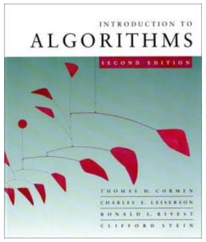
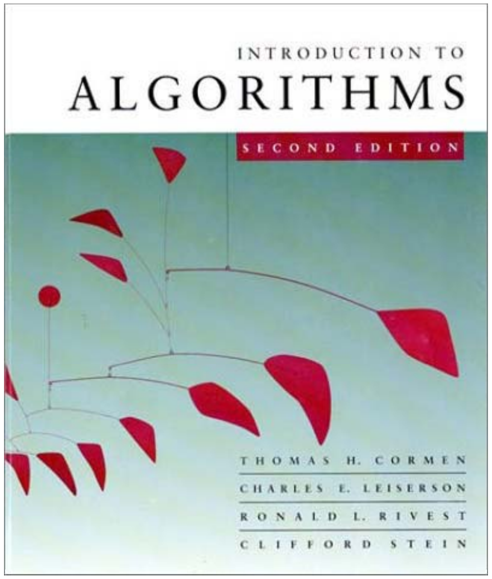
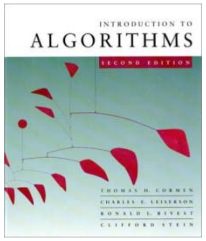



# CS 6463 -- Fall 2010




# *Range Searching and Windowing*

**Carola Wenk**



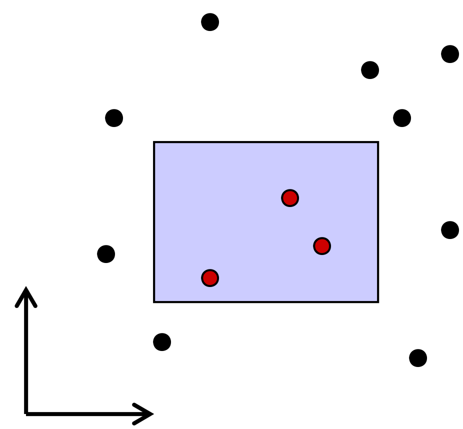

# Orthogonal range searching

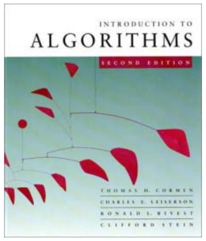
**Input:** $n$ points in $d$ dimensions

- E.g., representing a database of $n$ records each with $d$ numeric fields

**Query:** Axis-aligned ***box*** (in 2D, a rectangle)

- Report on the points inside the box:
  - Are there any points?
  - How many are there?
  - List the points.






# Orthogonal range searching

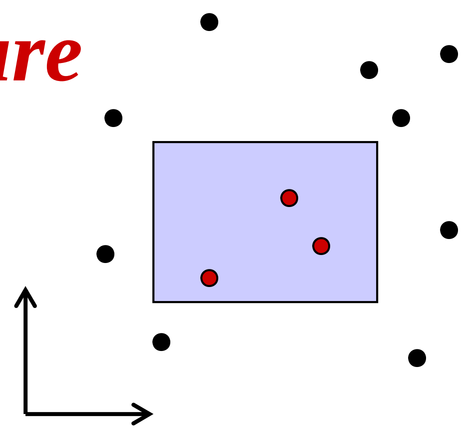
**Input:** *n* points in *d* dimensions

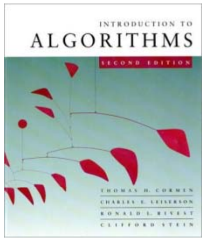
**Query:** Axis-aligned ***box*** (in 2D, a rectangle)

- Report on the points inside the box

**Goal:** Preprocess points into a data structure to support fast queries

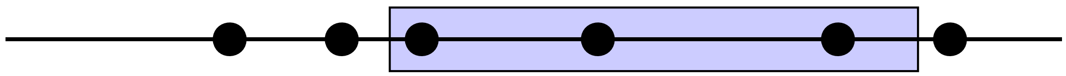
- Primary goal: ***Static data structure***
- In 1D, we will also obtain a dynamic data structure supporting insert and delete






# 1D range searching

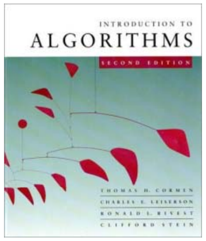
In 1D, the query is an interval:



First solution:

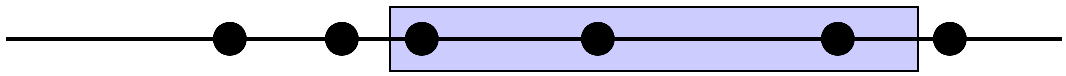
- Sort the points and store them in an array
  - Solve query by binary search on endpoints.
  - Obtain a static structure that can list $k$ answers in a query in $O(k + \log n)$ time.

**Goal:** Obtain a dynamic structure that can list $k$ answers in a query in $O(k + \log n)$ time.



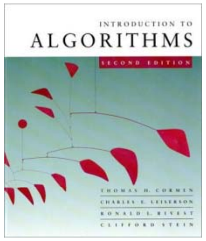

# 1D range searching

In 1D, the query is an interval:

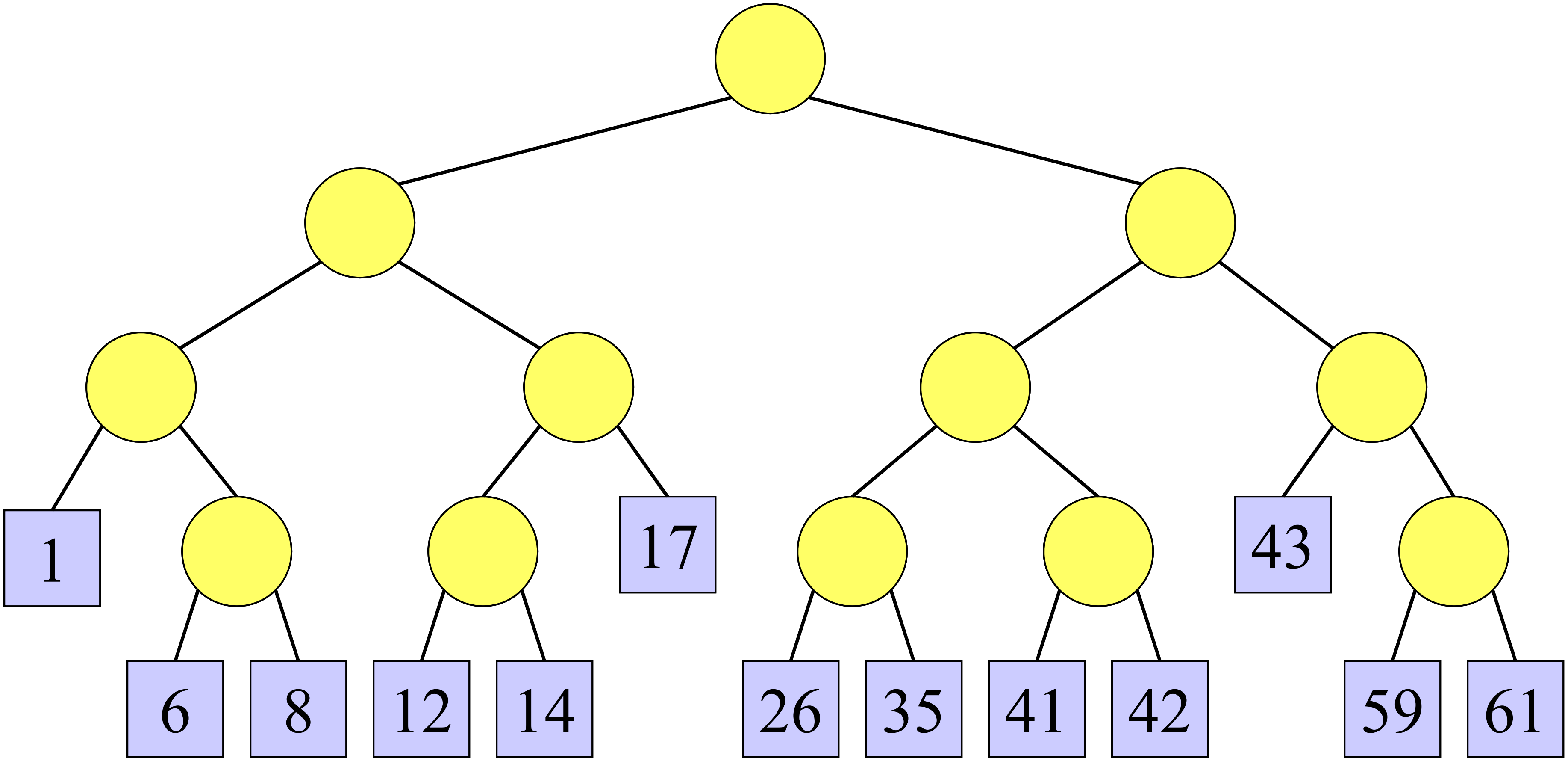


New solution that extends to higher dimensions:

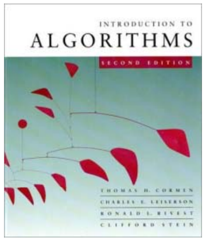
- Balanced binary search tree
  - New organization principle:
    Store points in the *leaves* of the tree.
  - Internal nodes store copies of the leaves to satisfy binary search property:
    - Node $x$ stores in $key[x]$ the maximum key of any leaf in the left subtree of $x$.



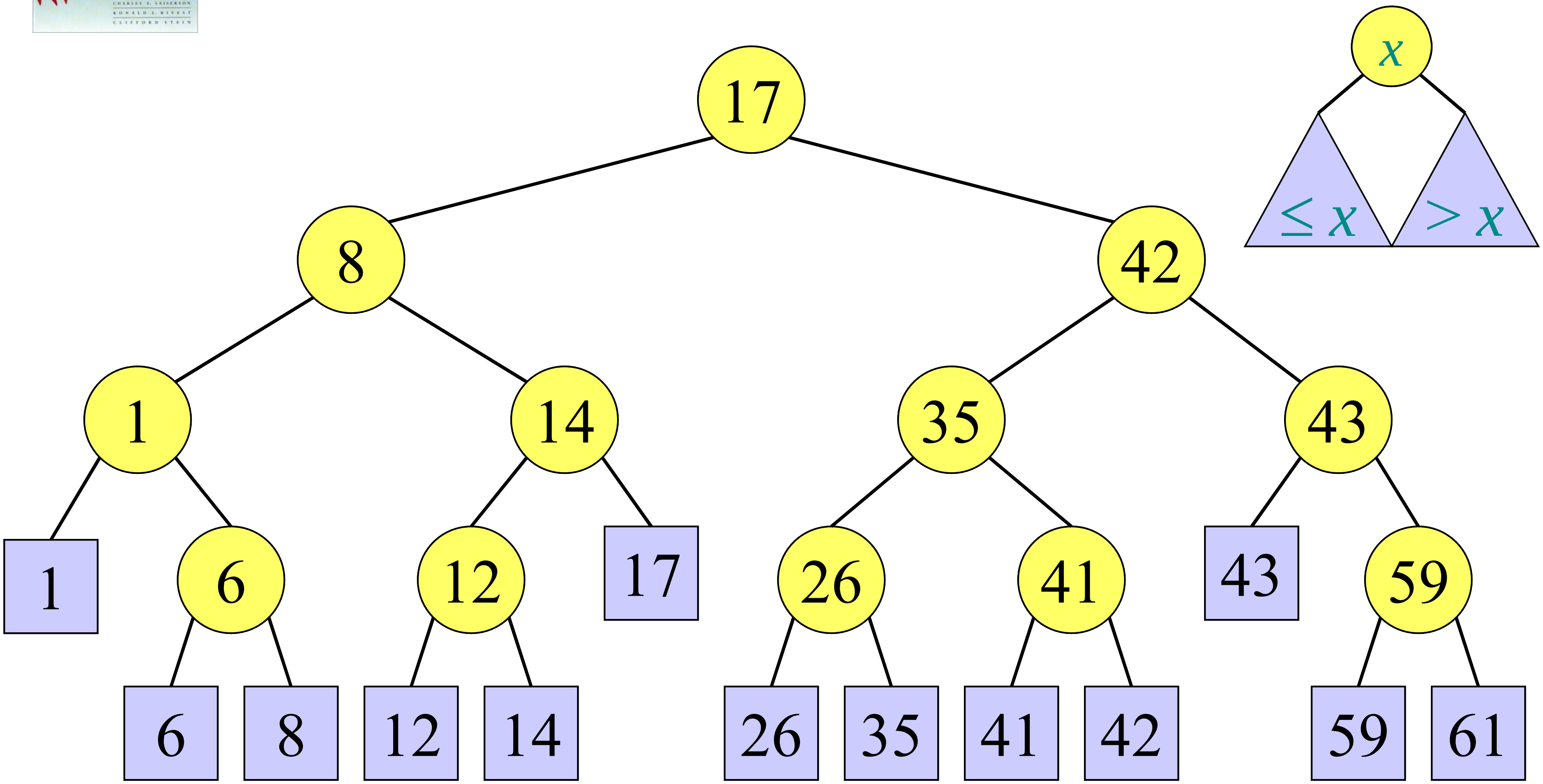

# Example of a 1D range tree



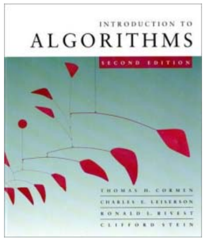

*key*[*x*] is the maximum key of any leaf in the left subtree of *x*.



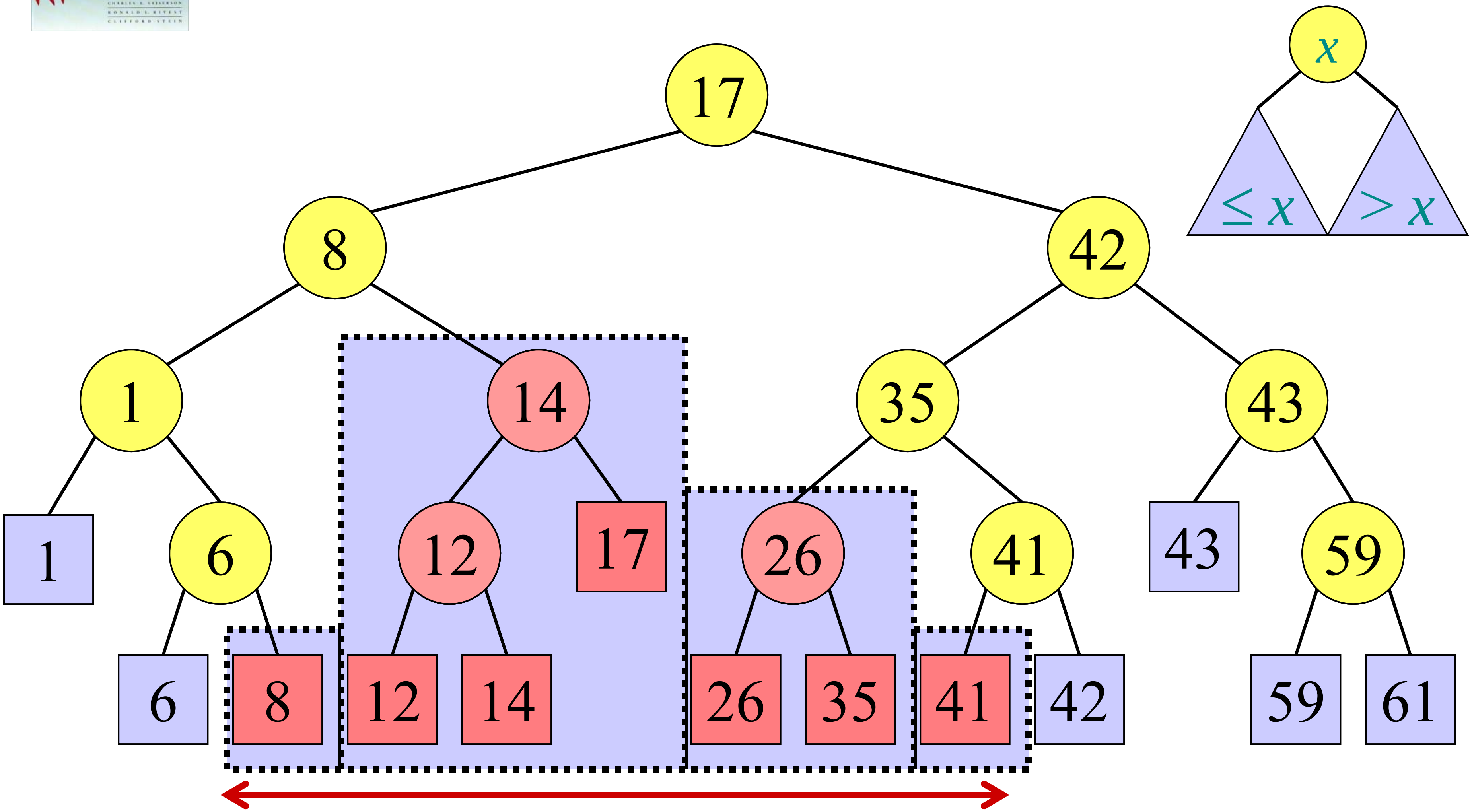

# Example of a 1D range tree



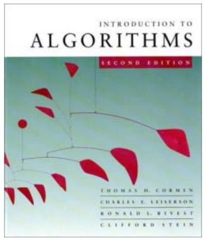

$key[x]$ is the maximum key of any leaf in the left subtree of $x$.



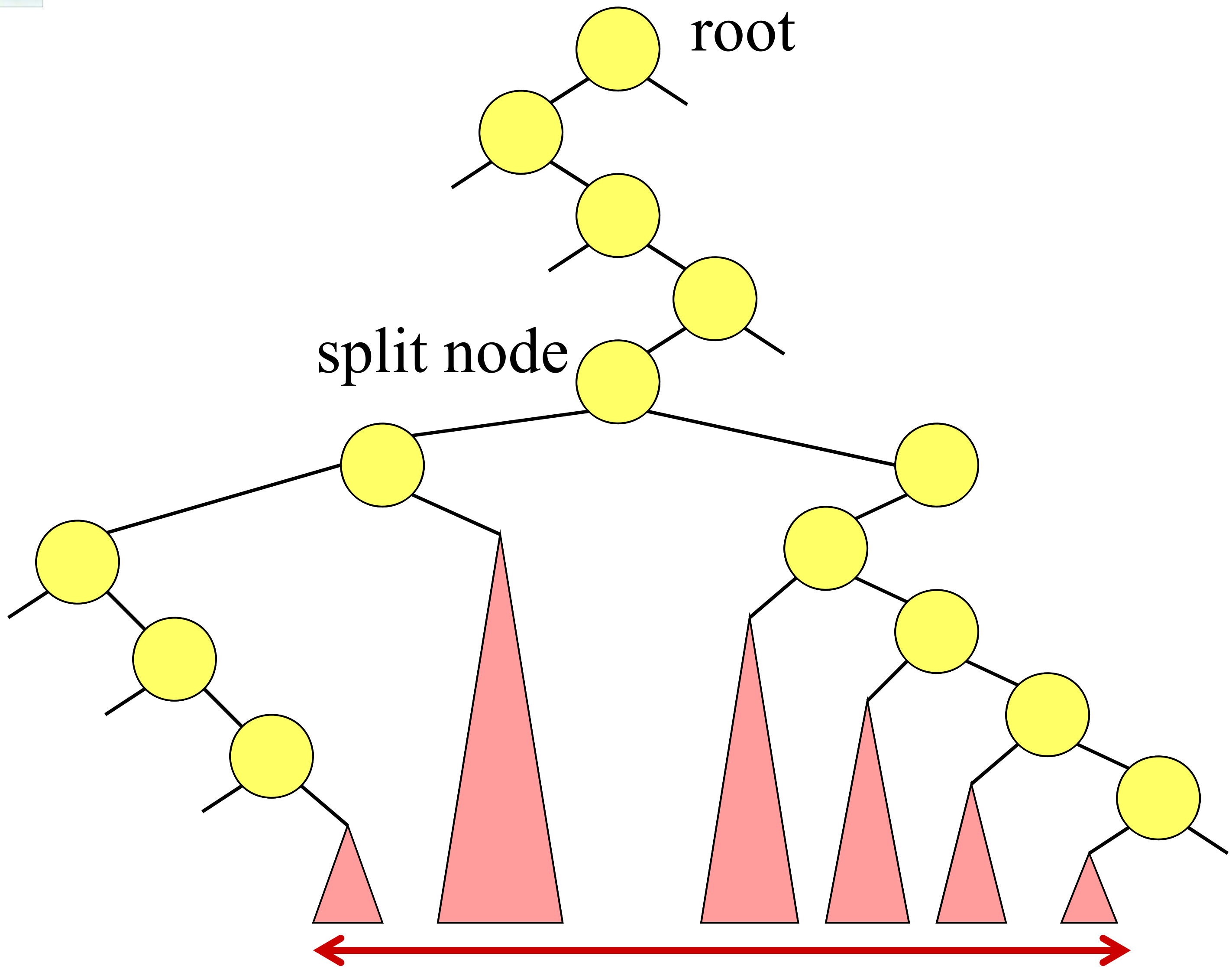

# Example of a 1D range query

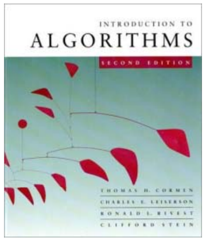



RANGE-QUERY([7, 41])




# General 1D range query






# Pseudocode, part 1: Find the split node

1D-RANGE-QUERY$(T, [x_1, x_2])$

    $w \leftarrow$ root$[T]$

    **while** $w$ is not a leaf and $(x_2 \leq key[w]$ or $key[w] < x_1)$

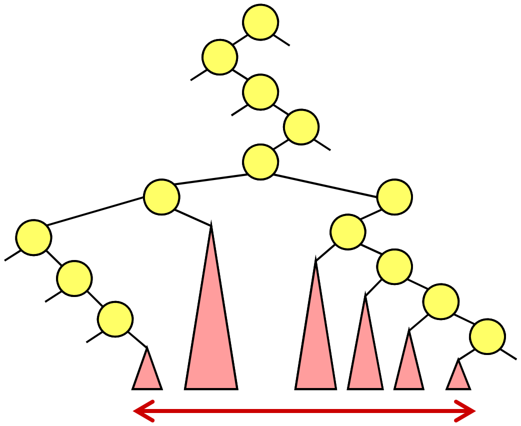
        **do if** $x_2 \leq key[w]$

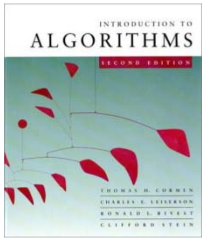
            **then** $w \leftarrow left[w]$

            **else** $w \leftarrow right[w]$

    // $w$ is now the split node

    [*traverse left and right from w and report relevant subtrees*]






# Pseudocode, part 2: Traverse left and right from split node

1D-RANGE-QUERY$(T, [x_1, x_2])$

*[find the split node]*

// $w$ is now the split node

**if** $w$ is a leaf

**then** output the leaf $w$ if $x_1 \leq key[w] \leq x_2$

**else** $v \leftarrow left[w]$ // Left traversal

**while** $v$ is not a leaf

**do if** $x_1 \leq key[v]$

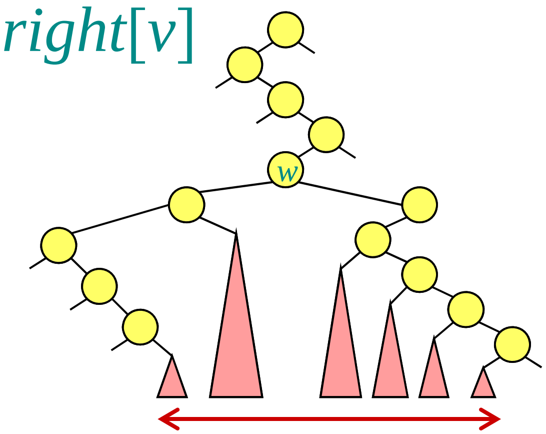
**then** output the subtree rooted at $right[v]$

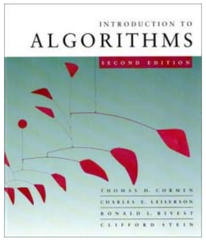
$v \leftarrow left[v]$

**else** $v \leftarrow right[v]$

output the leaf $v$ if $x_1 \leq key[v] \leq x_2$

*[symmetrically for right traversal]*



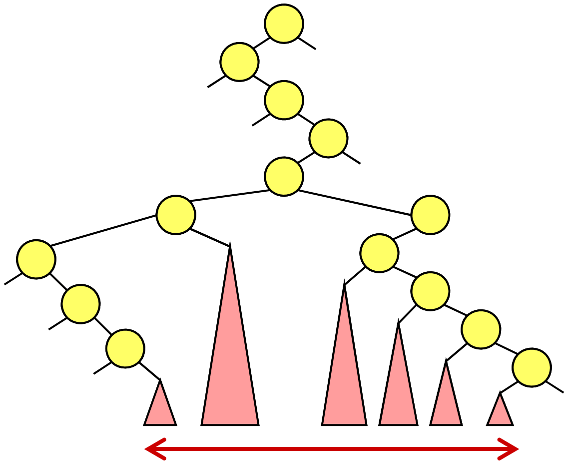



# Analysis of 1D-RANGE-QUERY

**Query time:** Answer to range query represented by O(log $n$) subtrees found in O(log $n$) time.

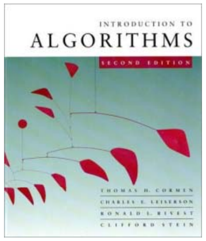
Thus:

- Can test for points in interval in O(log $n$) time.
- Can report all $k$ points in interval in O(k + log $n$) time.
- Can count points in interval in O(log $n$) time

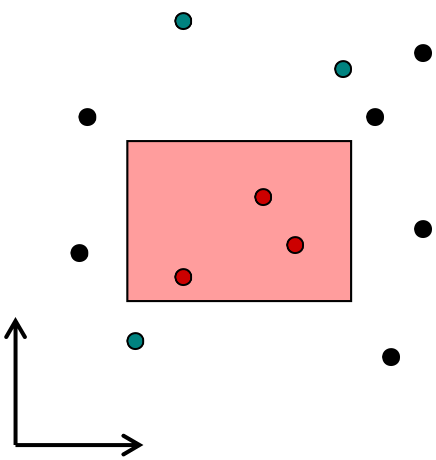


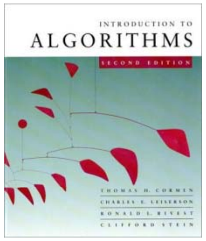
**Space:** O($n$)

**Preprocessing time:** O($n$ log $n$)




# 2D range trees



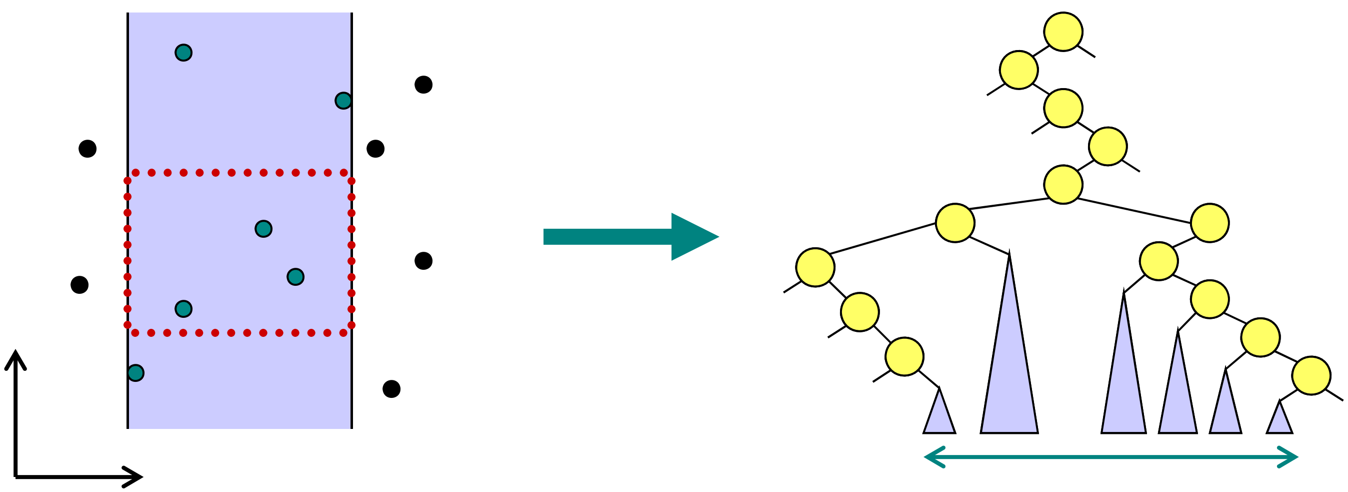


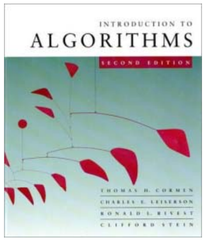

# 2D range trees

Store a ***primary*** 1D range tree for all the points based on *x*-coordinate.

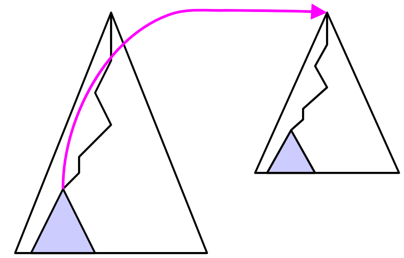
Thus in O(log *n*) time we can find O(log *n*) subtrees representing the points with proper *x*-coordinate.

How to restrict to points with proper *y*-coordinate?

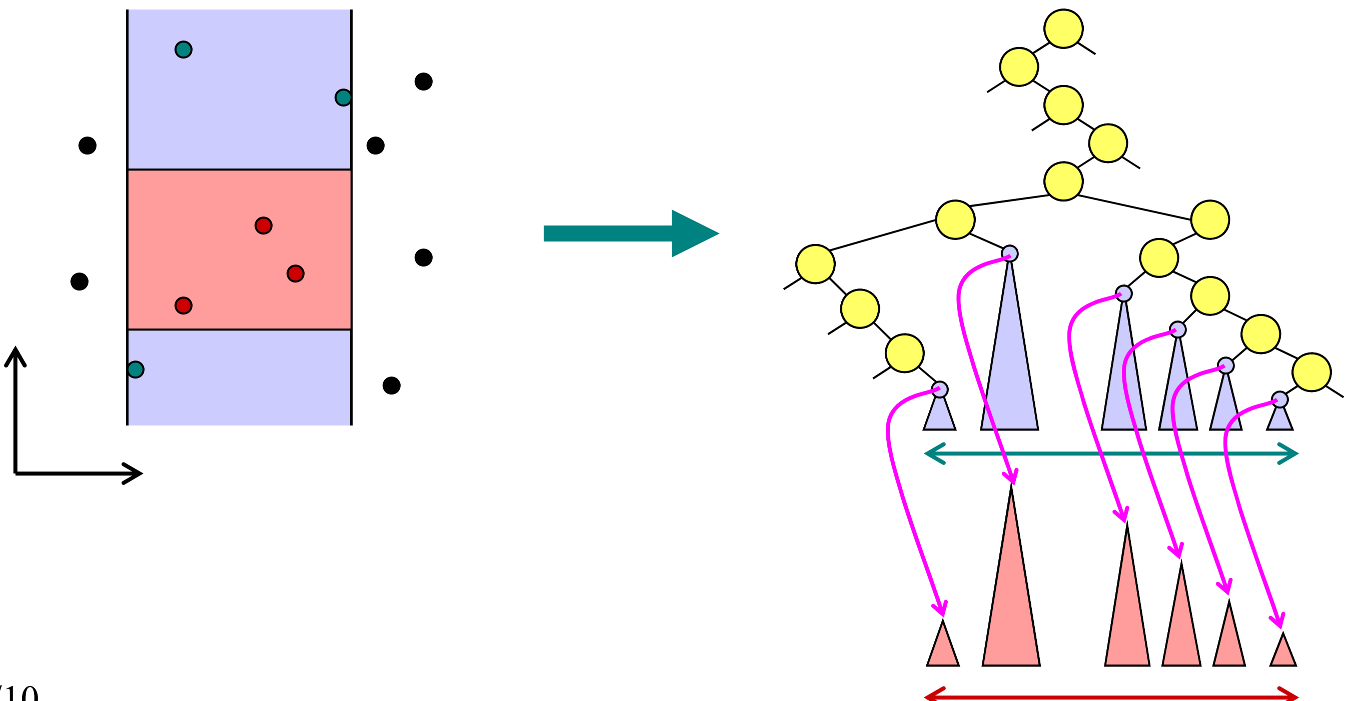


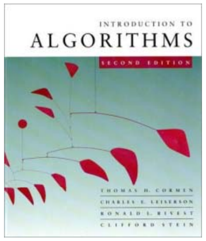



# 2D range trees

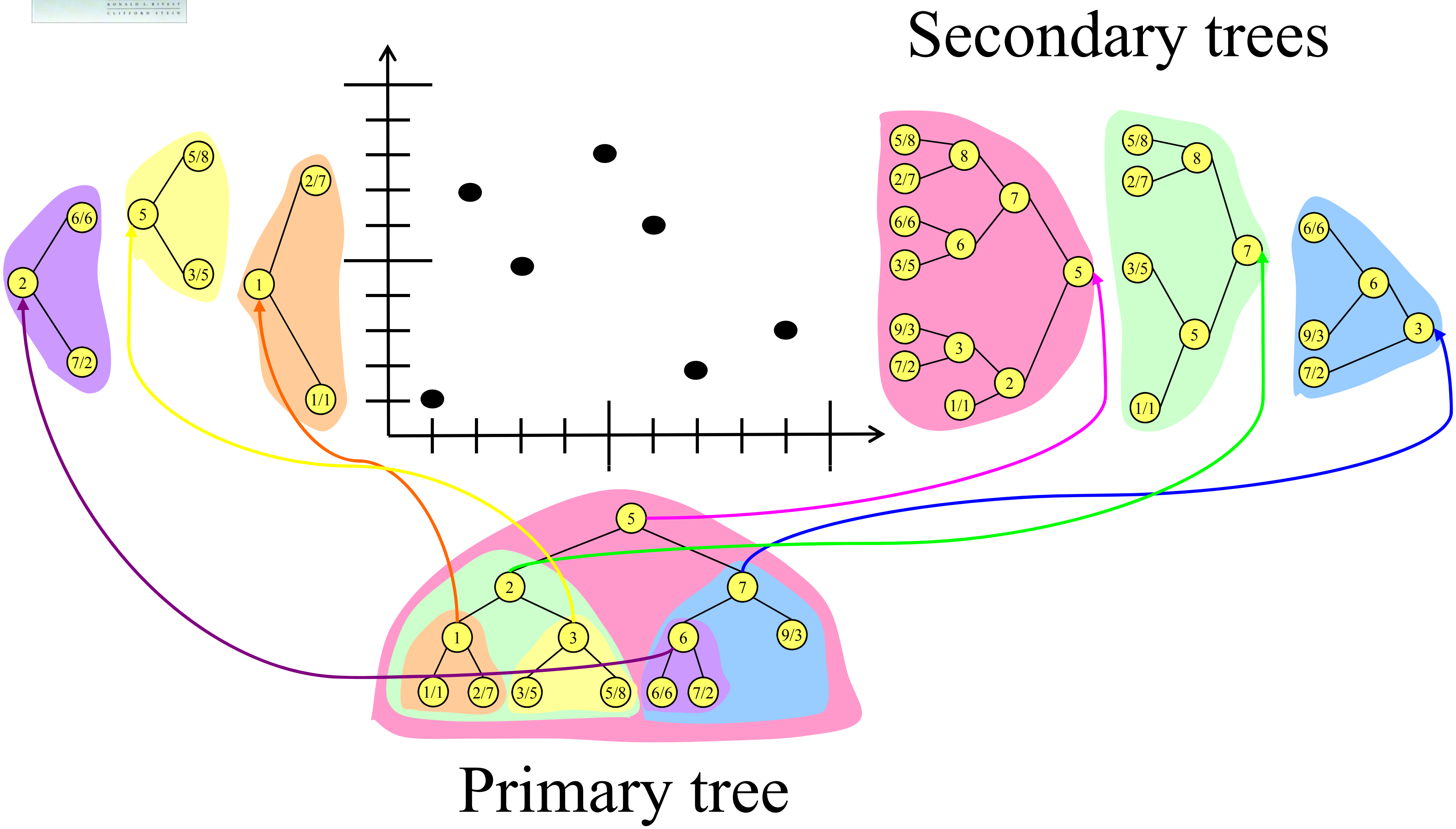


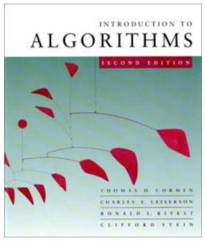
**Idea:** In primary 1D range tree of *x*-coordinate, **<u>every</u>** node stores a ***secondary*** 1D range tree based on *y*-coordinate for all points in the subtree of the node. Recursively search within each.





# 2D range tree example



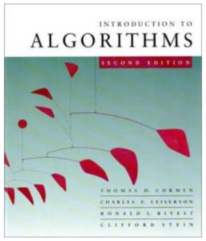



# Analysis of 2D range trees

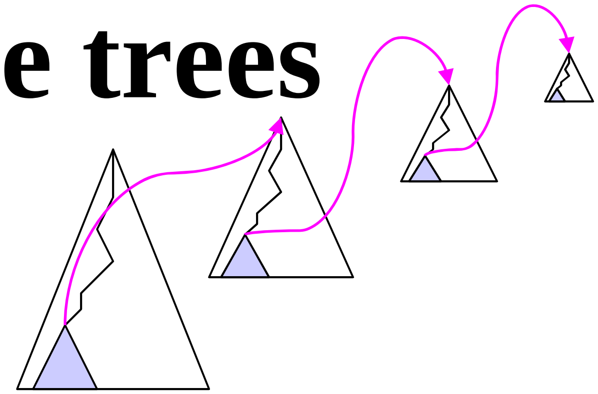
**Query time:** In $O(\log^2 n) = O((\log n)^2)$ time, we can represent answer to range query by $O(\log^2 n)$ subtrees. Total cost for reporting $k$ points: $O(k + (\log n)^2)$.

**Space:** The secondary trees at each level of the primary tree together store a copy of the points. Also, each point is present in each secondary tree along the path from the leaf to the root. Either way, we obtain that the space is $O(n \log n)$.

**Preprocessing time:** $O(n \log n)$



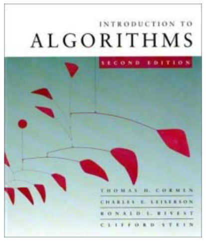

# *$d$*-dimensional range trees



Each node of the secondary *$y$*-structure stores a tertiary *$z$*-structure representing the points in the subtree rooted at the node, etc.

Save one log factor using fractional cascading

**Query time:** $O(k + \log^d n)$ to report $k$ points.
**Space:** $O(n \log^{d-1} n)$
**Preprocessing time:** $O(n \log^{d-1} n)$




# Search in Subsets

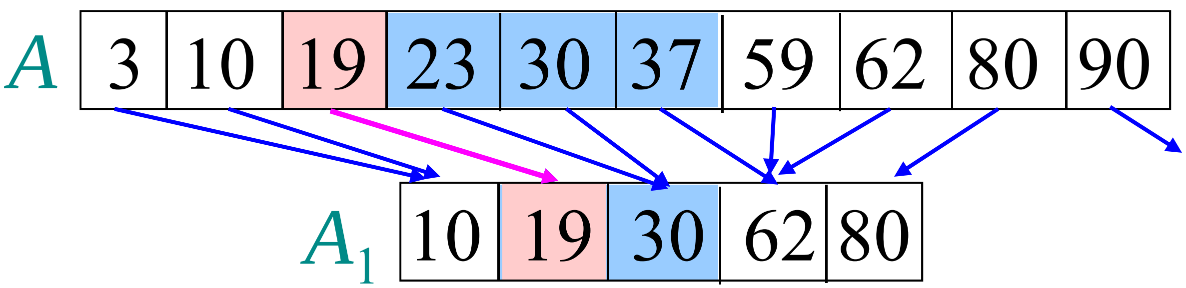
**Given:** Two sorted arrays $A_1$ and $A$, with $A_1 \subseteq A$
A query interval $[l,r]$

**Task:** Report all elements $e$ in $A_1$ and $A$ with $l \leq e \leq r$

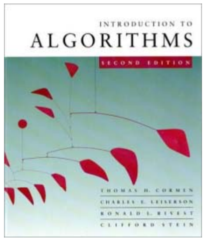
**Idea:** Add pointers from $A$ to $A_1$:
$\rightarrow$ For each $a \in A$ add a pointer to the smallest element $b \in A_1$ with $b \geq a$

**Query:** Find $l \in A$, follow pointer to $A_1$. Both in $A$ and $A_1$ sequentially output all elements in $[l,r]$.

Query:
[15,40]



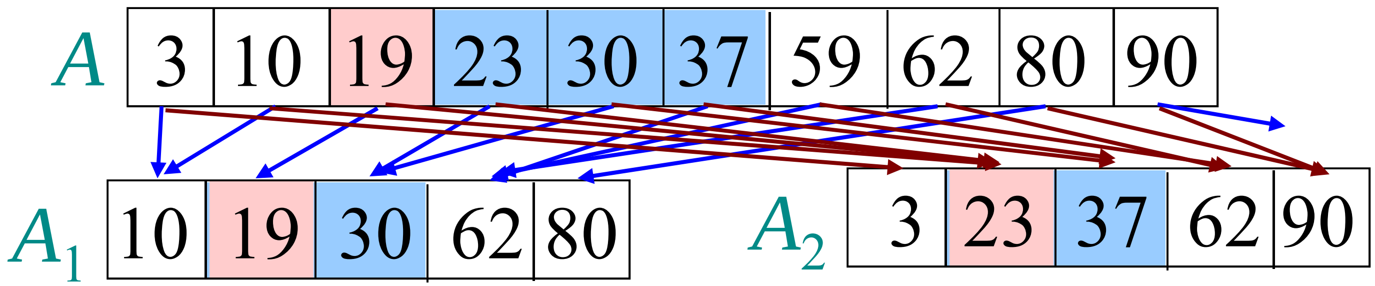

**Runtime:** $O((\log n + k) + (1 + k)) = O(\log n + k)$




# Search in Subsets (cont.)

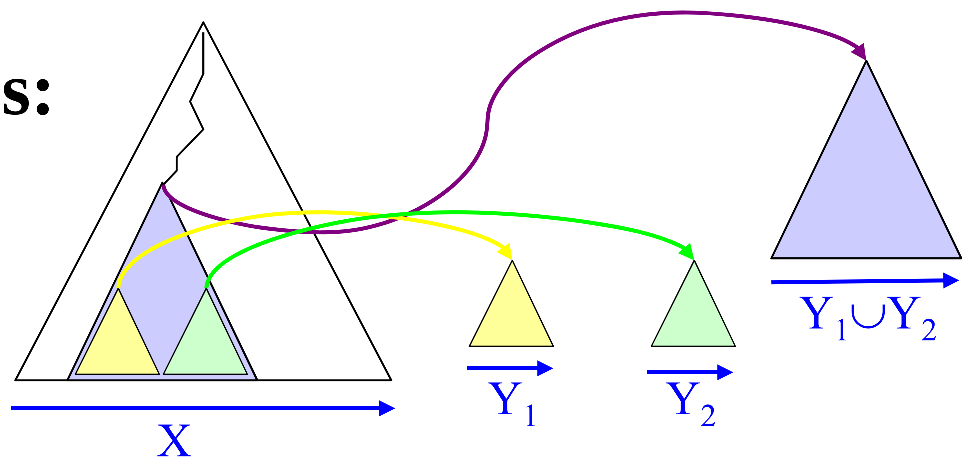
**Given:** Three sorted arrays $A_1$, $A_2$, and $A$, with $A_1 \subseteq A$ and $A_2 \subseteq A$

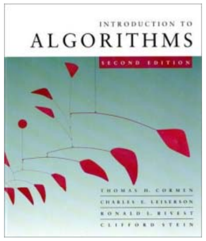
Query: [15,40]




**Runtime:** $O((\log n + k) + (1+k) + (1+k)) = O(\log n + k)$

**Range trees:**






# Fractional Cascading: Layered Range Tree

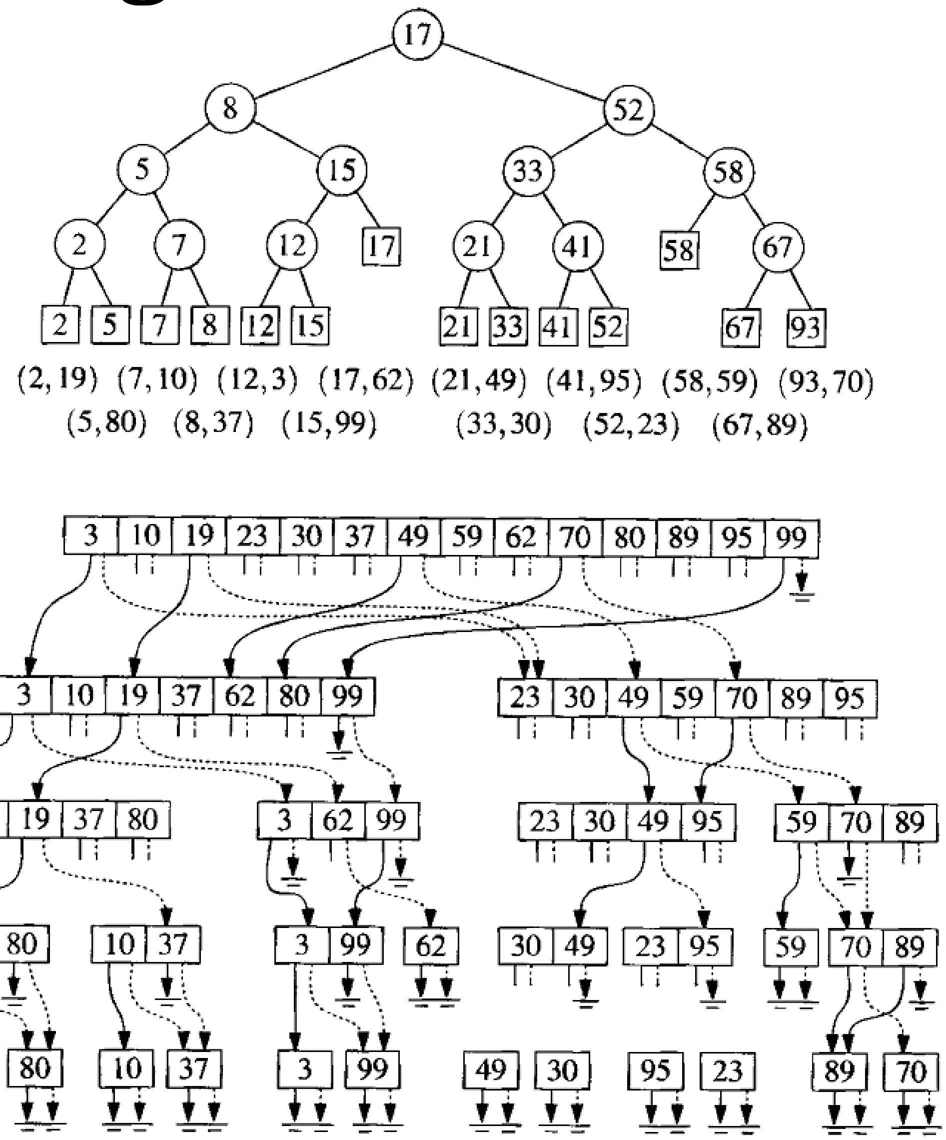
Replace 2D range tree with a layered range tree, using sorted arrays and pointers instead of the secondary range trees.

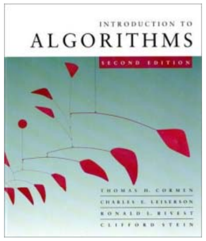
Preprocessing:

$O(n \log n)$

Query:

$O(\log n + k)$






# *d*-dimensional range trees

**Query time:** $O(k + \log^{d-1} n)$ to report $k$ points, uses fractional cascading in the last dimension

**Space:** $O(n \log^{d-1} n)$

**Preprocessing time:** $O(n \log^{d-1} n)$

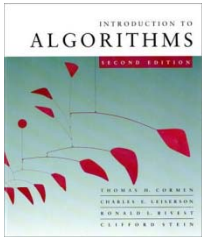
---

**Best data structure to date:**

**Query time:** $O(k + \log^{d-1} n)$ to report $k$ points.

**Space:** $O(n (\log n / \log \log n)^{d-1})$

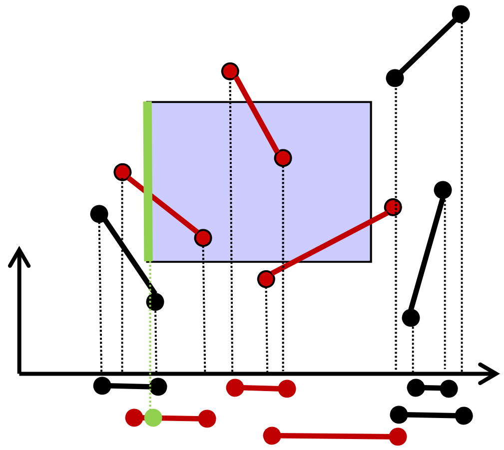
**Preprocessing time:** $O(n \log^{d-1} n)$



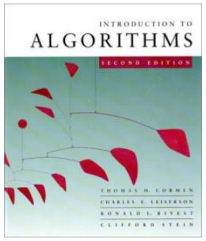

# Windowing

**Input:** A set *S* of *n* line segments in the plane

**Query:** Report all segments in *S* that intersect a given query window

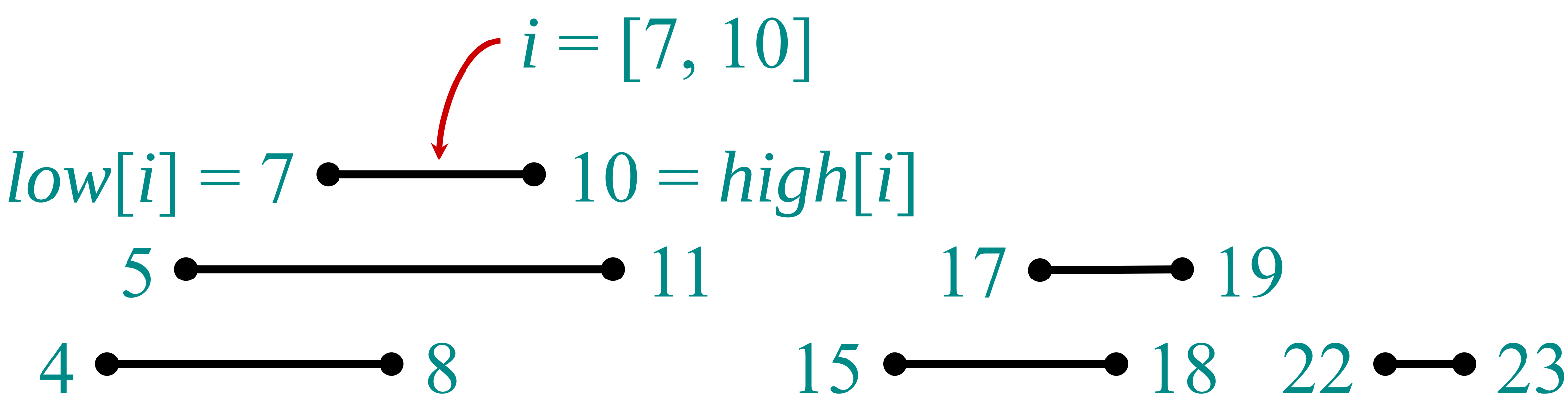


**Subproblem:** Process a set of intervals on the line into a data structure which supports queries of the type: Report all intervals that contain a query point.

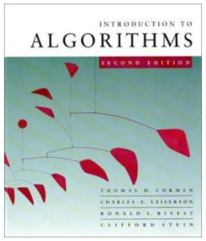



# Interval trees

**Goal:** To maintain a dynamic set of intervals, such as time intervals.

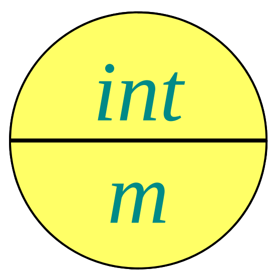


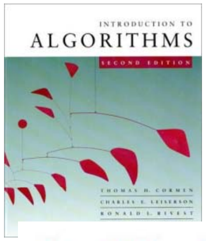

**Query:** For a given query interval *i*, find an interval in the set that overlaps *i*.



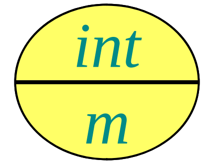

# Following the methodology

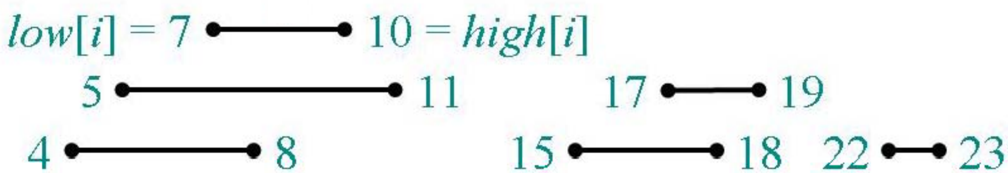
1. *Choose an underlying data structure.*
   - Red-black tree keyed on low (left) endpoint.
2. *Determine additional information to be stored in the data structure.*
   - Store in each node *x* the interval *int*[*x*] corresponding to the key, as well as the largest value *m*[*x*] of all right interval endpoints stored in the subtree rooted at *x*.

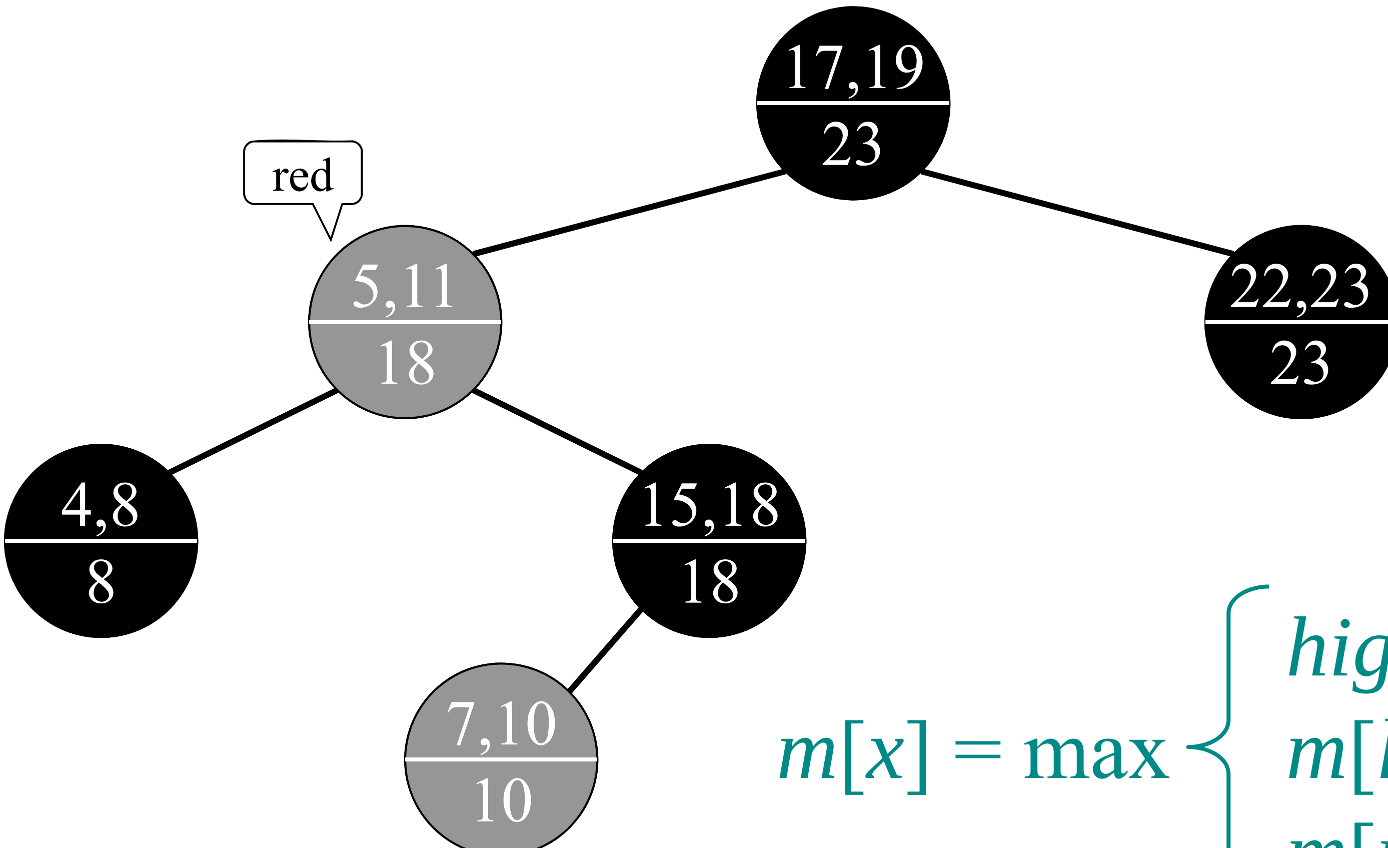




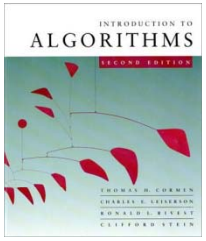
# Example interval tree



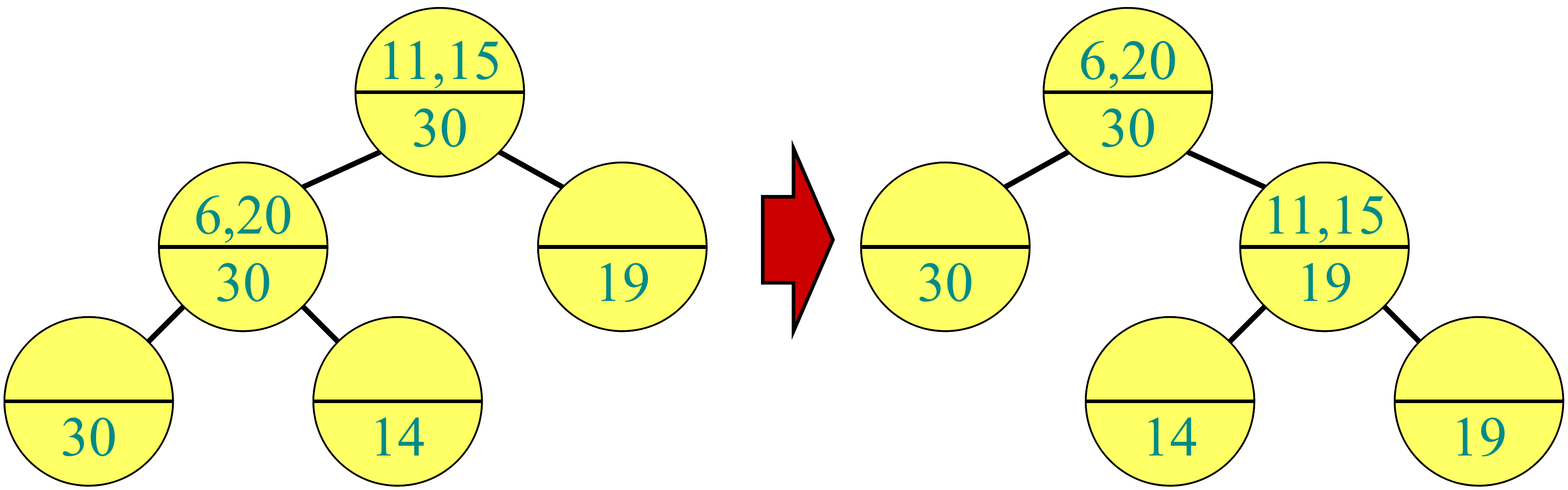




$$m[x] = \max \begin{cases} high[int[x]] \\ m[left[x]] \\ m[right[x]] \end{cases}$$

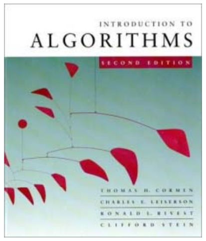



# Modifying operations

3. *Verify that this information can be maintained for modifying operations.*
   - INSERT: Fix $m$'s on the way down.
   - Rotations — Fixup $= O(1)$ time per rotation:




Total INSERT time $= O(\log n)$; DELETE similar.




# New operations

4. Develop new dynamic-set operations that use the information.

INTERVAL-SEARCH($i$)

$x \leftarrow root$

**while** $x \neq$ NIL and ($low[i] > high[int[x]]$ or $low[int[x]] > high[i]$)

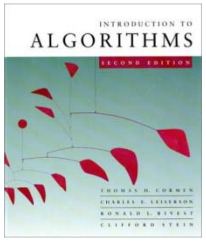
**do** ▷ $i$ and $int[x]$ don't overlap

**if** $left[x] \neq$ NIL and $low[i] \leq m[left[x]]$

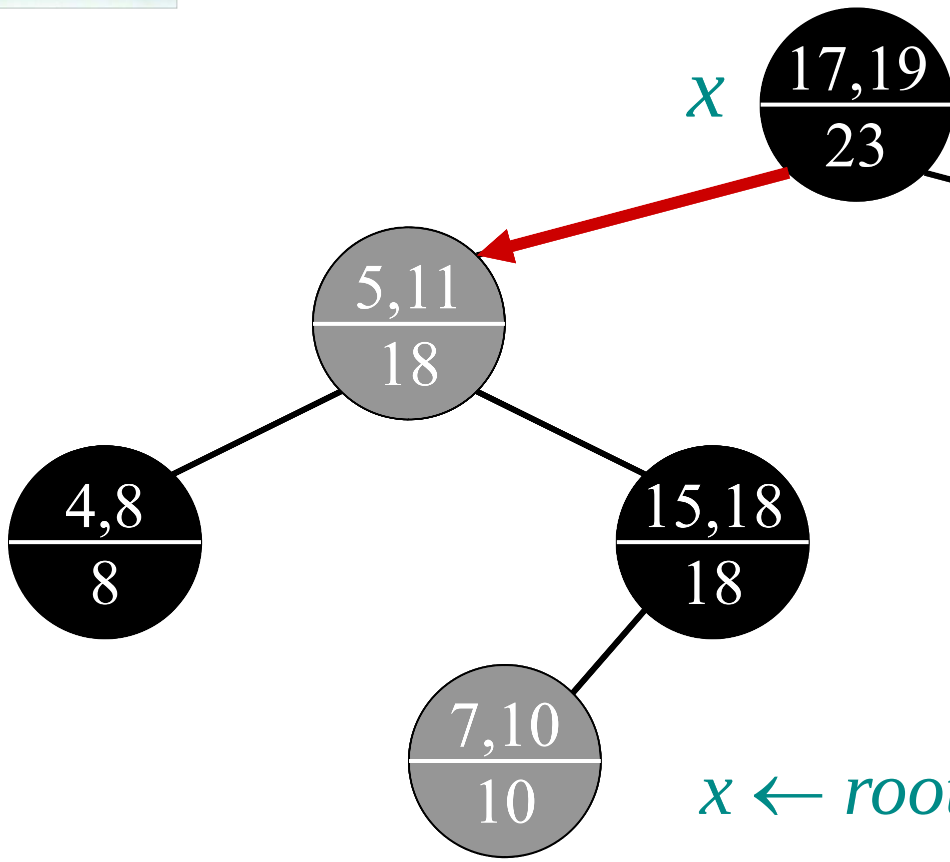
**then** $x \leftarrow left[x]$

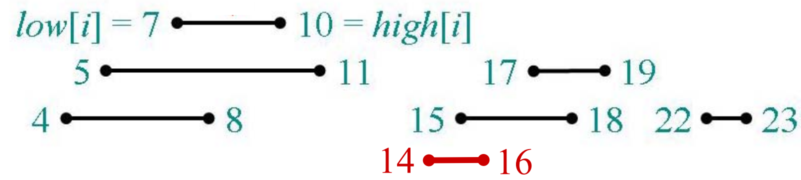
**else** $x \leftarrow right[x]$

**return** $x$




# Example 1: INTERVAL-SEARCH([14,16])



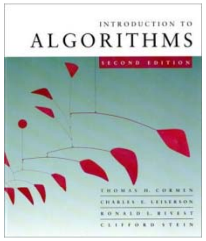




**while** $x \neq$ NIL and ($low[i] > high[int[x]]$
or $low[int[x]] > high[i]$)
**do** ▷ $i$ and $int[x]$ don't overlap
**if** $left[x] \neq$ NIL and $low[i] \leq m[left[x]]$
**then** $x \leftarrow left[x]$
**else** $x \leftarrow right[x]$

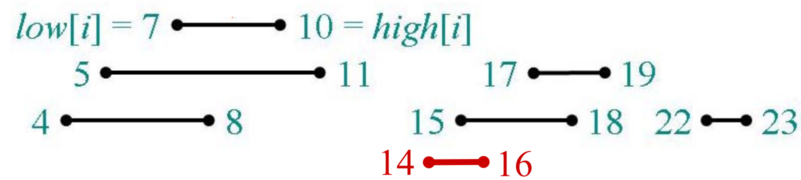
$x \leftarrow root$

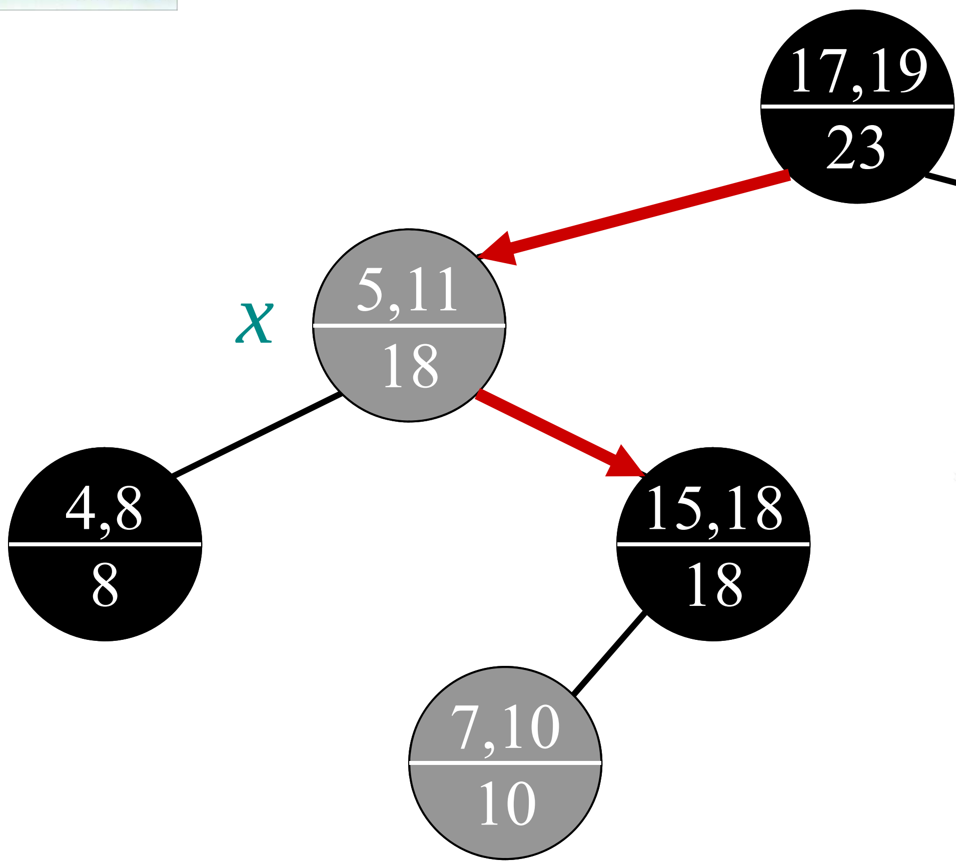
[14,16] and [17,19] don't overlap

$14 \leq 18 \Rightarrow x \leftarrow left[x]$




# Example 1: INTERVAL-SEARCH([14,16])

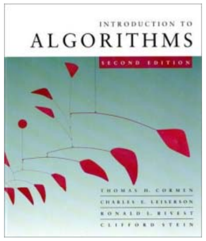


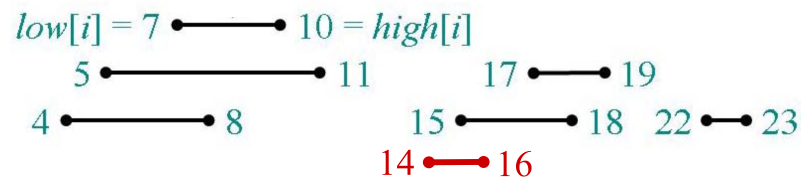

**while** $x \neq$ NIL and ($low[i] > high[int[x]]$
or $low[int[x]] > high[i]$)
**do** ▹ $i$ and $int[x]$ don't overlap
**if** $left[x] \neq$ NIL and $low[i] \leq m[left[x]]$
**then** $x \leftarrow left[x]$
**else** $x \leftarrow right[x]$

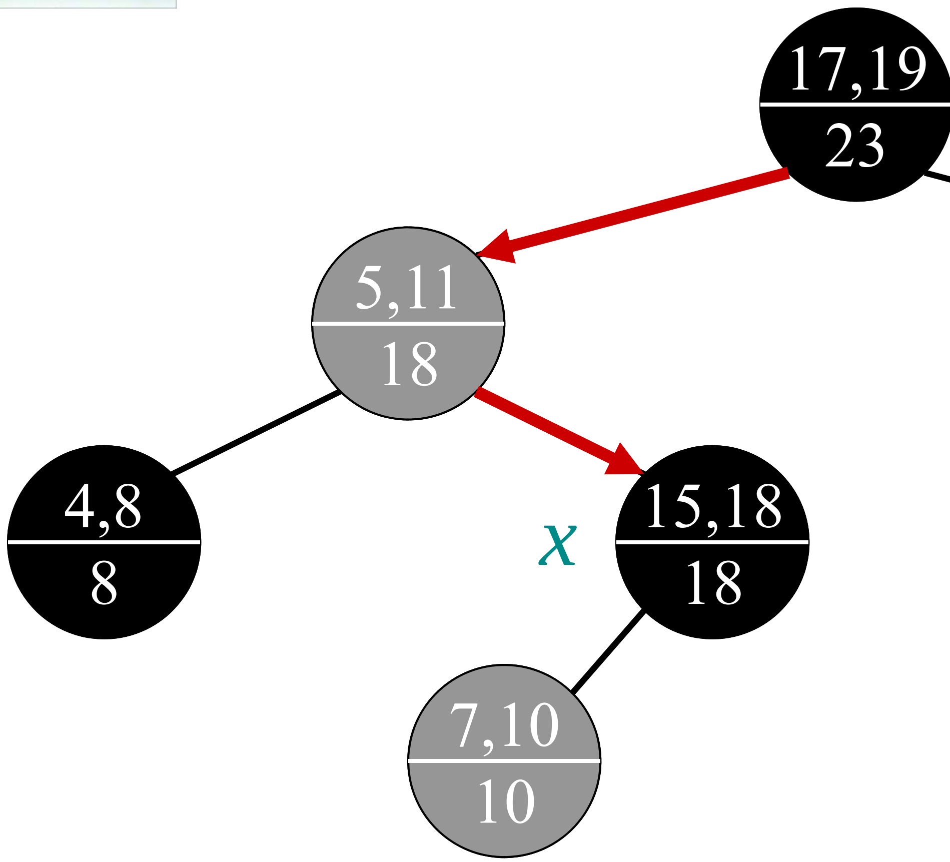
[14,16] and [5,11] don't overlap

$14 > 8 \Rightarrow x \leftarrow right[x]$




# Example 1: INTERVAL-SEARCH([14,16])



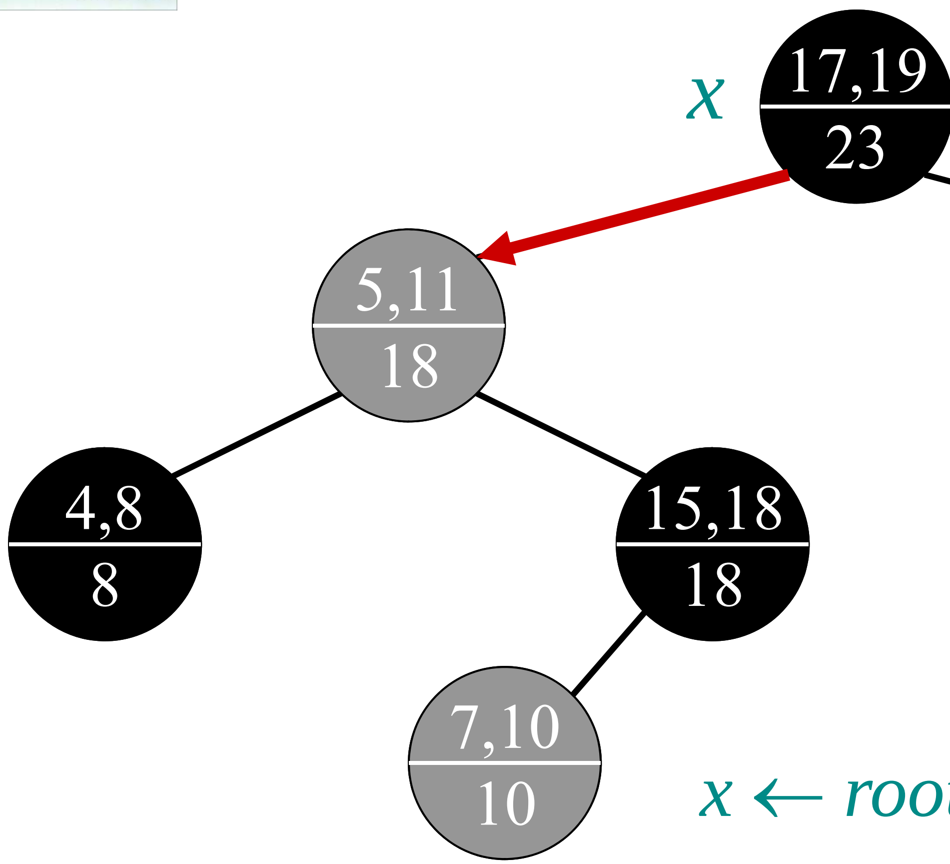



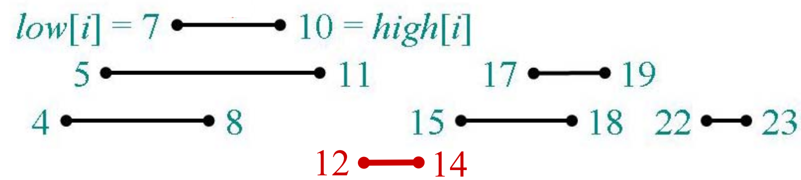

**while** $x \neq$ NIL and $(low[i] > high[int[x]]$
or $low[int[x]] > high[i])$
**do** ▷ $i$ and $int[x]$ don't overlap
**if** $left[x] \neq$ NIL and $low[i] \leq m[left[x]]$
**then** $x \leftarrow left[x]$
**else** $x \leftarrow right[x]$

[14,16] and [15,18] overlap

**return** [15,18]

# Example 2: INTERVAL-SEARCH([12,14])



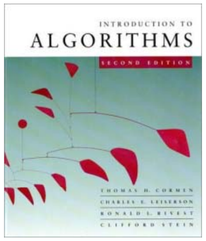




**while** $x \neq$ NIL and ($low[i] > high[int[x]]$
or $low[int[x]] > high[i]$)
**do** ▷ $i$ and $int[x]$ don't overlap
**if** $left[x] \neq$ NIL and $low[i] \leq m[left[x]]$
**then** $x \leftarrow left[x]$
**else** $x \leftarrow right[x]$

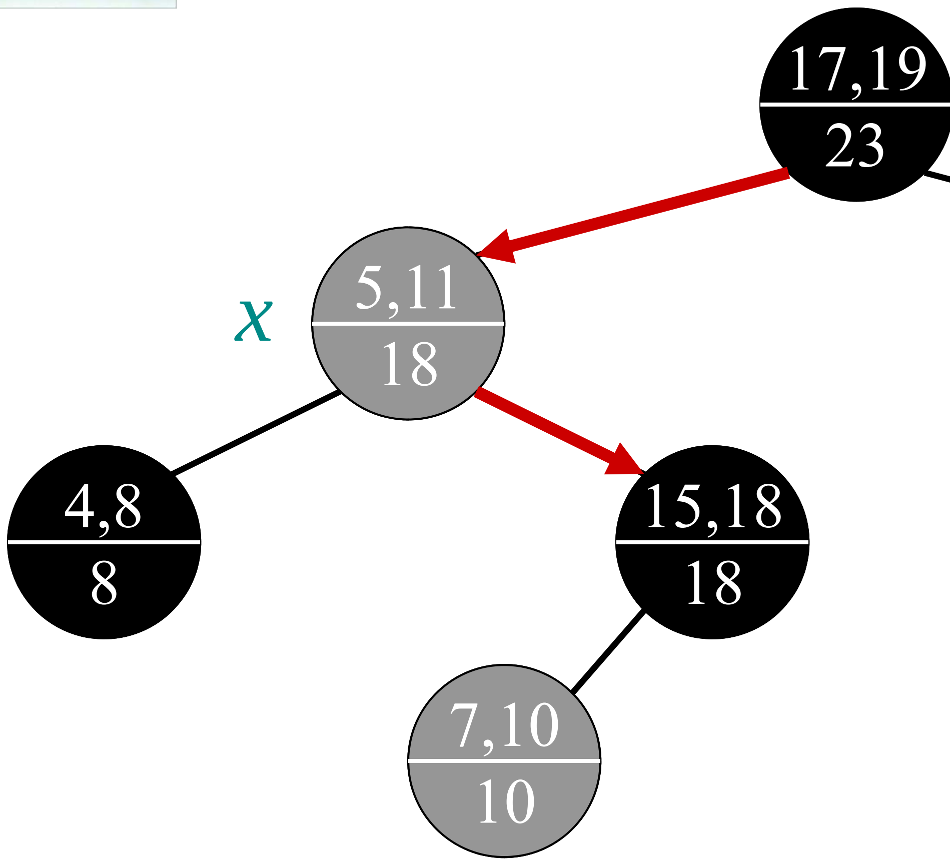
$x \leftarrow root$

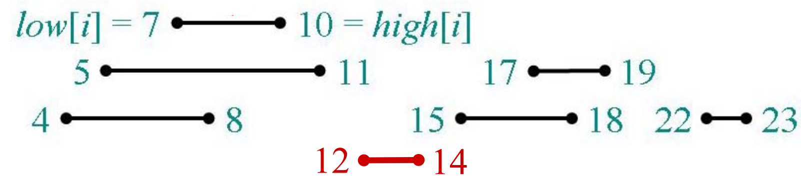
[12,14] and [17,19] don't overlap

$12 \leq 18 \Rightarrow x \leftarrow left[x]$




# Example 2: INTERVAL-SEARCH([12,14])

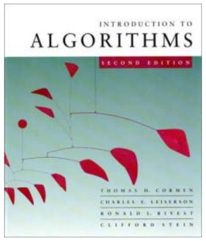


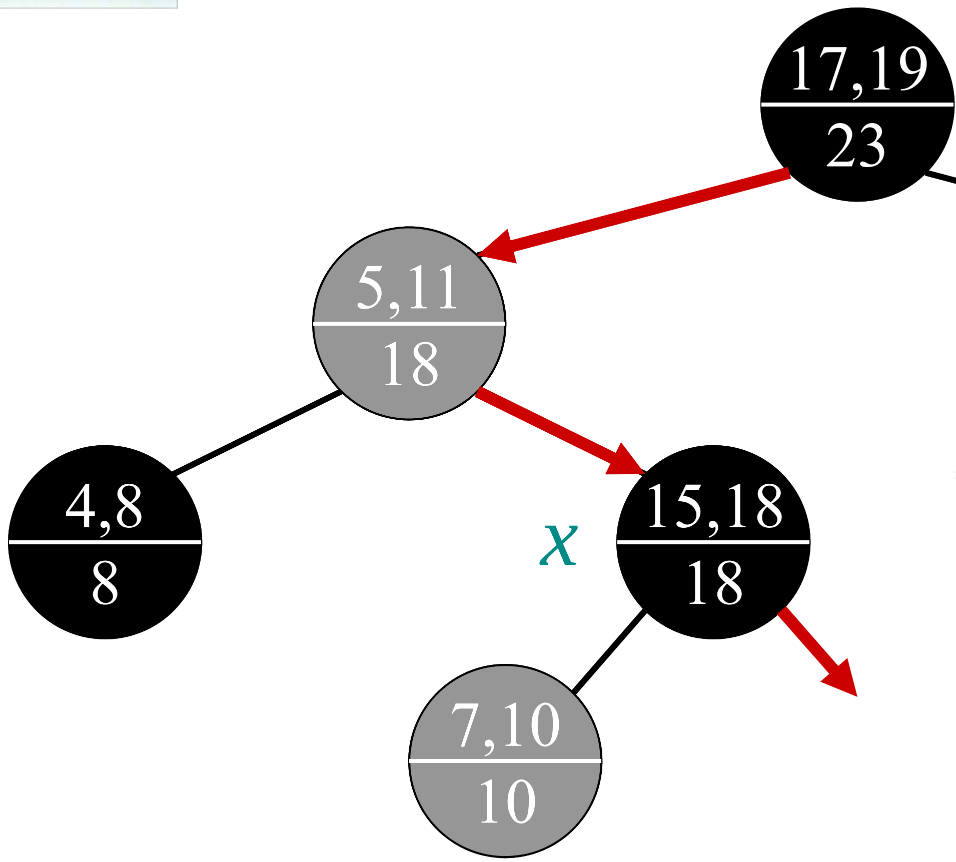

**while** $x \neq$ NIL and $(low[i] > high[int[x]]$
or $low[int[x]] > high[i])$
**do** ▷ $i$ and $int[x]$ don't overlap
**if** $left[x] \neq$ NIL and $low[i] \leq m[left[x]]$
**then** $x \leftarrow left[x]$
**else** $x \leftarrow right[x]$

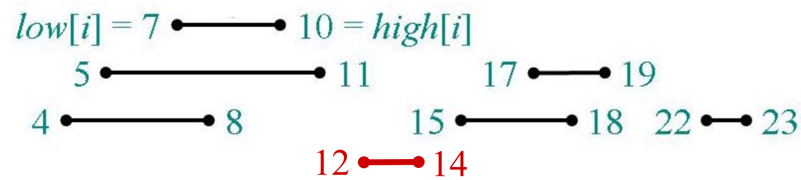
[12,14] and [5,11] don't overlap

$12 > 8 \Rightarrow x \leftarrow right[x]$




# Example 2: INTERVAL-SEARCH([12,14])

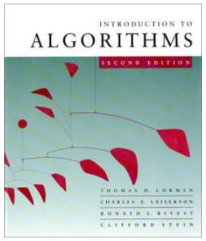


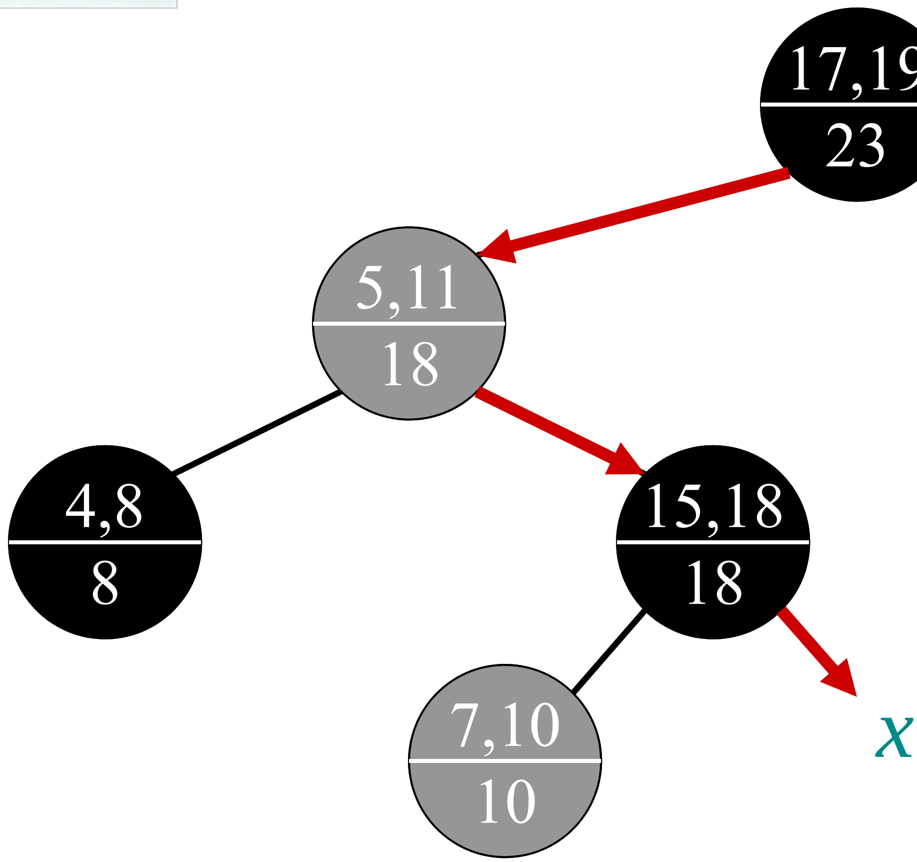

**while** $x \neq$ NIL and $(low[i] > high[int[x]]$
or $low[int[x]] > high[i])$
**do** ▷ $i$ and $int[x]$ don't overlap
**if** $left[x] \neq$ NIL and $low[i] \leq m[left[x]]$
**then** $x \leftarrow left[x]$
**else** $x \leftarrow right[x]$

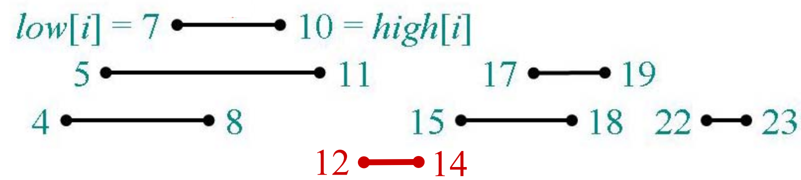
[12,14] and [15,18] don't overlap

$12 > 10 \Rightarrow x \leftarrow right[x]$



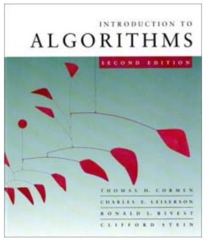

# Example 2: INTERVAL-SEARCH([12,14])




**while** $x \neq$ NIL and ($low[i] > high[int[x]]$
or $low[int[x]] > high[i]$)
**do** ▷ $i$ and $int[x]$ don't overlap
**if** $left[x] \neq$ NIL and $low[i] \leq m[left[x]]$
**then** $x \leftarrow left[x]$
**else** $x \leftarrow right[x]$

$x =$ NIL $\Rightarrow$ no interval that overlaps [12,14] exists




# Analysis

Time $= O(h) = O(\log n)$, since INTERVAL-SEARCH does constant work at each level as it follows a simple path down the tree.

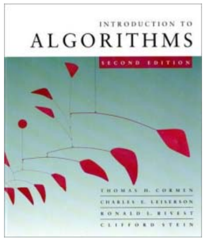
List *all* overlapping intervals:

- Search, list, delete, repeat.
- Insert them all again at the end.

Time $= O(k \log n)$, where $k$ is the total number of overlapping intervals.

This is an ***output-sensitive*** bound.

Best algorithm to date: $O(k + \log n)$.




# Correctness

**Theorem.** Let $L$ be the set of intervals in the left subtree of node $x$, and let $R$ be the set of intervals in $x$'s right subtree.

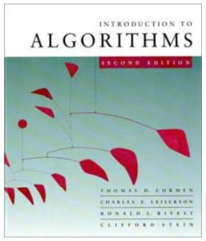
- If the search goes right, then

$$\{ i' \in L : i' \text{ overlaps } i \} = \emptyset.$$

- If the search goes left, then

$$\{i' \in L : i' \text{ overlaps } i \} = \emptyset$$
$$\Rightarrow \{i' \in R : i' \text{ overlaps } i \} = \emptyset.$$

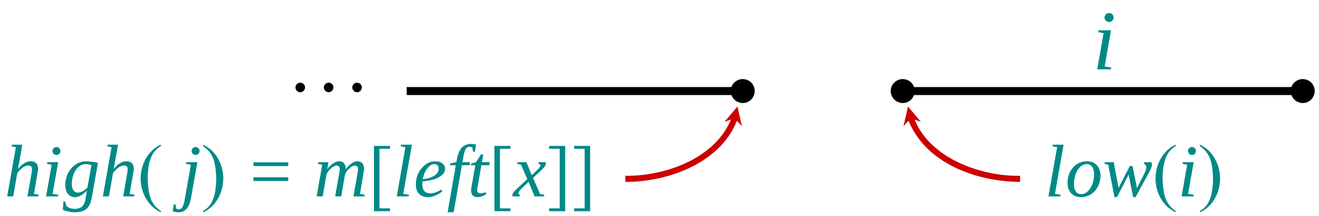
*In other words, it's always safe to take only 1 of the 2 children: we'll either find something, or nothing was to be found.*



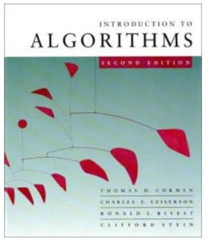

# Correctness proof

*Proof.* Suppose first that the search goes right.

- If $left[x] = \text{NIL}$, then we're done, since $L = \emptyset$.
- Otherwise, the code dictates that we must have $low[i] > m[left[x]]$. The value $m[left[x]]$ corresponds to the right endpoint of some interval $j \in L$, and no other interval in $L$ can have a larger right endpoint than $high(j)$.




- Therefore, $\{i' \in L : i' \text{ overlaps } i\} = \emptyset$.




# Proof (continued)

Suppose that the search goes left, and assume that

$$\{i' \in L : i' \text{ overlaps } i\} = \emptyset.$$

- Then, the code dictates that $low[i] \le m[left[x]] = high[j]$ for some $j \in L$.
- Since $j \in L$, it does not overlap $i$, and hence $high[i] < low[j]$.
- But, the binary-search-tree property implies that for all $i' \in R$, we have $low[j] \le low[i']$.
- But then $\{i' \in R : i' \text{ overlaps } i\} = \emptyset$. ∎