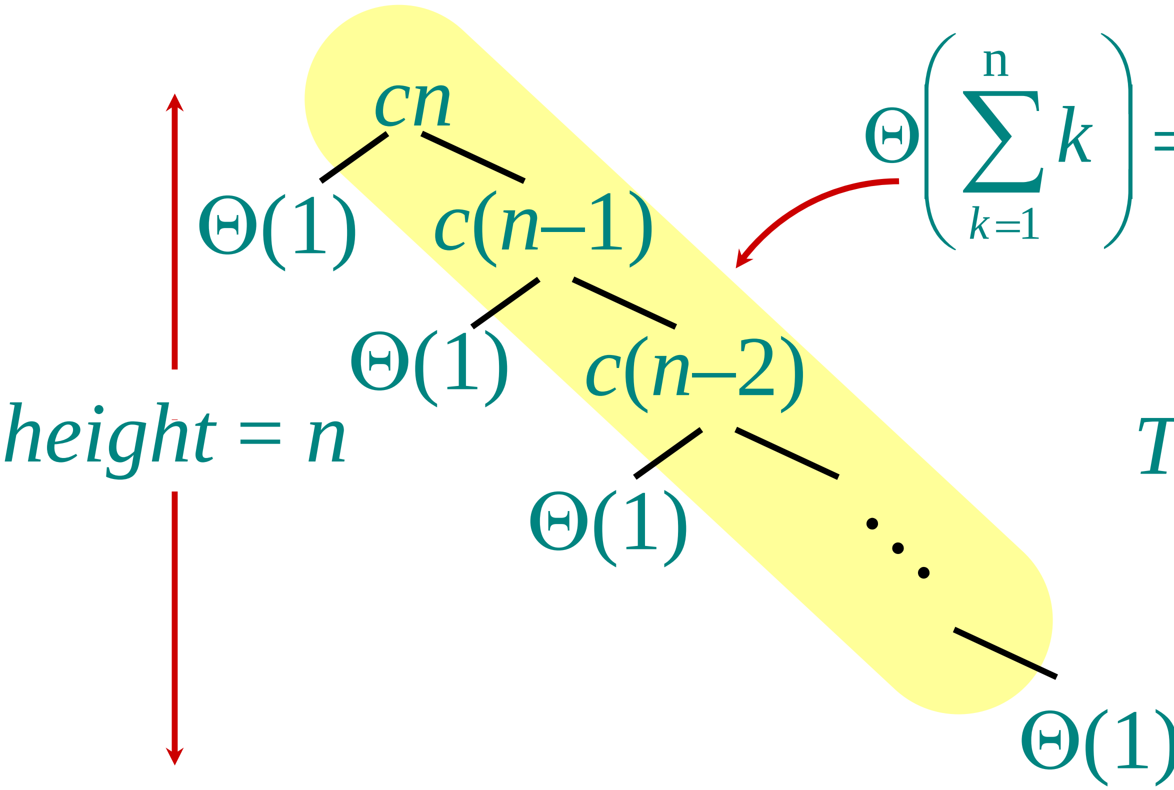
Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



$$T(n) = \Theta(n) + \Theta(n^2)$$
$$= \Theta(n^2)$$

Best-case analysis

(For intuition only!)

If we're lucky, PARTITION splits the array evenly:

$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \log n) \quad \text{(same as merge sort)} \end{aligned}$$

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

Analysis of "almost-best" case

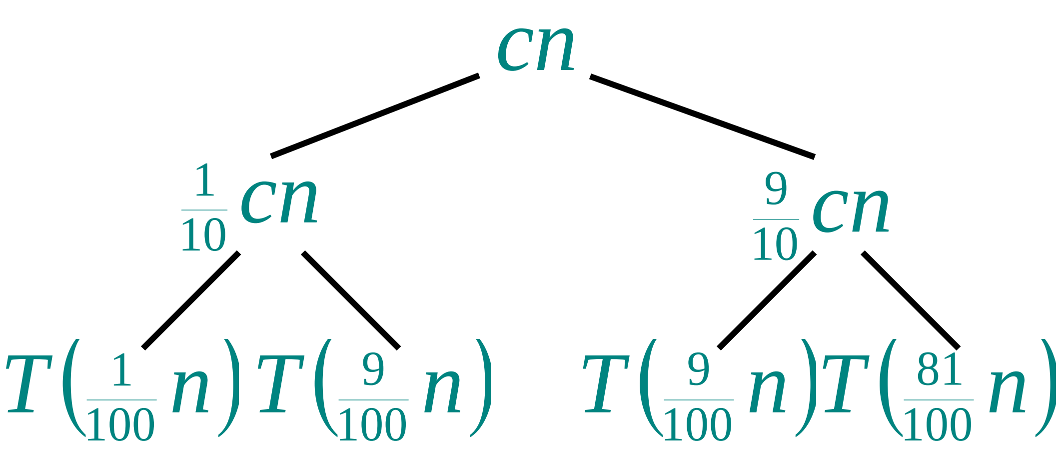
$T(n)$

Analysis of "almost-best" case

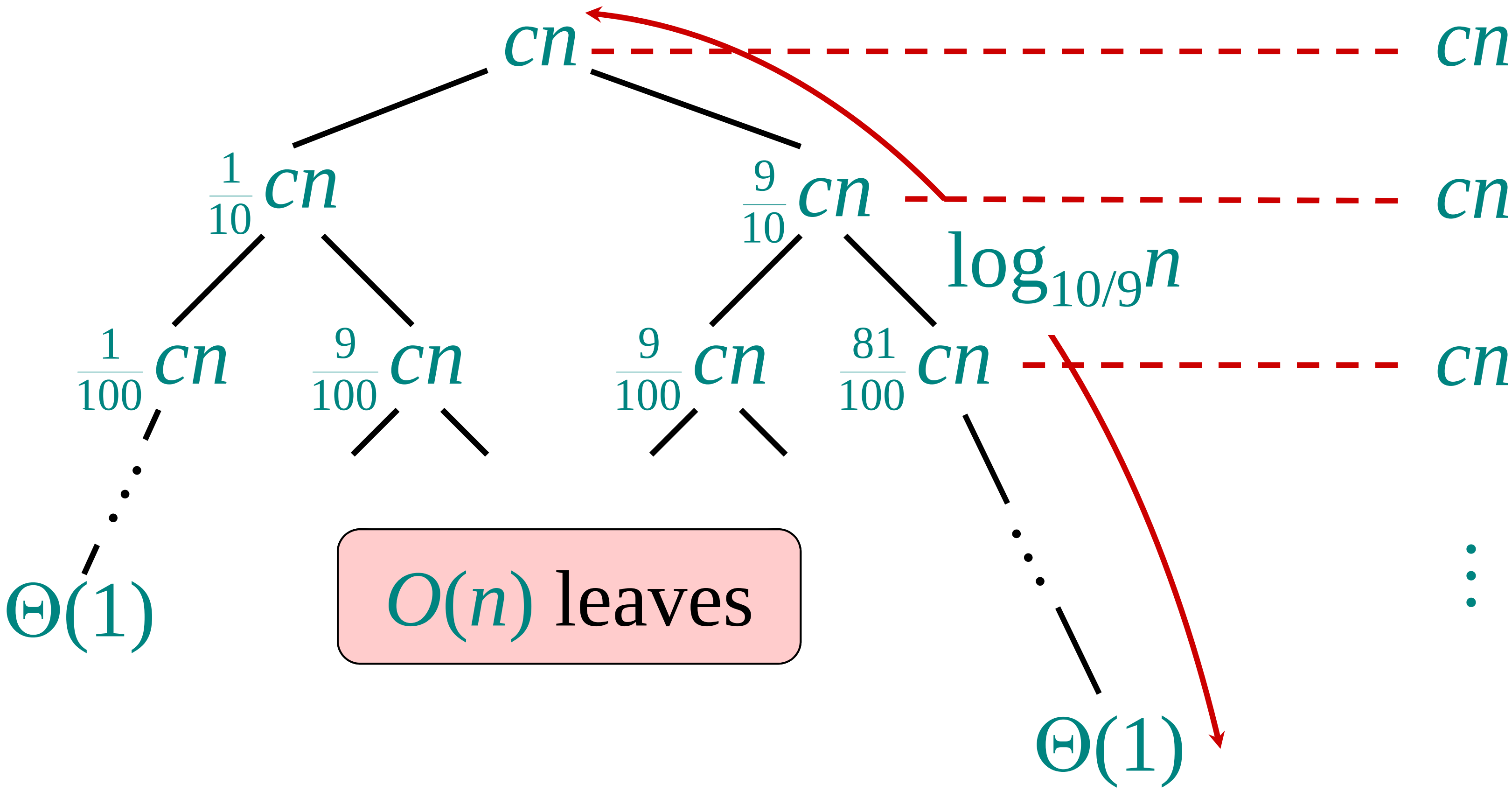
$$cn$$

$$T\left(\tfrac{1}{10}n\right) \qquad\qquad T\left(\tfrac{9}{10}n\right)$$

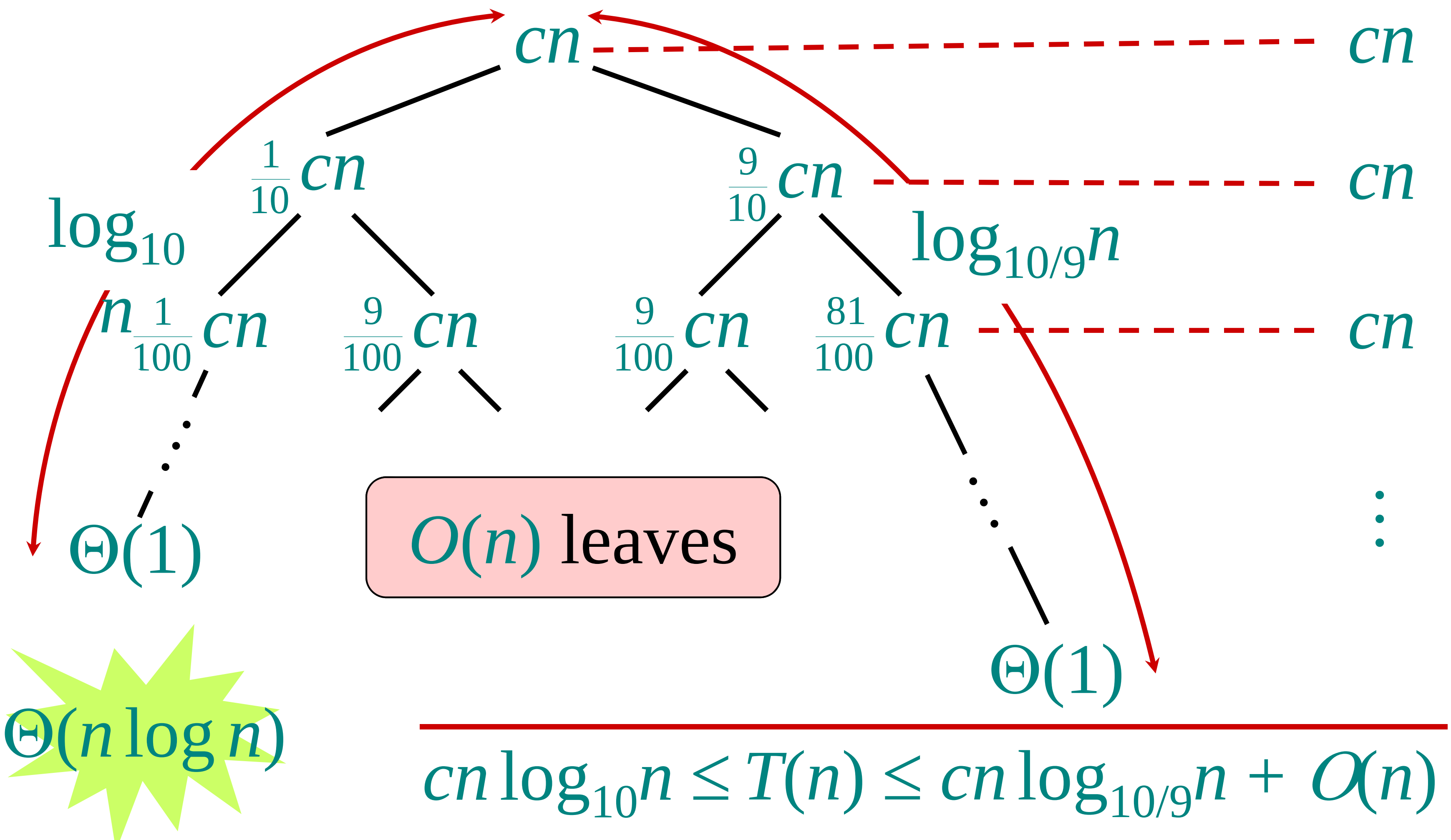
Analysis of “almost-best” case



Analysis of “almost-best” case



Analysis of "almost-best" case



$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$

Quicksort Runtimes

- Best case runtime $T_{best}(n) \in O(n \log n)$
- Worst case runtime $T_{worst}(n) \in O(n^2)$

- Worse than mergesort? Why is it called quicksort then?
- Its average runtime $T_{avg}(n) \in O(n \log n)$
- Better even, the expected runtime of **randomized quicksort** is $O(n \log n)$

Average Runtime

The **average runtime** $T_{avg}(n)$ for Quicksort is the average runtime over **all possible inputs** of length n.

- $T_{avg}(n)$ has to average the runtimes over all $n!$ different input permutations.
- There are still worst-case inputs that will have a $O(n^2)$ runtime

$\Rightarrow$ **Better:** Use randomized quicksort

Randomized quicksort

IDEA: Partition around a ***random*** element.

- Running time is independent of the input order. It depends only on the sequence s of random numbers.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the sequence s of random numbers.

Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.

Average Runtime vs. Expected Runtime

- Average runtime is averaged over all inputs of a deterministic algorithm.
- Expected runtime is the expected value of the runtime random variable of a randomized algorithm. It effectively "averages" over all sequences of random numbers.

- De facto both analyses are very similar. However in practice the randomized algorithm ensures that not one single input elicits worst case behavior.