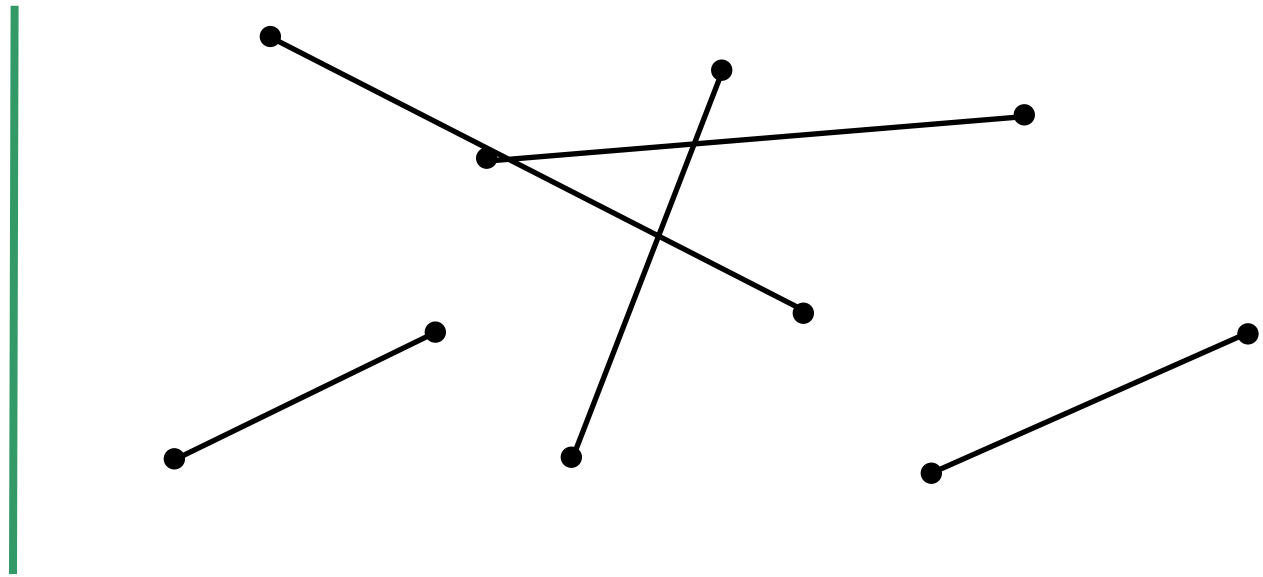
CS 6463: AT Computational Geometry
Fall 2010

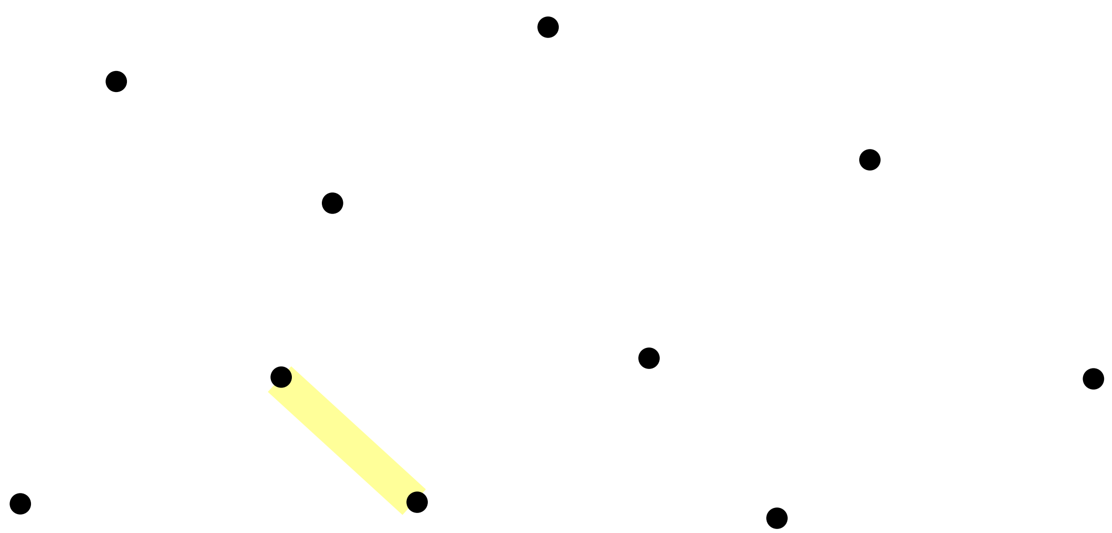


Plane Sweep Algorithms and Segment Intersection

Carola Wenk

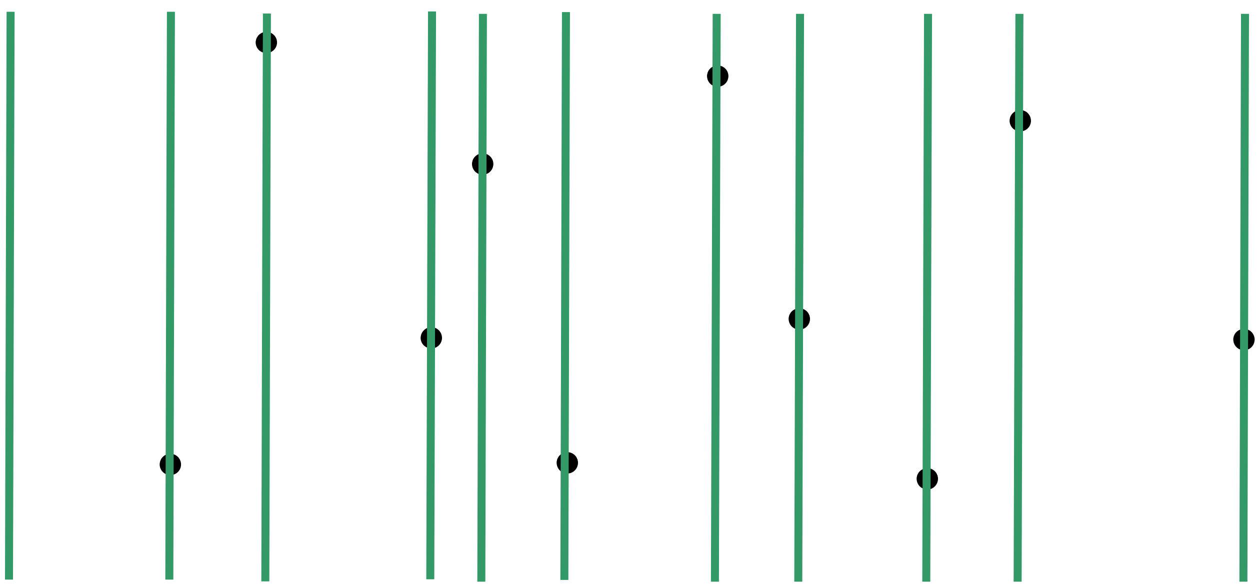
Closest Pair

- **Problem:** Given $P \subseteq \mathbf{R}^2$, $|P|=n$, find the distance between the closest pair in P



Plane Sweep: An Algorithm Design Technique

- Simulate sweeping a vertical line from left to right across the plane.
- Maintain **cleanliness property**: At any point in time, to the left of sweep line everything is clean, i.e., properly processed.
- **Sweep line status**: Store information along sweep line
- **Events**: Discrete points in time when sweep line status needs to be updated



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Algorithm Generic_Plane_Sweep:

Initialize **sweep line status** S at time $x=-\infty$

Store initial events in **event queue** Q, a priority queue ordered by x-coordinate

while $Q \neq \emptyset$

    // extract next event e:
    e = Q.extractMin();
    // handle event:
    Update sweep line status
    Discover new upcoming events and insert them into Q

Plane sweep for Closest Pair

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- **Problem:** Given $P \subseteq \mathbf{R}^2$, $|P|=n$, find the distance of the closest pair in P
- **Sweep line status:**
  Cleanliness property
  - Store current distance Δ of closest pair of points to the left of sweep line
  - Store points in Δ-strip left of sweep line
  - Store pointer to leftmost point in strip
- **Events:** All points in P. No new events will be added during the sweep.
  $\rightarrow$ Presort P by x-coordinate.

Plane sweep for Closest Pair, II

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$O(n \log n)$

- Presort P by x-coordinate
- How to store points in Δ-strip?
  - Store points in Δ-strip left of sweep line in a balanced binary search tree, ordered by y-coordinate
    $\rightarrow$ Add point, delete point, and search in $O(\log n)$ time
- **Event handling:**
  - New event: Sweep line advances to point $p \in P$
  - Update sweep line status:
    - Delete points outside Δ-strip from search tree by using previous leftmost point in strip and x-order on P

$O(n \log n)$ total

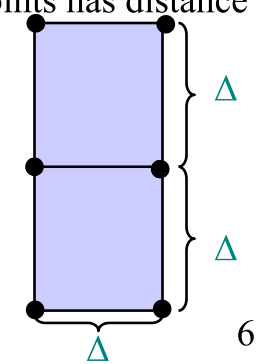
    - Compute candidate points that may have distance $\leq \Delta$ from p:

$O(n \log n + 6n)$ total

      - Perform a search in the search tree to find points in Δ–strip whose y-coordinates are at most Δ away from $p.y$.
        $\rightarrow \Delta$ x 2Δ box
      - Because of the cleanliness property each pair of these points has distance $\leq\Delta$.
        $\rightarrow$ A Δ x 2Δ box can contain at most 6 such points.

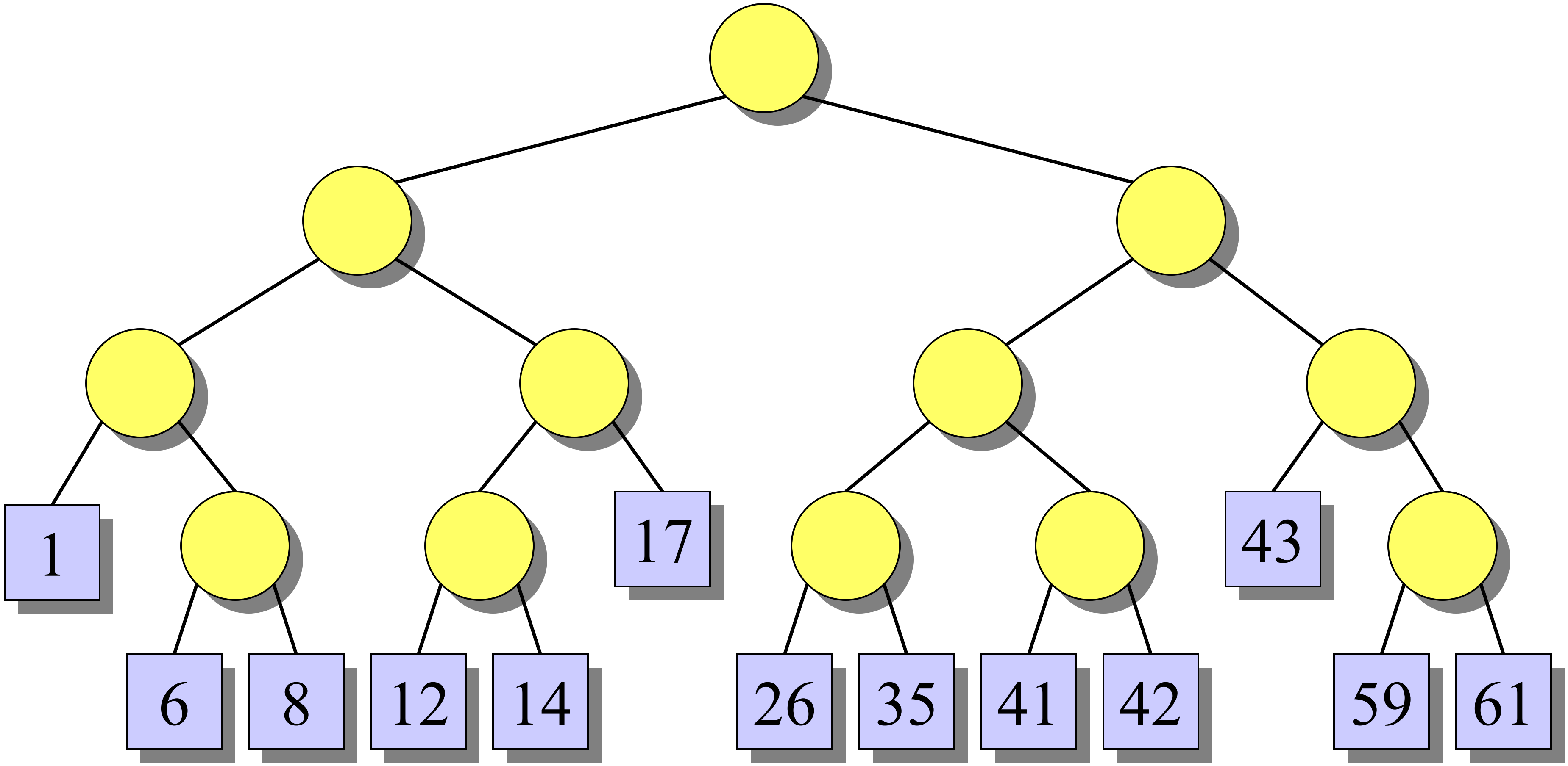
$O(6n)$ total

    - Check distance of these points to p, and possibly update Δ
  - No new events necessary to discover



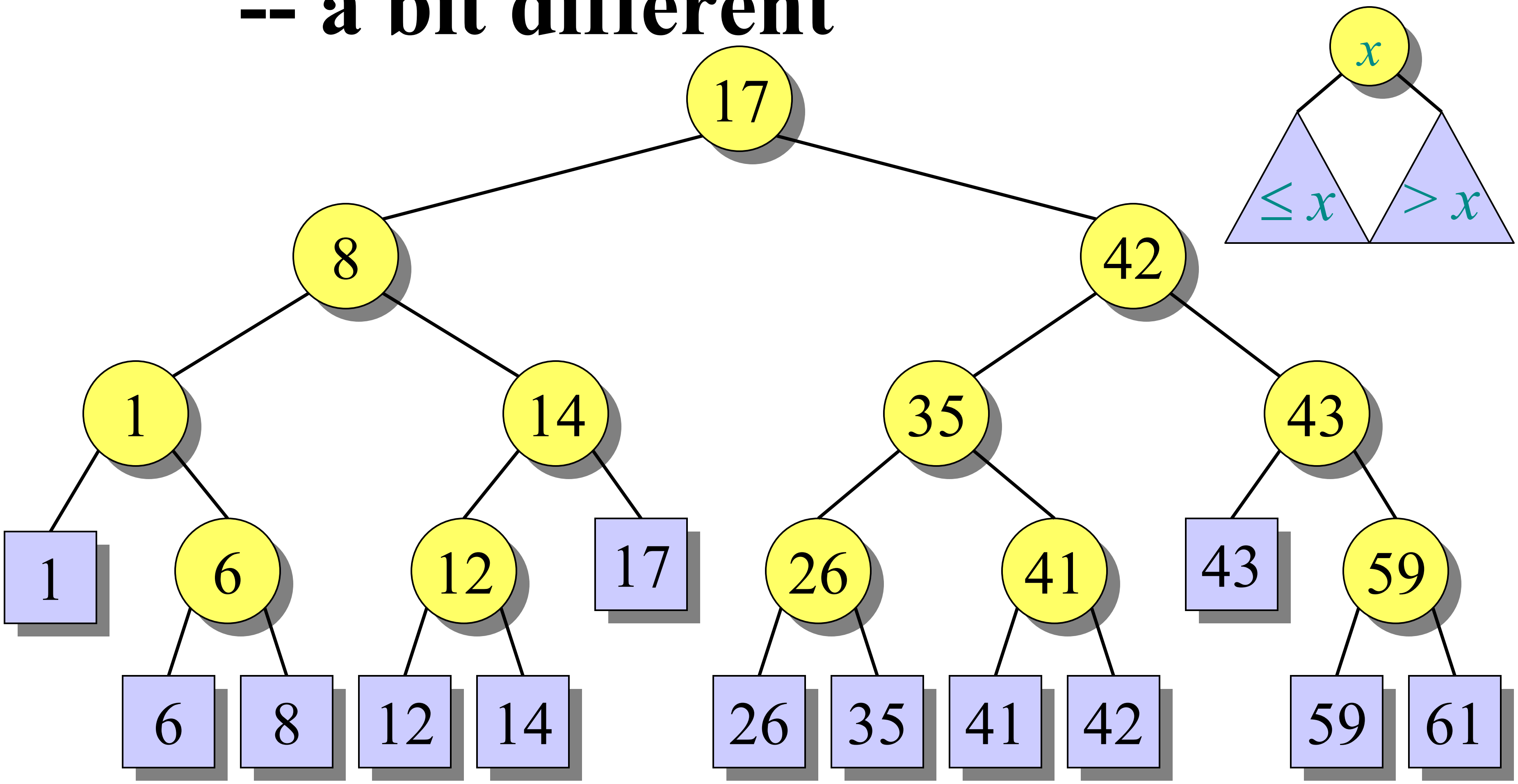
Total runtime: $O(n \log n)$

Balanced Binary Search Tree -- a bit different



key[*x*] is the maximum key of any leaf in the left subtree of *x*.

Balanced Binary Search Tree -- a bit different



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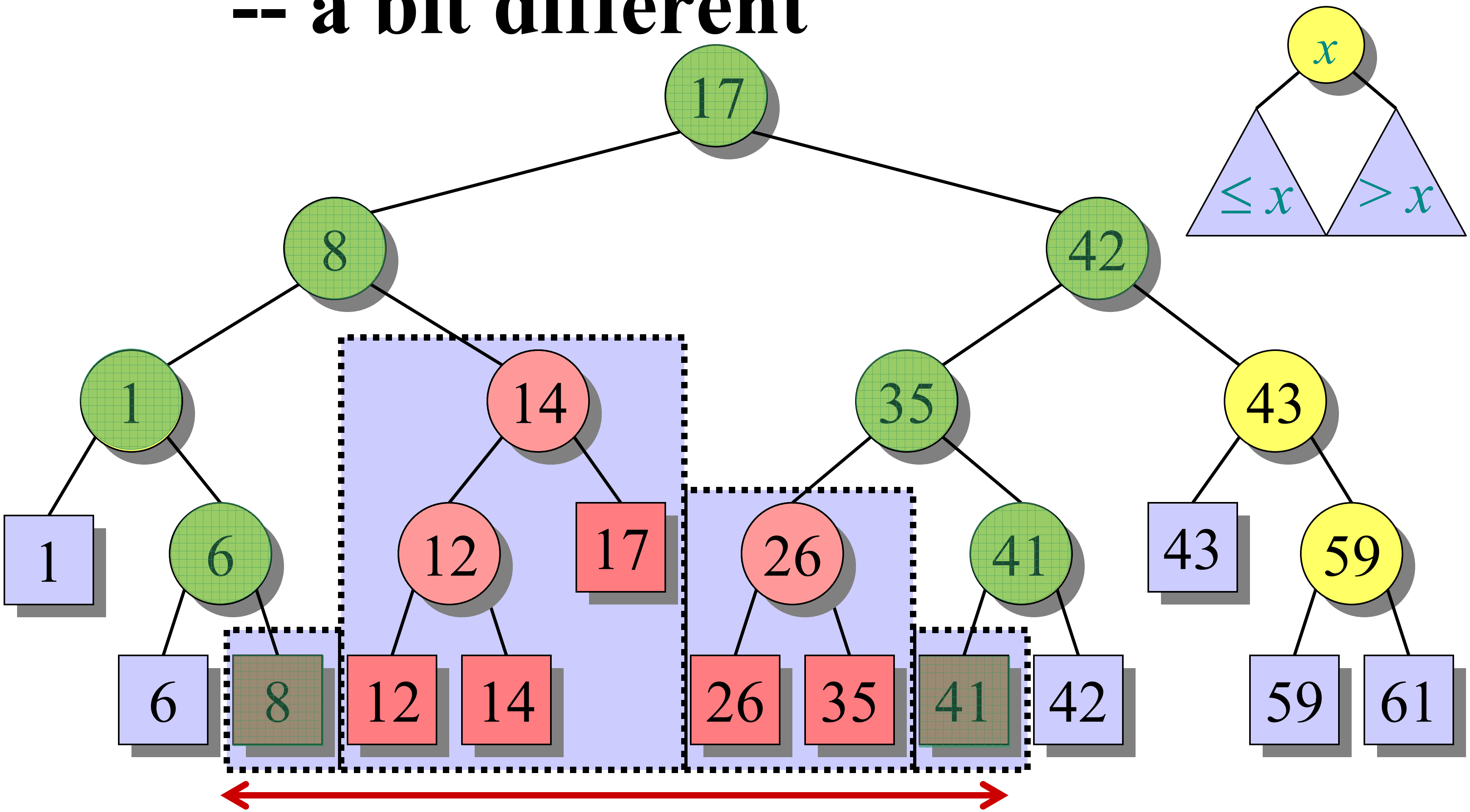


RANGE-QUERY([7, 41])

Plane Sweep: An Algorithm Design Technique

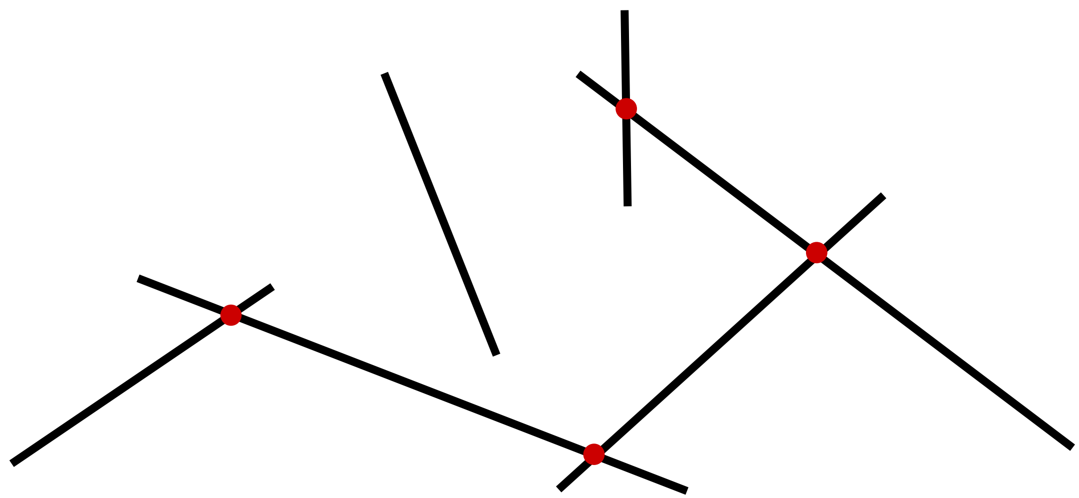
- Plane sweep algorithms (also called sweep line algorithms) are a special kind of incremental algorithms
- Their correctness follows inductively by maintaining the cleanliness property
- *Common* runtimes in the plane are $O(n \log n)$:
  - n events are processed
  - Update of sweep line status takes $O(\log n)$
  - Update of event queue: $O(\log n)$ per event

Geometric Intersections

- Important and basic problem in Computational Geometry
- Solid modeling: Build shapes by applying set operations (intersection, union).
- Robotics: Collision detection and avoidance
- Geographic information systems: Overlay two subdivisions (e.g., road network and river network)
- Computer graphics: Ray shooting to render scenes

Line Segment Intersection

- Input: A set $S=\{s_1, \ldots, s_n\}$ of (closed) line segments in $\mathbf{R}^2$
- Output: All **intersection points** between segments in S



Line Segment Intersection

- n line segments can intersect as few as 0 and as many as $\binom{n}{2} = O(n^2)$ times
- Simple algorithm: Try out all pairs of line segments
  $\rightarrow$ Takes $O(n^2)$ time
  $\rightarrow$ Is optimal in worst case
- Challenge: Develop an **output-sensitive algorithm**
  - Runtime depends on size k of the output
  - Here: $0 \leq k \leq c\, n^2$, where c is a constant
  - Our algorithm will have runtime: $O(\ (n+k) \log n)$
  - Best possible runtime: $O(\ n \log n + k)$
    $\rightarrow$ $O(n^2)$ in worst case, but better in general

Complexity

- Why is runtime $O(n \log n + k)$ optimal?
- The element uniqueness problem requires $\Omega(n \log n)$ time in algebraic decision tree model of computation (Ben-Or '83)
- Element uniqueness: Given n real numbers, are all of them distinct?
- Solve element uniqueness using line segment intersection:
  - Take n numbers, convert into vertical line segments. There is an intersection iff there are duplicate numbers.
  - If we could solve line segment intersection in $o(n \log n)$ time, i.e., strictly faster than $\Theta(n \log n)$, then element uniqueness could be solved faster. Contradiction.

Intersection of two line segments

- Two line segments ab and cd
- Write in terms of convex combinations:

  $p(s) = (1\text{-}s)\ a + s\ b$ for $0 \leq s \leq 1$

  $q(t) = (1\text{-}t)\ c + t\ d$ for $0 \leq t \leq 1$

  Intersection if $p(s)=q(t)$

$\Rightarrow$ Equation system

$$(1\text{-}s)\ a_x + s\ b_x = (1\text{-}t)\ c_x + t\ d_x$$
$$(1\text{-}s)\ a_y + s\ b_y = (1\text{-}t)\ c_y + t\ d_y$$

- Solve for s and t. In division, if divisor $= 0$ then line segments are parallel (or collinear). Otherwise get rational numbers for s and t. Either use floating point arithmetic or exact arithmetic.

Plane sweep algorithm

```
Algorithm Generic_Plane_Sweep:
Initialize sweep line status S at time x=-∞
Store initial events in event queue Q, a priority queue ordered by x-coordinate
while Q ≠ ∅
    // extract next event e:
    e = Q.extractMin();
    // handle event:
    Update sweep line status
    Discover new upcoming events and insert them into Q
```

- **Cleanliness property:**
  - All intersections to the left of sweep line l have been reported
- **Sweep line status:**
  - Store segments that intersect the sweep line l, ordered along the intersection with l .
- **Events:**
  - Points in time when sweep line status changes combinatorially (i.e., the order of segments intersecting l changes)

  → Endpoints of segments (insert in beginning)

  → Intersection points (compute on the fly during plane sweep)

General position

Assume that “nasty” special cases don’t happen:

- No line segment is vertical
- Two segments intersect in at most one point
- No three segments intersect in a common point

Event Queue

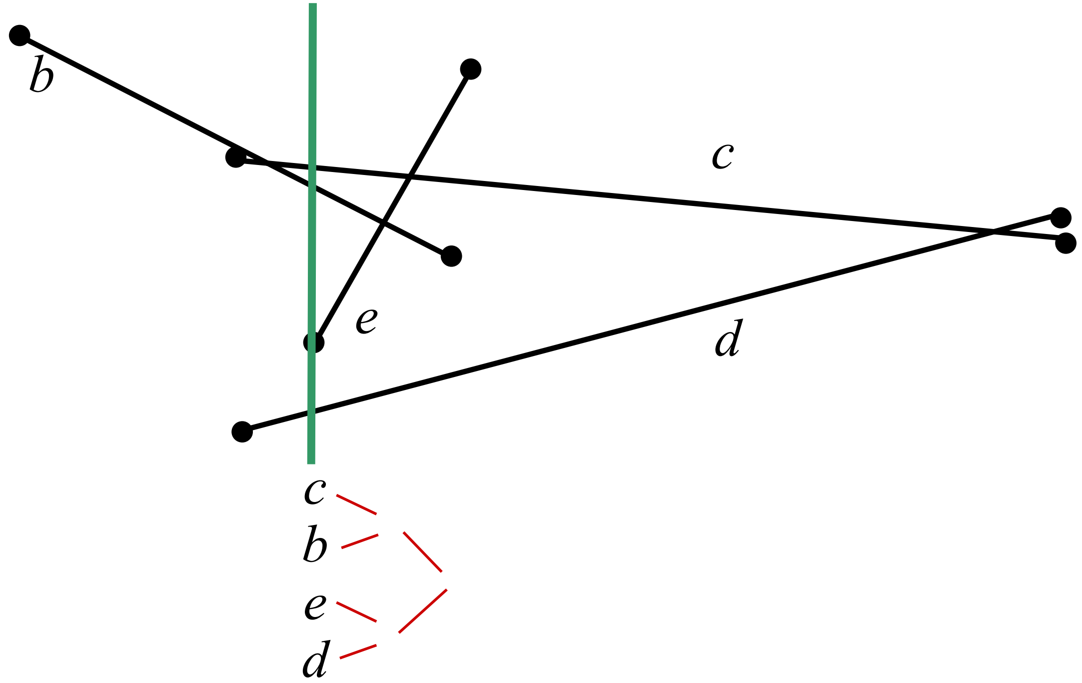
- Need to keep events sorted:
  - Lexicographic order (first by x-coordinate, and if two events have same x-coordinate then by y-coordinate)
- Need to be able to remove next point, and insert new points in $O(\log n)$ time
- Need to make sure not to process same event twice

$\Rightarrow$ Use a priority queue (heap), and possibly extract multiples

$\Rightarrow$ Or, use balanced binary search tree

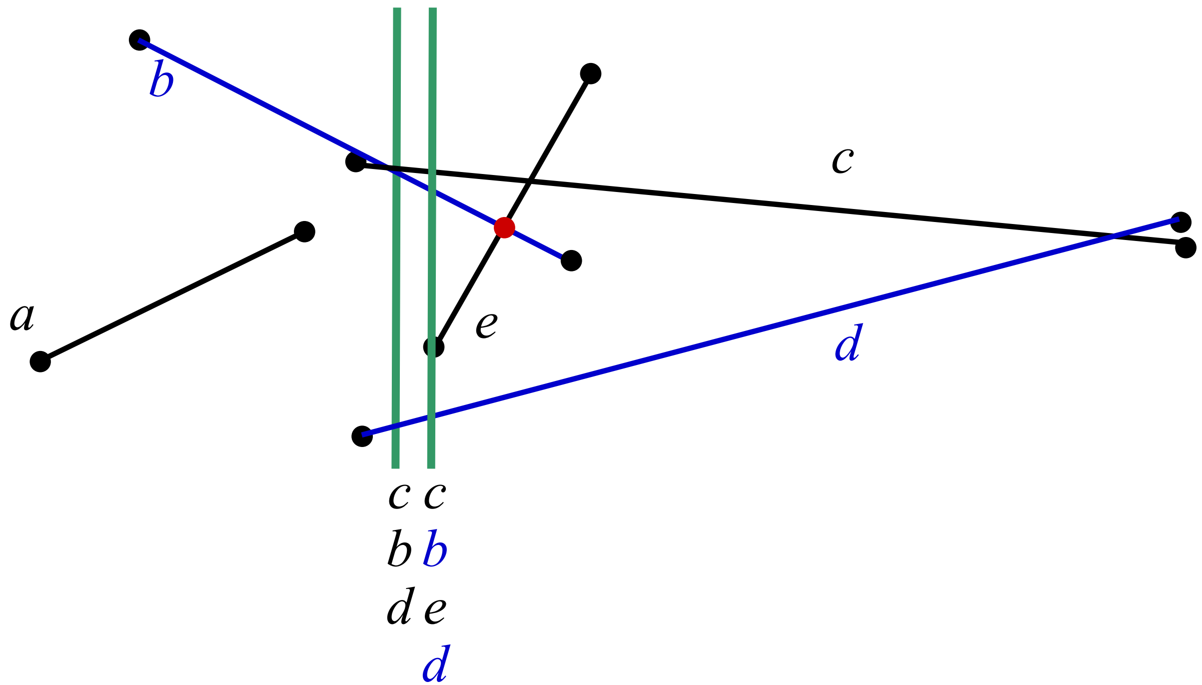
Sweep Line Status

- Store segments that intersect the sweep line *l*, ordered along the intersection with *l* .
- Need to insert, delete, and find adjacent neighbor in O(log *n*) time
- Use **balanced binary search** tree, storing the order in which segments intersect *l* in leaves



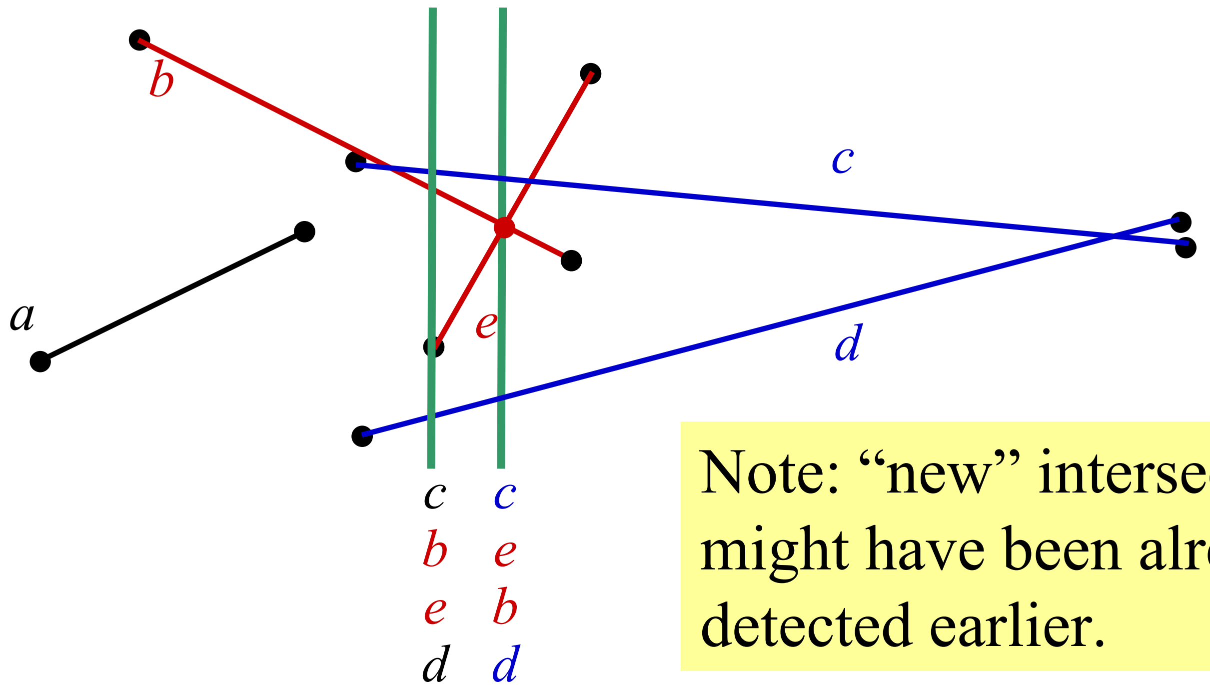
Event Handling

1. Left segment endpoint
   - Add segment to sweep line status
   - Test **adjacent segments** on sweep line *l* for intersection with new segment (see Lemma)
   - Add **new intersection points** to event queue



Event Handling

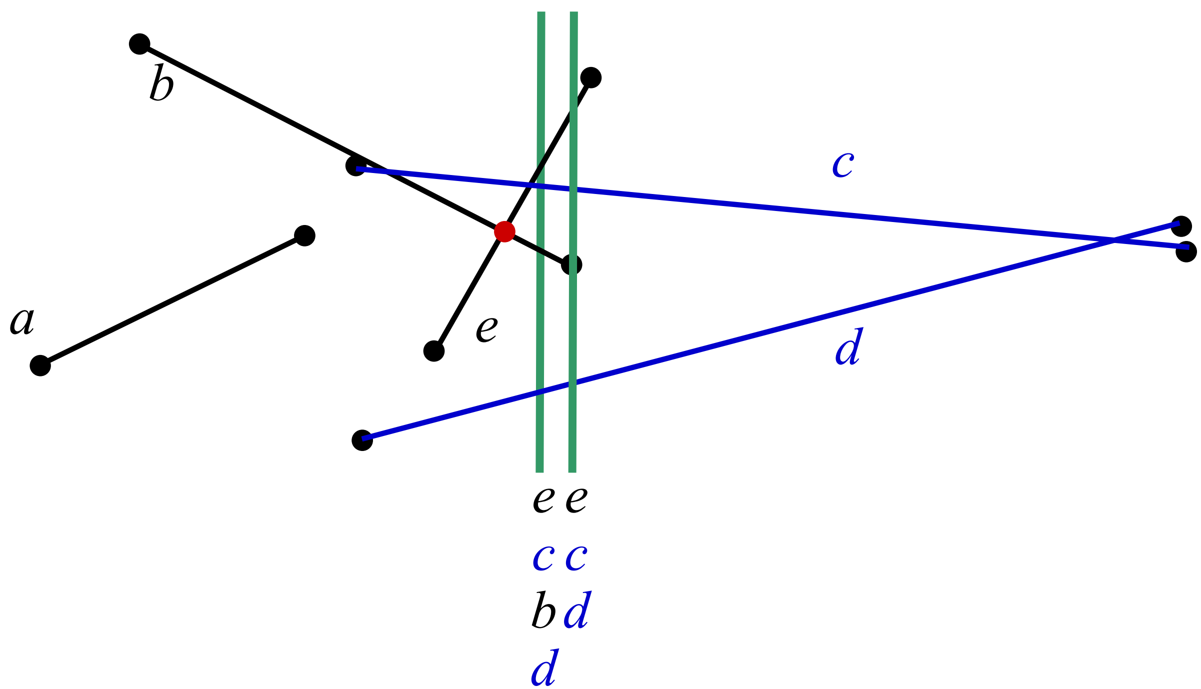
2. Intersection point
   - Report new intersection point
   - Two segments **change order** along l
     → Test new adjacent segments for new intersection points (to insert into event queue)



Note: "new" intersection might have been already detected earlier.

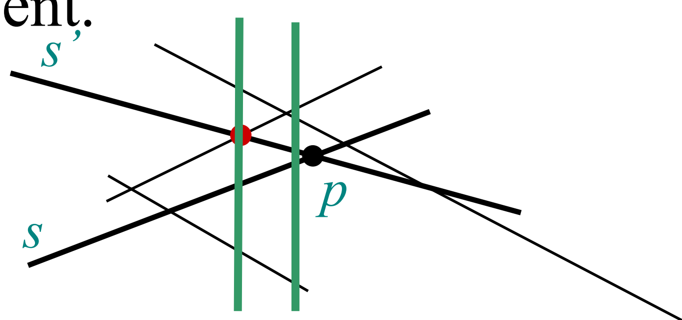
Event Handling

3. Right segment endpoint
   - Delete segment from sweep line status
   - **Two segments become adjacent**. Check for intersection points (to insert in event queue)



Intersection Lemma

- **Lemma:** Let *s*, *s'* be two non-vertical segments whose interiors intersect in a single point *p*. Assume there is no third segment passing through *p*. Then there is an event point to the left of *p* where *s* and *s'* become adjacent (and hence are tested for intersection).
- **Proof:** Consider placement of sweep line infinitesimally left of *p*. *s* and *s'* are adjacent along sweep line. Hence there must have been a **previous event point** where *s* and *s'* become adjacent.



Runtime

- Sweep line status updates: $O(\log n)$
- Event queue operations: $O(\log n)$, as the total number of stored events is $\leq 2n + k$, and each operation takes time $O(\log(2n+k)) = O(\log n^2) = O(\log n)$

  $k = O(n^2)$

- There are $O(n+k)$ events. Hence the total runtime is $O((n+k) \log n)$