Comparing Embedded and Immersed Graphs

Fréchet-like distances for graphs

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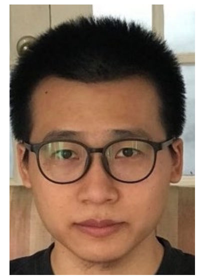

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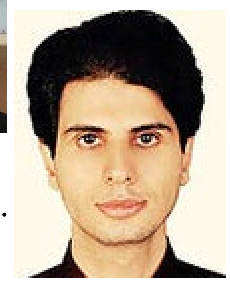

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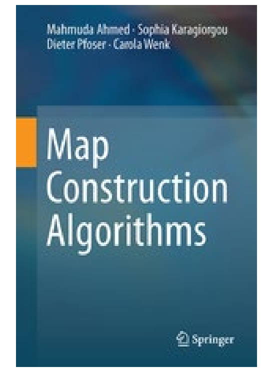

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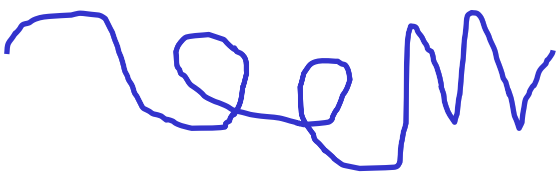
Upcoming paper:

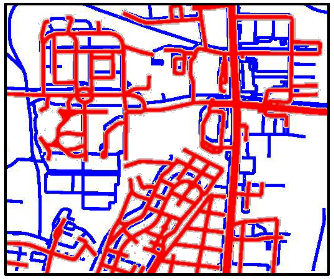
"Distances Between Immersed Graphs: Metric Properties", M. Buchin, E. Chambers, P. Fan, B.T. Fasy, E. Gasparovic, E. Munch, C. Wenk; in revision to La Matematica, 2022

Outline

1. 1D embedded data: Curves and embedded & immersed graphs
2. Hausdorff and Fréchet-like distances:
   - Hausdorff distance
   - Fréchet distance
   - Path-based distance
   - Traversal distance
   - Strong/weak graph distance
   - Contour tree distance
3. Other distances
   - Edit distance for geometric graphs
   - Point sampling distance

1. 1D Embedded Data



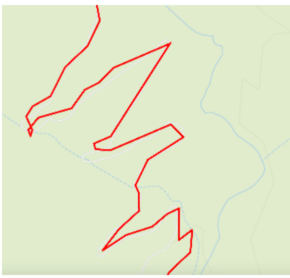


1D Embedded Data

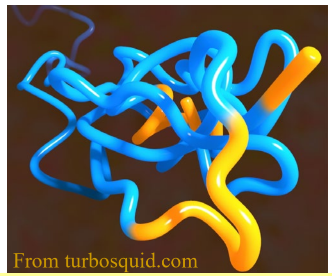
embedded in abient (usually Euclidean) space

C u r v e s

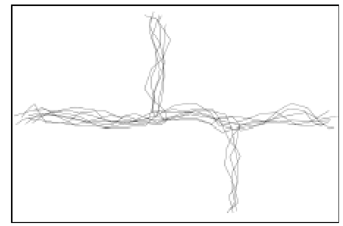
GPS trajectories



Protein chains

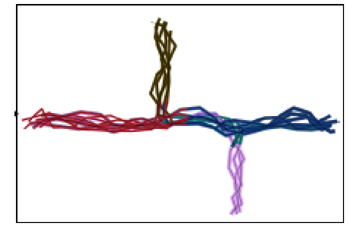


Set of trajectories



⇒

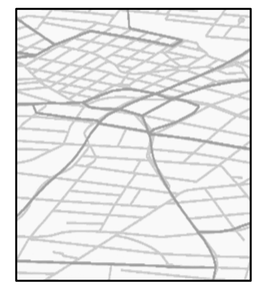
Sub-trajectory clusters



⇒Want to
order to fi

- Want to compare such 1D embedded data ⇒ Geometric shapes
- There are lots of distance measures and algorithms for comparing curves, and some for trees. But not so many for embedded (geometric) graphs.
- Graphs are the most general 1D shapes.

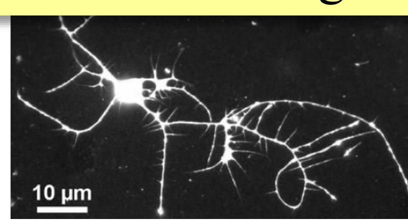
⇒



Constructed roadmap

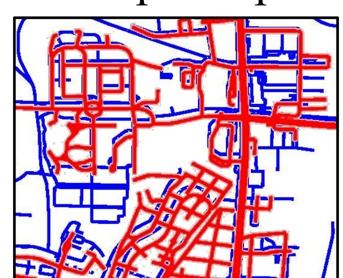
T r e e s

Plant mor



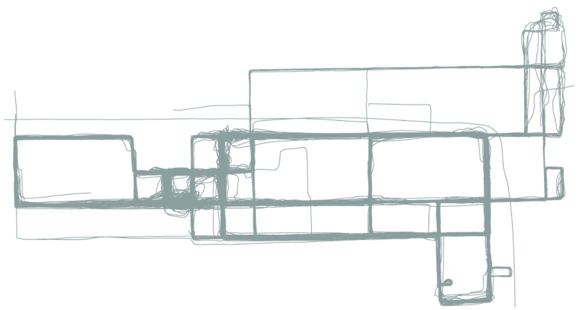
a p h s

Roadmap comparison

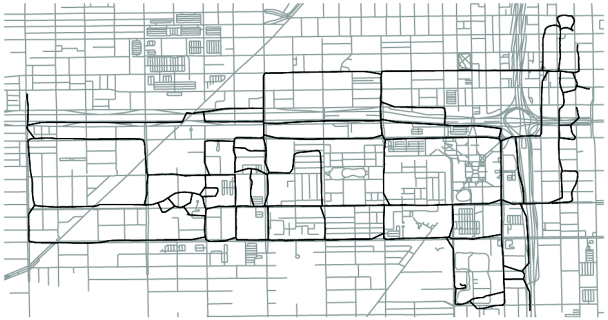


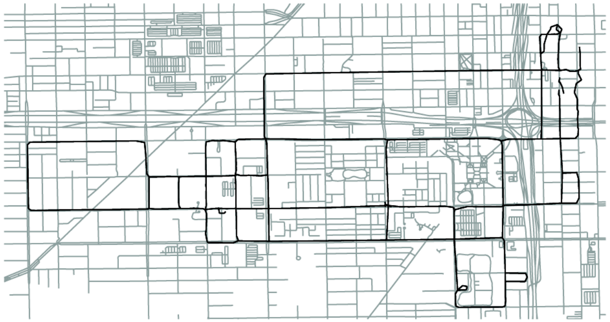
Compare Reconstructed Roadmaps

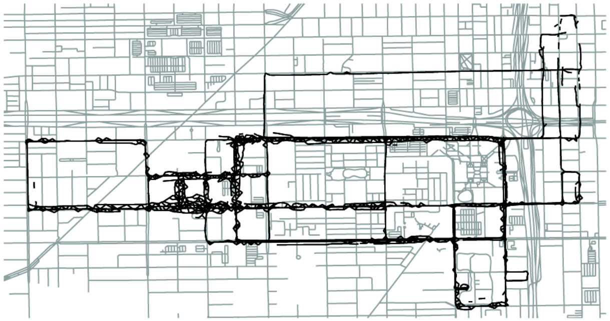
GPS Trajectory Data

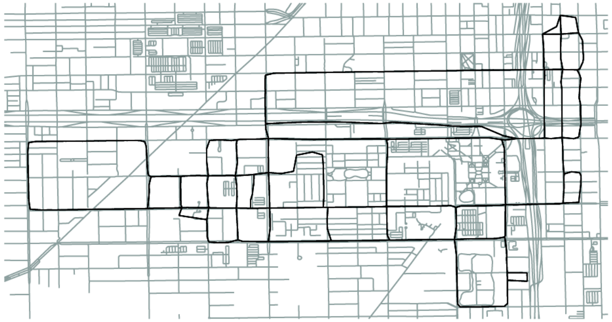


Reconstructed Roadmaps



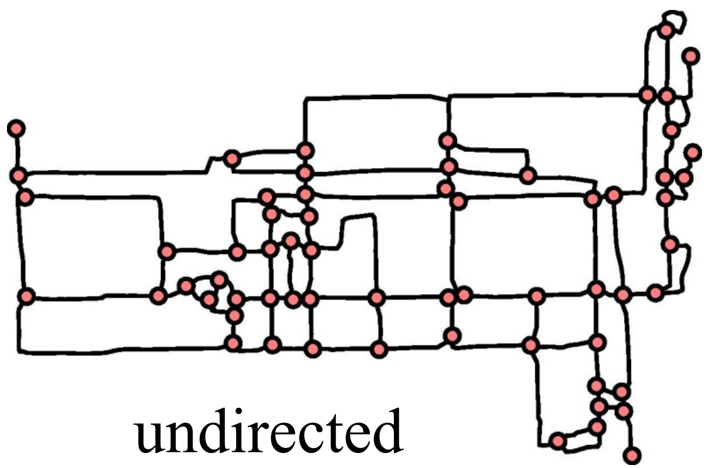


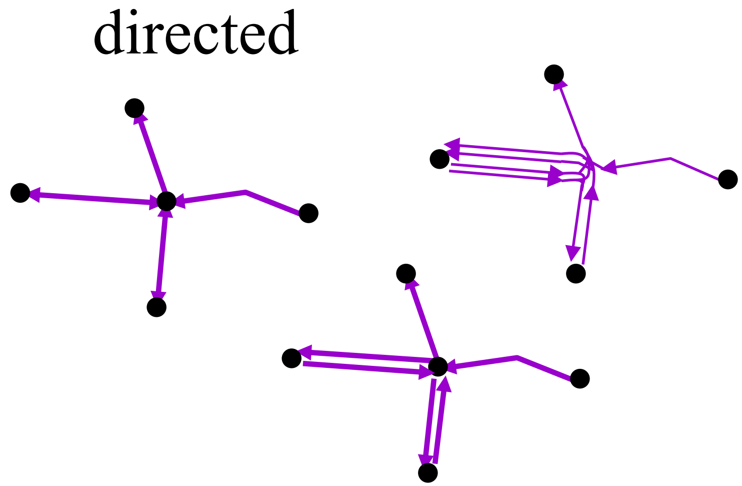




Embedded/Immersed Graphs

- Graph $G = (V, E)$ with a set of vertices V and edges E.
- Road network: Planar embedded





- Can consider G as a topological space (e.g., 1D simplicial complex)
- **Embedded graph**: Have a continuous function $\phi: G \to \mathbb{R}^d$, $d \geq 2$, that is homeomorphic onto its image.
- **Immersed graph**: $\phi: G \to \mathbb{R}^d$ is only **locally** homeomorphic onto its image.

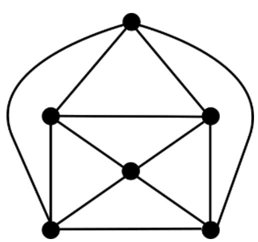
Homeomorphism: A continuous, bijective map whose inverse is continuous.

Embedded/Immersed Graphs

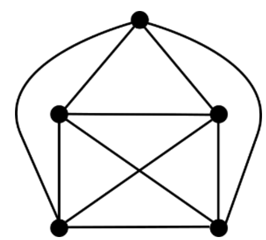
- **Embedded graph**: Have a continuous function $\phi: G \to \mathbb{R}^d$, $d \geq 2$, that is homeomorphic onto its image.
- **Immersed graph**: $\phi: G \to \mathbb{R}^d$ is only **locally** homeomorphic onto its image.

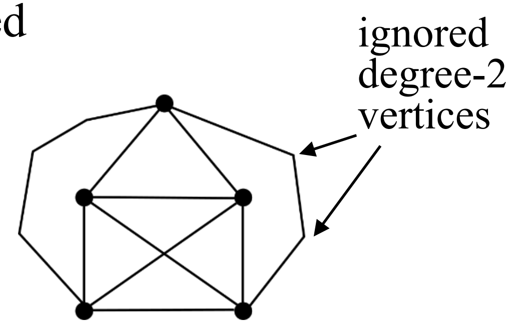
=> Each vertex is mapped to a point and edges are mapped to curves in $\mathbb{R}^d$ in such a way that the graph structure is maintained.

Embedding:
all edge-curves are non-crossing
(every crossing is a vertex)



Immersion:
“Bridges” are allowed



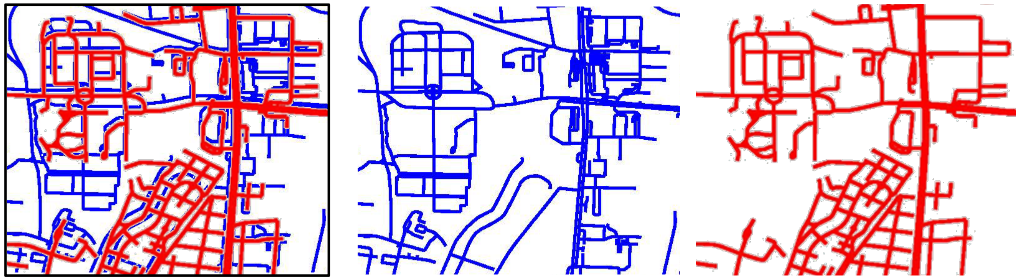


- Assume edge curves are piecewise linear, and may ignore deg-2 vertices

Immersed Graph Comparison

Given two immersed graphs (G, ϕ_G) and (H, ϕ_H), we want to compare them.

- How similar / different are they?
- What does it mean to be similar?
  - Depends on the application.
  - Graph isomorphism?



Here: Assume G and H are embedded in the same space and aligned.

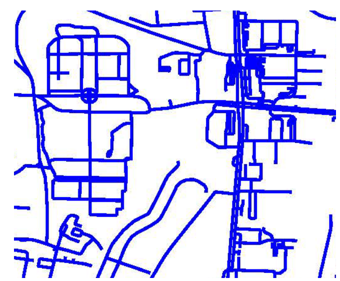
1. Define different distances between G and H, and study their properties (e.g., metric) and computational complexities.
2. Compute correspondences between portions of G and H.

When considering spaces of (immersed or embedded) graphs, we will consider underlying graphs up to homeomorphism.

Graph Isomorphism

- An isomorphism of $G = (V_G, E_G)$ and $\mathrm{H} = (V_H, E_H)$ is a
  - bijective map $f: V_G \to V_H$ for which holds
  - $\{u, v\} \in E_G \Leftrightarrow \{f(u), f(v)\} \in E_H$

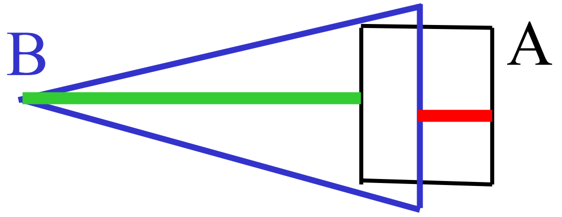
  Can be computed in linear time for planar graphs [HW74]
- Subgraph isomorphism: An isomorphism between G and a subgraph of H
  - NP-complete
  - Can be computed in linear time if G and H are planar and G has constant complexity [E95]

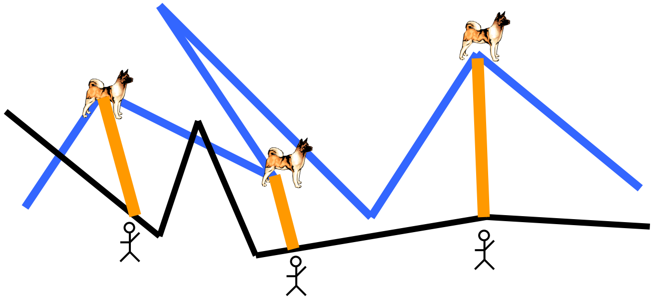



- Isomorphisms are bijective (1-to-1). However, we may want to allow 1-to-many assignments.
- We may also want to allow partial matchings.
- Isomorphisms are combinatorial in nature and don't take the embeddings/immersions into account.

[E95] D. Eppstein, Subgraph isomorphism in planar graphs and related problems, SODA: 632–640, 1995.
[HW74] J. Hopcroft, J. Wong, Linear time algorithm for isomorphism of planar graphs, STOC: 172–184, 1974.

2. Hausdorff and Fréchet-Like Distances

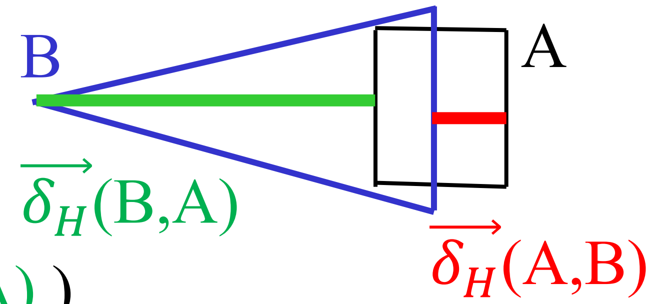




Hausdorff Distance

- Directed Hausdorff distance

$$\overrightarrow{\delta_H}(A,B) = \max_{a\in A} \min_{b\in B} \| a-b \|$$

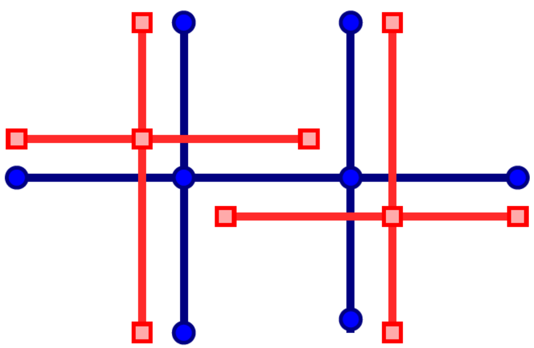


- Undirected Hausdorff-distance

$$\delta_H(A,B) = \max\left(\overrightarrow{\delta_H}(A,B)\,,\ \overrightarrow{\delta_H}(B,A)\right)$$

- Can be computed in polynomial time; O(N log N) in the plane.

- **Con:** When applied to graph comparison, δ_H only compares the geometry but not the topology
- **Pro:** $\overrightarrow{\delta_H}$ allows for partial comparison of one graph
- δ_H is a metric on the set of compact subsets of $\mathbb{R}^d$



Metric Properties

Definition 1 (Key Properties of Dissimilarity Functions). Let $\mathbb{X}$ be a set. Consider a function $d\colon \mathbb{X} \times \mathbb{X} \to \mathbb{R}_{\geq 0}$. We define the following properties:

1. Identity: $d(x,x) = 0$.
2. Symmetry: for all $x, y \in \mathbb{X}$, $d(x,y) = d(y,x)$.
3. Separability: for all $x, y \in \mathbb{X}$, $d(x,y) = 0$ implies $x = y$.
4. Subadditivity (Triangle Inequality): for all $x, y, z \in \mathbb{X}$, $d(x,y) \leq d(x,z) + d(z,y)$.

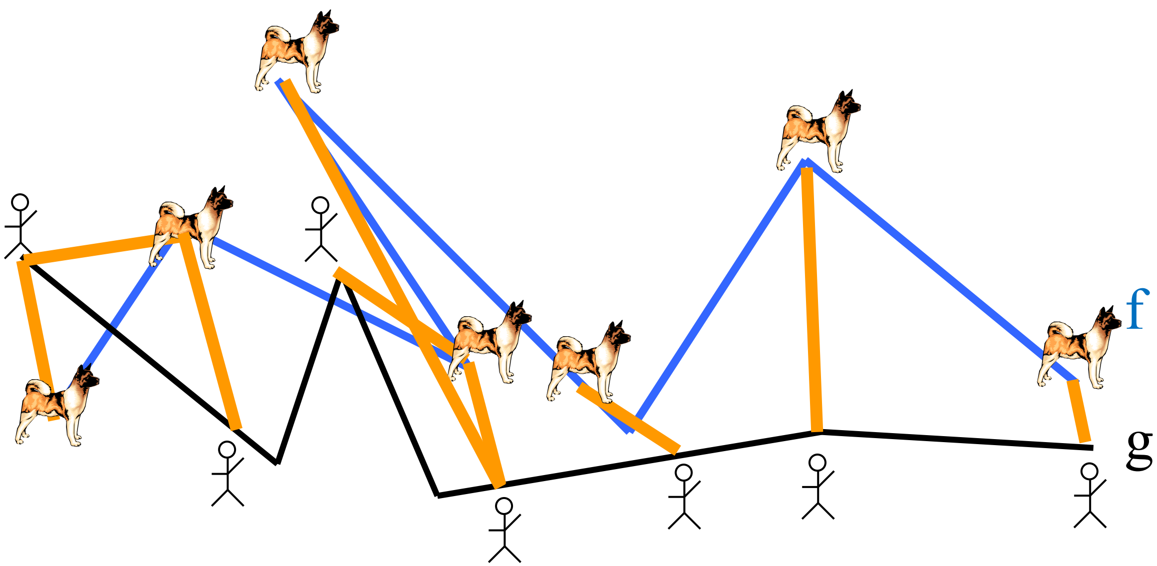
- **Metric:** Fulfills 1.-4.
- **Directed (as a modifier):** Does not fulfill 2.
- **Pseudo-metric:** 1., 2., 4.
- **Semi-metric:** 1., 2., 3.
- **Quasi-metric:** 1., 3., 4. (a directed metric)

$\Rightarrow \overrightarrow{\delta_H}$ is a directed pseudo-metric

Fréchet Distance for Curves

$$\delta_F(f,g) = \inf_{\alpha,\beta:[0,1]\to[0,1]} \; \max_{t\in[0,1]} \|f(\alpha(t)) - g(\beta(t))\|$$

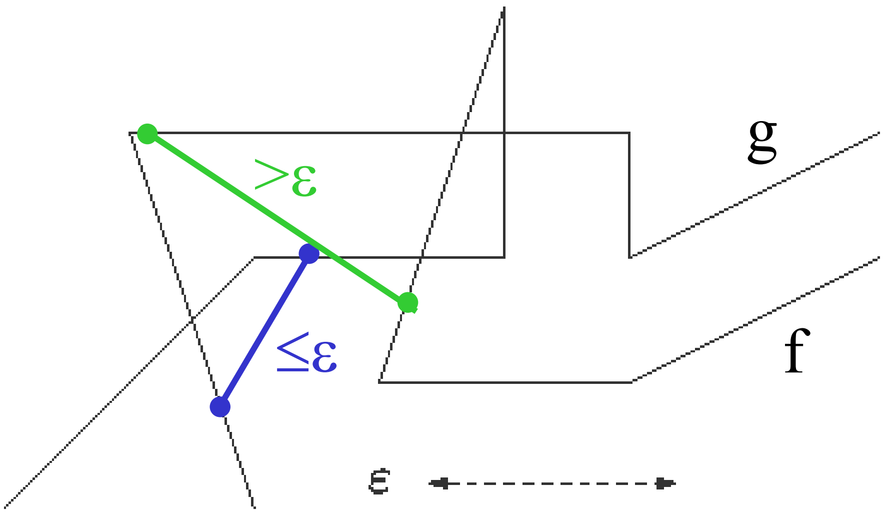
where α and β range over continuous monotone increasing reparameterizations only.

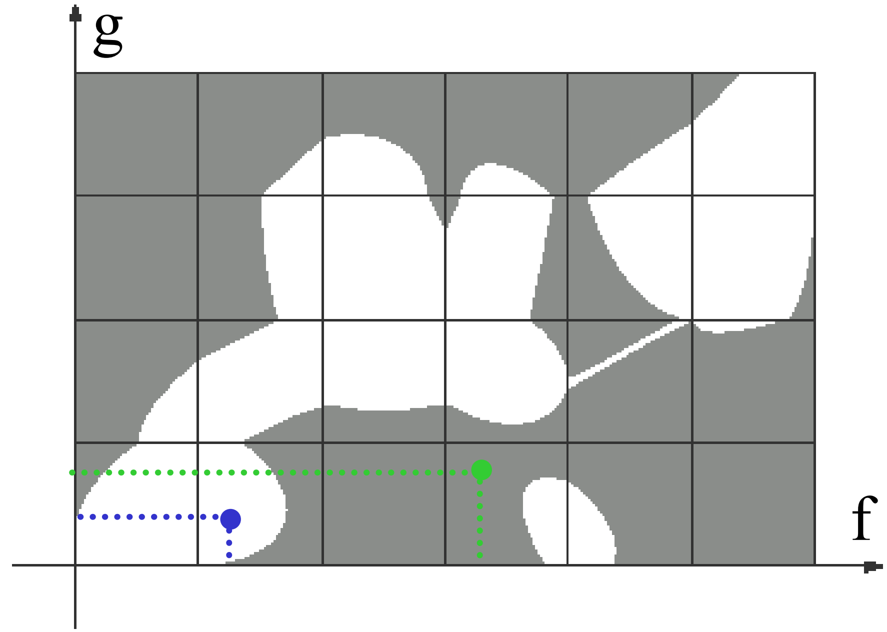


- Man and dog walk on one curve each
- They hold each other at a **leash**
- They are only allowed to go forward
- δ_F is the minimal possible leash length

[F06] M. Fréchet, Sur quelques points de calcul fonctionel, *Rendiconti del Circolo Mathematico di Palermo* 22: 1-74, 1906.

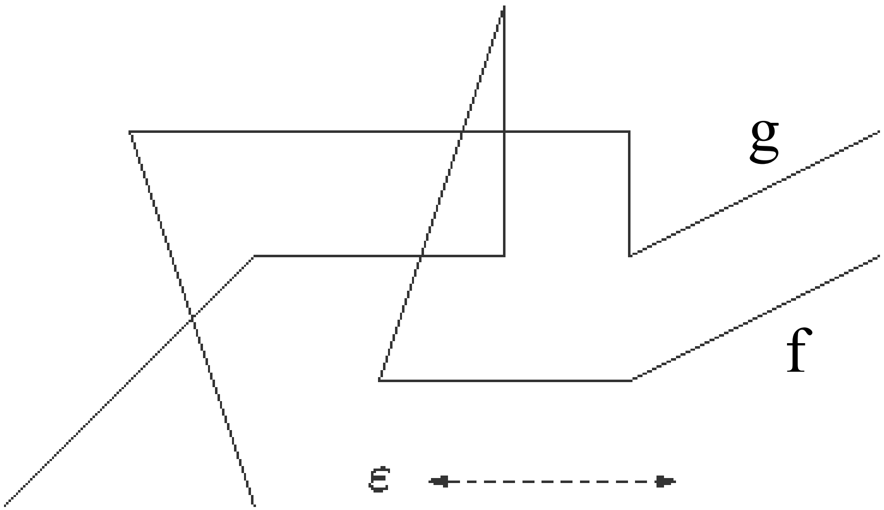
Free Space Diagram

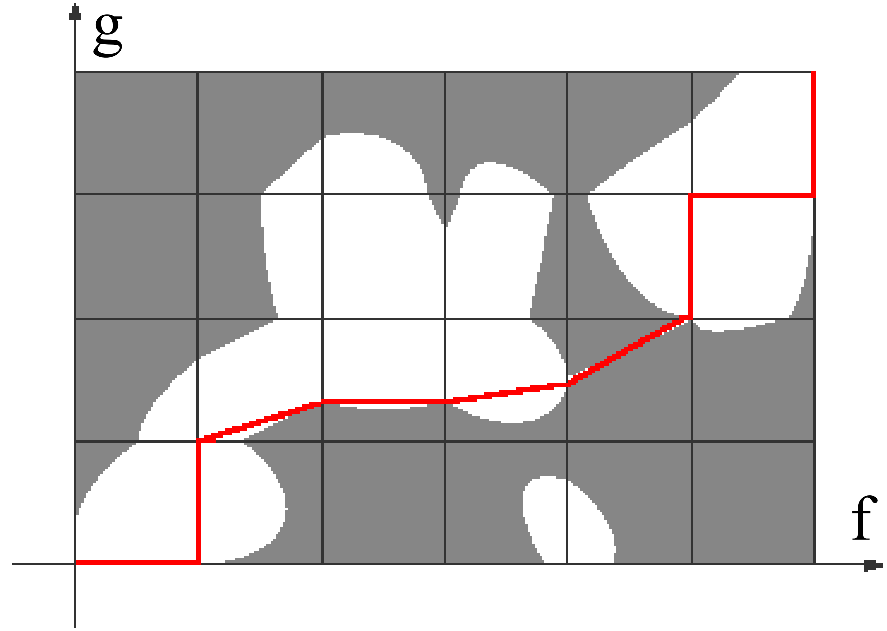




- Let ε > 0 fixed (eventually solve decision problem)
- $F_\varepsilon(f,g) = \{ (s,t) \in [0,1]^2 \mid \| f(s) - g(t) \| \leq \varepsilon \}$ *white points*
  free space of f and g
- The free space in one cell is an ellipse.

Free Space Diagram





- Monotone path encodes reparametrizations of f and g
- $\delta_F(f,g) \leq \varepsilon$ iff there is a monotone path in the free space from (0,0) to (1,1)
- Such a path can be computed using DP in O(*mn*) time

Fréchet Distance, General

Let $A, B \subseteq \mathbb{R}^k$ be two oriented manifolds. And let $f: A \to \mathbb{R}^d$ and g: $B \to \mathbb{R}^d$ be two immersions. Then

$$\delta_F(f, g) = \inf_{\alpha} \max_{t \in A} \|f(t) - g(\alpha(t))\|,$$

where $\alpha: A \to B$ ranges over all orientation-preserving homeomorphisms.

- The Fréchet distance is a metric (up to orientation-preserving homeomorphism)
- Originally defined for oriented manifolds, but can be generalized even further.

[M24] M. Fréchet. Sur la distance de deux surfaces. Ann. Soc. Polonaise Math. 3, 4–19, 1924.

Fréchet Distance, Immersed Graphs

Let (G, ϕ_G) and (H, ϕ_H) be two immersed graphs.

- We can apply the Fréchet distance definition in principle on the maps ϕ_G and ϕ_G.
- Drop the „orientation-preserving“ requirement.
- Equivalent definition:

$$\delta_F(G, H) = \min_{\alpha} \max_{e \in E_G} \delta_F(\phi_G(e), \phi_H(\alpha(e))),$$

where α ranges over all edge mappings corresponding to **isomorphisms** of G and H.

- Is graph-isomorphism hard. Can be computed in poly time for trees and for graphs of bounded tree-width. [BKN20]
- For connected planar graphs, can enumerate orientation-preserving isomorphisms in O(mn log(mn)) time*. [FW21]

[BKN20] M. Buchin, A. Krivosija, A. Neuhaus. Computing the Fréchet distance of trees and graphs of bounded tree width. EuroCG. 2020
[F**W**21] P. Fang, **C. Wenk**. The Fréchet distance for plane graphs. EuroCG 21.

Path-Based Distance

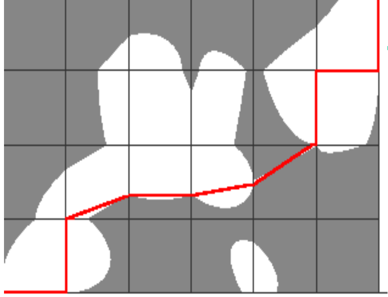
- Directed Hausdorff distance on path-sets:

$$\overrightarrow{d_{path}}(G,H) = \max_{p \in \Pi_G} \min_{q \in \Pi_H} \delta_F(\phi_G(p), \phi_H(q))$$

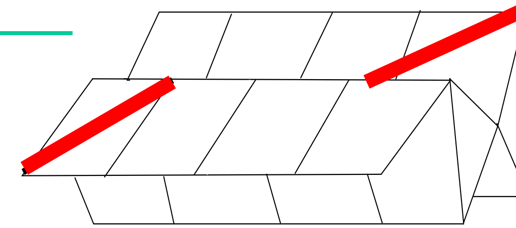
Fréchet distance

- Π_G path-set in G, and Π_H path-set in H
- Asymmetry of distance definition is desirable, if G is a reconstructed map and H a ground-truth map.

[AFH**W**14] M. Ahmed, B. Fasy, K. Hickmann, **C. Wenk**, Path-based distance for street map comparison, arXiv:1309.6131, 2014.



Path-Based Distance

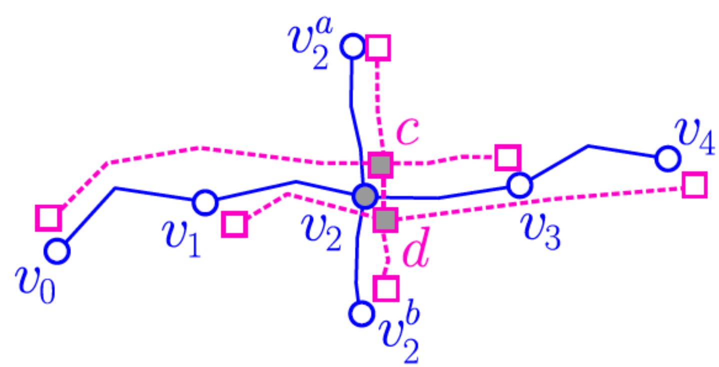


- Ideally, Π_G and Π_H are the set of all paths in G and H

$$\overrightarrow{d_{path}}(G,H) = \max_{p \in \Pi_G} \underbrace{\min_{q \in \Pi_H} \delta_F(\phi_G(p), \phi_H(q))}_{\text{map-matching}}$$

- It is a directed pseudo-metric.
- One can use the set of paths of link-length three to **approximate** the overall distance in polynomial time, if vertices in G are well-separated and have **degree ≠ 3**.

→ Stitch link-length three paths together to form longer paths



[AFHW14] M. Ahmed, B. Fasy, K. Hickmann, **C. Wenk**, Path-based distance for street map comparison, TSAS 1(1): article 3, 28 pages, 2015.

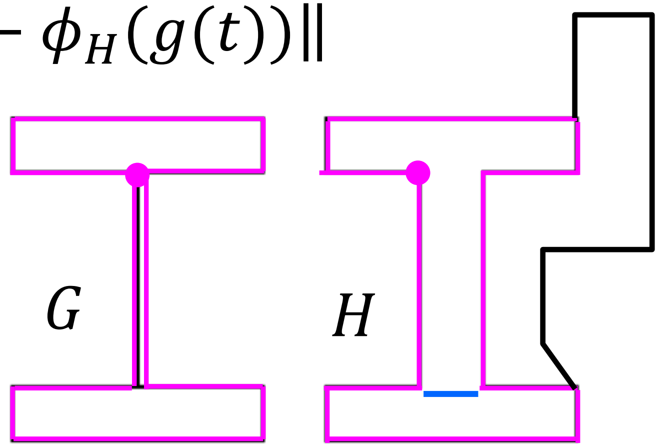
Traversal Distance

Let (G, ϕ_G) and (H, ϕ_H) be two immersed graphs.

- Represent G by traversals $f: [0,1] \to G$ (continuous, surjective) and H by partial traversals $\mathrm{g}: [0,1] \to H$:

$$\overrightarrow{d_T}(G,H) = \inf_{f,g} \max_{t \in [0,1]} \|\phi_G(f(t)) - \phi_H(g(t))\|$$

- Can be computed in O(mn log mn) time using free space diagram.
- Is a directed distance, but fulfills neither separability nor triangle inequality.
- Concides with the weak Fréchet distance when G and H are polygonal curves.



Small traversal distance

[AERW03] **H. Alt**, A. Efrat, **G. Rote**, **C. Wenk**, Matching Planar Maps, Journal of Algorithms 49: 262-283, 2003.

Strong and Weak Graph Distances

Let (G, ϕ_G) and (H, ϕ_H) be two immersed graphs.

- A graph mapping is a **continuous map** $s: G \rightarrow H$.

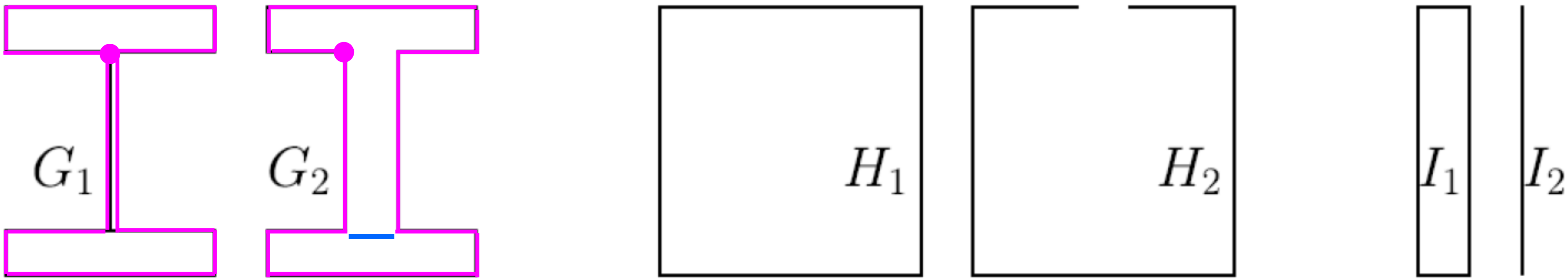
  Can be combinatorially represented as:
  - s sends each $v \in V_G$ to a point $s(v) \in H$
  - s sends each $\mathrm{e} \in E_G$ to a path from $s(u)$ to $s(v)$ in H.
- Then the strong graph distance is

$$\vec{\delta}(G, H) = \inf_{s:G \rightarrow H} \max_{e \in E_G} \delta_F(\phi_G(e), \phi_H(s(e)))$$

- The weak graph distance $\overrightarrow{\delta_w}$ uses δ_{wF} instead of δ_F.
- We have $\overrightarrow{d_T}(G, H) \leq \overrightarrow{\delta_w}(G, H) \leq \vec{\delta}(G, H)$
- NP-hard to decide, but poly time for trees. The weak distance can be computed in poly time for (spike-free) plane graphs.

[ABKS**W**21] H.A. Akitaya, M. Buchin, B. Kilgus, S. Sijben, **C. Wenk**, Distance Measures for Embedded Graphs, CGTA 95, 101743, 2021.

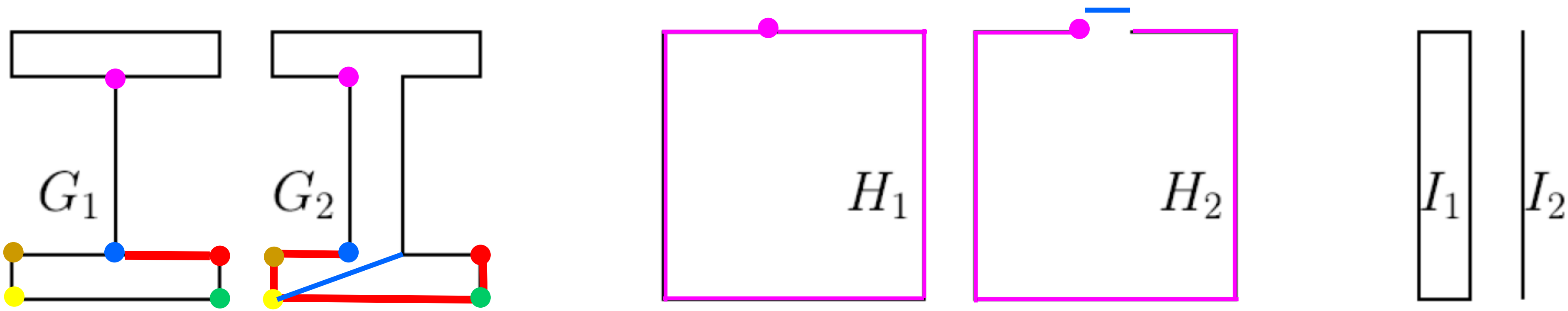
Traversal and Graph Distance



- Small traversal distance

[ABKSW21] H.A. Akitaya, M. Buchin, B. Kilgus, S. Sijben, **C. Wenk**, Distance Measures for Embedded Graphs, CGTA 95, 101743, 2021.

Traversal and Graph Distance

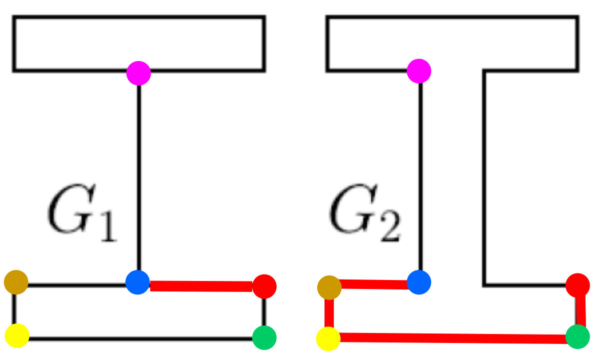


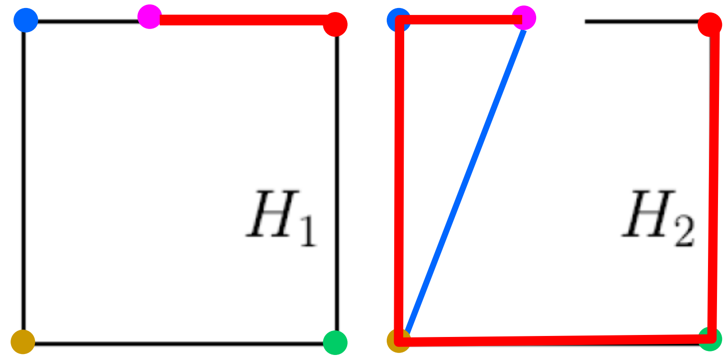
- Small traversal distance
- Large graph distance

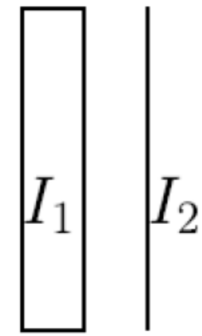
- Small traversal distance

[ABKSW21] H.A. Akitaya, M. Buchin, B. Kilgus, S. Sijben, **C. Wenk**, Distance Measures for Embedded Graphs, CGTA 95, 101743, 2021.

Traversal and Graph Distance







- Small traversal distance
- Large graph distance

- Small traversal distance
- Large graph distance

- Both small

Stay tuned for traversal distance code!




[ABKS**W**21] H.A. Akitaya, M. Buchin, B. Kilgus, S. Sijben, **C. Wenk**, Distance Measures for Embedded Graphs, CGTA 95, 101743, 2021.

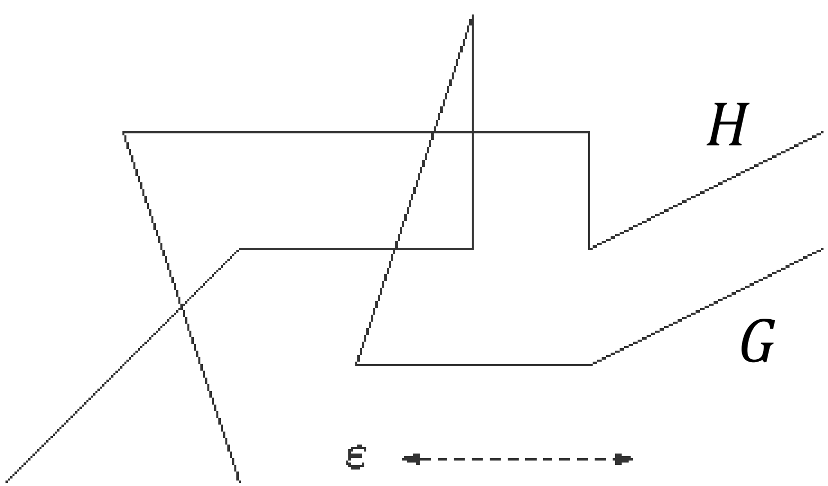
Contour Tree Distance

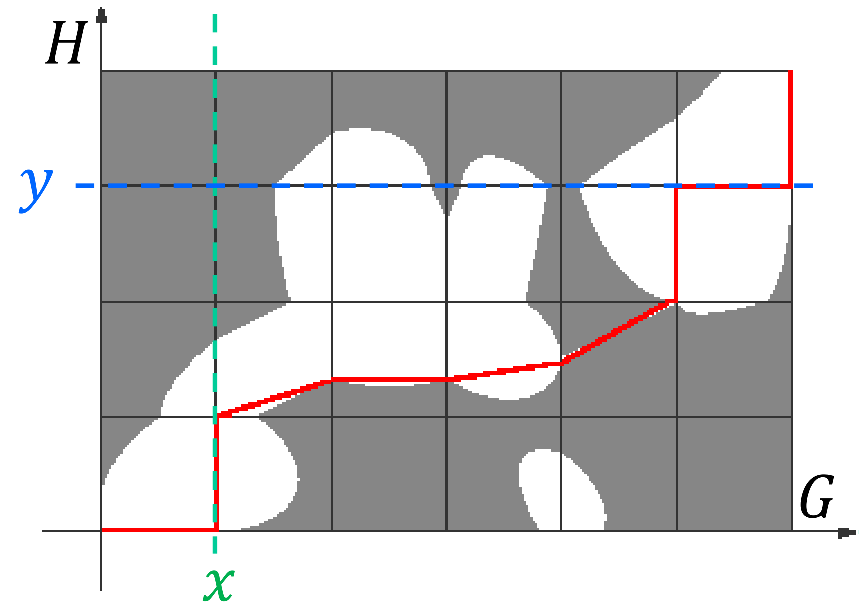
Let (G, ϕ_G) and (H, ϕ_H) be two connected immersed graphs.

$$d_C(G,H) = \inf_{\tau} \sup_{(x,y)\in\tau} \|\phi_G(x) - \phi_H(y)\|,$$

where G ranges over all correspondences τ between G and H such that

1. $\tau \subseteq G \times H$ is connected
2. For each $x \in G$: The set $\tau \cap (\{x\} \times H)$ is non-empty and connected
3. For each $\mathrm{y} \in H$: The set $\tau \cap (G \times \{y\})$ is non-empty and connected





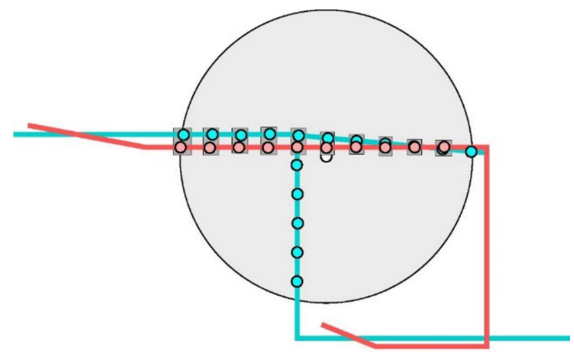
[BOS17] K. Buchin, T. Ophelders, B. Speckmann. Computing the Fréchet distance between real-valued surfaces. SODA: 2443–2455, 2017.

Contour Tree Distance

- The contour tree distance is a metric on connected graphs.
- But it is NP-complete, already for trees.
- This distance seems to correspond to a symmetric version of the (strong or weak) graph distances.

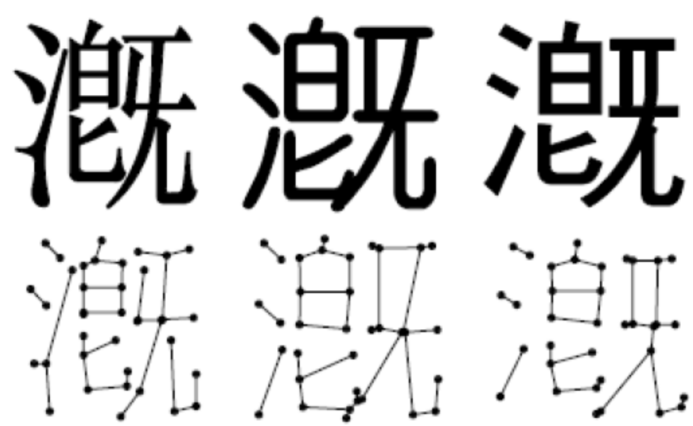
[BOS17] K. Buchin, T. Ophelders, B. Speckmann. Computing the Fréchet distance between real-valued surfaces. SODA: 2443–2455, 2017.
[BK**W**21] M. Buchin, B. Kilgus, **C. Wenk**. Ongoing work.

3. Other Distances



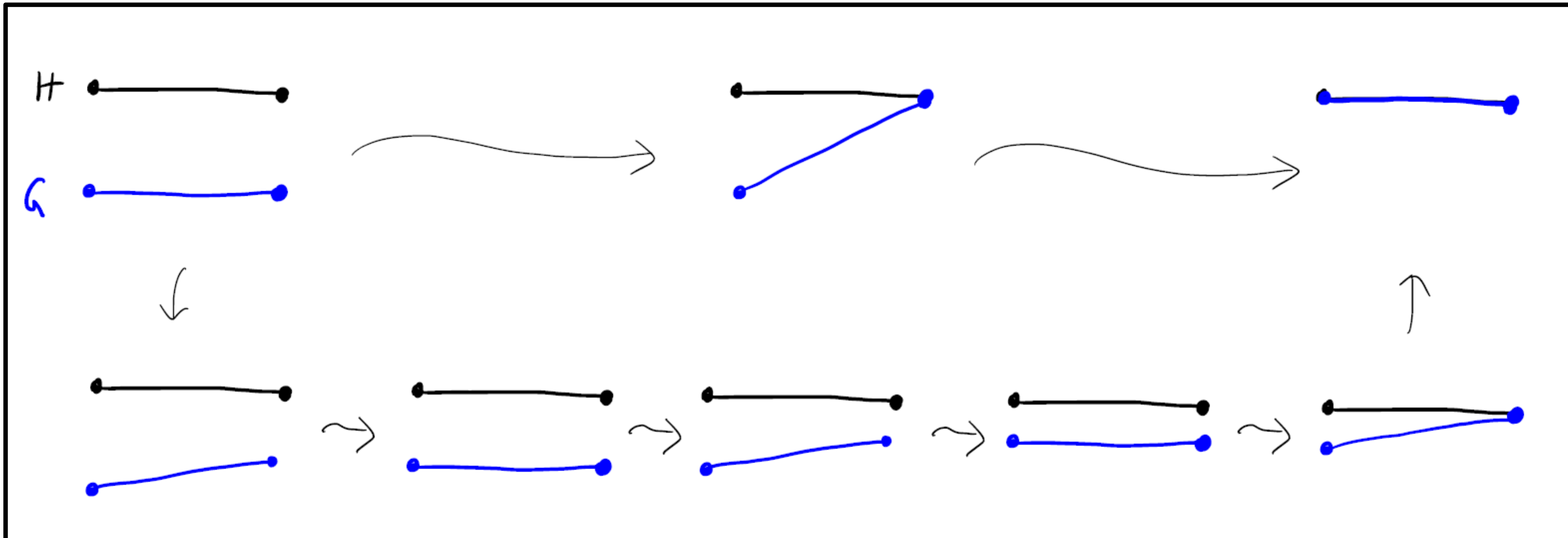
Geometric Edit Distance

- Geometric Edit Distance in $\mathbb{R}^2$:
  - Defined for straight-line embedded graphs.
  - Motivated by Chinese character comparison
  - Perform the following edit operations in this order:
    Edge deletion, vertex deletion,
    vertex translation,
    vertex insertion, edge insertion
  - Costs are proportional to edge length differences and to the distance a vertex has moved.
  - Is a metric. But NP-hard.



[CGKSS09] O. Cheong, J. Gudmundsson, H.-S. Kim, D. Schymura, F. Stehn, Measuring the similarity of geometric graphs, SEA: 101–112, 2009.

Geometric Edit Distance



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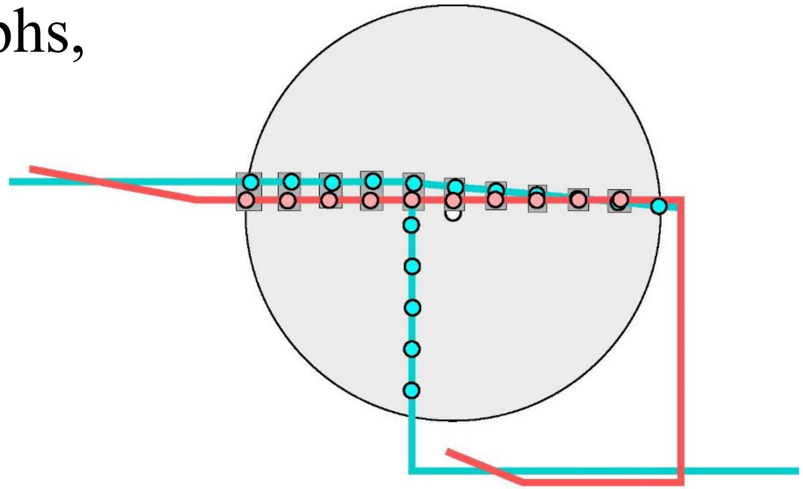
[CGKSS09] O. Cheong, J. Gudmundsson, H.-S. Kim, D. Schymura, F. Stehn, Measuring the similarity of geometric graphs, SEA: 101–112, 2009.

Graph Sampling Distance

- In a local neighborhood of both graphs, traverse the graphs (from random seeds) and place point samples. (Only graph edges of length ≤ τ.)

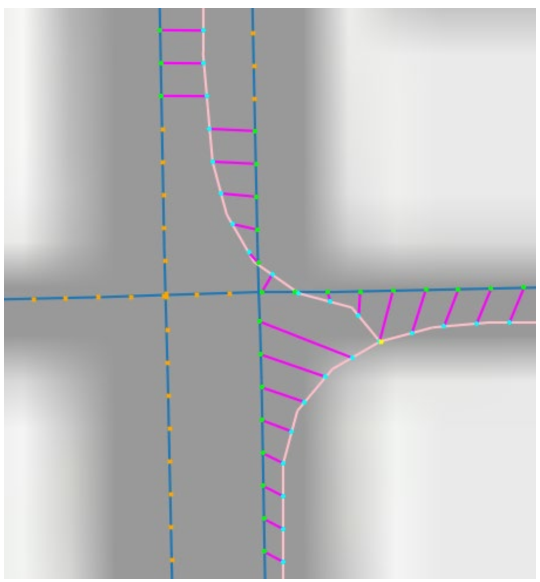


- $\boldsymbol{\tau}$: match_distance threshold
  $\boldsymbol{m}=m(\tau)$: #samples in G
  $\boldsymbol{n}=n(\tau)$: #samples in H
  $\boldsymbol{k}=k(\tau)$ = #matched samples (1-1) within distance τ
- **Precision:** $p = k/n$ **Recall:** $r = k/m$ **F-score:** $2pr/(p+r) = 2k/(n+m)$

[BE12b] J. Biagioni, J. Eriksson, Inferring road maps from global… TRR: J. of the Transportation Research Board 2291, 61-71, 2012.

Graph Sampling Distance

- Lacks theoretical foundation but is practical.
- Does not work well if the reconstructed graph is compared to a more detailed ground-truth graph (e.g., OSM).
- Provides a matching (1-to-1) between a subset of points in G and H



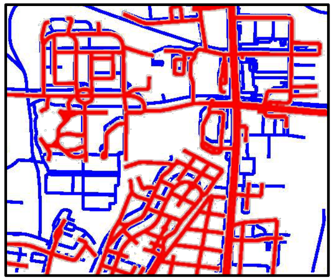
- What is a good matching? [ABBHSW21]
- Graph sampling toolkit
github.com/Erfanh1995/GraphSamplingToolkit
- Continuous definition via length-preserving Fréchet correspondence [BFHW21]

[ABBHS**W**21] J. Aguilar, K.&M Buchin, E. Hosseini S., R.I. Silveira, **C. Wenk**, Graph Sampling for Map Comparison, Spatial Gems, 2021

[BFH**W**21] K. Buchin, B.T. Fasy, E. Hosseini S., **C. Wenk**, Measuring Length-Preserving Fréchet Correspondence for Graphs in R^2, FWCG, 2021

Conclusion & Discussion

1. We've seen a lot of distances for immersed graphs.
   - Are they useful in practice? (Noisy input, runtimes)
   - What are their mathematical properties? (Metric, topological) [CFHMW22]
2. Would like to compute a correspondence / mapping between the two graphs efficiently.
   - An application: Merge multiple road networks



3. What's a good edit distance definition for immersed graphs?
4. Optimize under transformations

[CFHM**W**22] E. Chambers, B.T. Fasy, B. Holmgren, S. Majhi, **C. Wenk**, Path-Connectivity of Fréchet Spaces of Graphs, YRF 2022