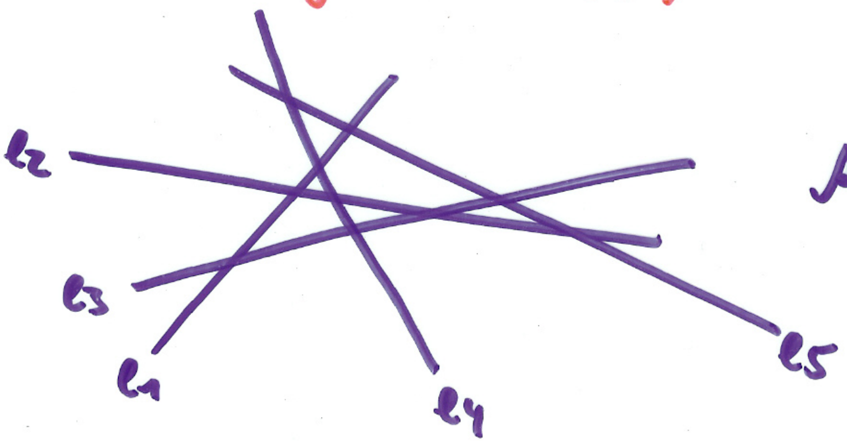
Arrangements of lines

Let $L = \{\ell_1, \dots, \ell_n\}$ be a set of lines in the plane. Then the subdivision of the plane induced by L is called the arrangement $\mathcal{A}(L)$



$\mathcal{A}(\{\ell_1, \dots, \ell_5\})$

- $\mathcal{A}(L)$ consists of vertices, edges, faces
- The combinatorial complexity of an arrangement of lines is # vertices + # edges + # faces
- $\mathcal{A}(L)$ is simple iff no three lines pass through the same point and no two lines are parallel
- Lemma:
  (i) # vertices of $\mathcal{A}(L)$ is at most $\frac{n(n-1)}{2}$
  (ii) # edges " " " " " n^2
  (iii) # faces " " " " " $\frac{n^2+n+2}{2}$
  Equality holds $\Leftrightarrow$ $\mathcal{A}(L)$ is simple

Proof:
(i) vertex = intersection of two lines $\rightsquigarrow \le \binom{n}{2} = \frac{n(n-1)}{2}$ line pai
(ii) On a line at most $n-1$ vertices $\rightsquigarrow \le n$ edges $\rightsquigarrow n^2$ total
(iii) $\mathcal{A}(\{\ell_1, \dots, \ell_{i-1}\}) \xrightarrow{\text{add } \ell_i} \mathcal{A}(\{\ell_1, \dots, \ell_i\})$
Every edge on ℓ_i splits face of $\mathcal{A}(\{\ell_1, \dots, \ell_{i-1}\})$ in two.
i edges on ℓ_i $\rightsquigarrow$ total # faces $= 1 + \sum_{i=1}^{n} i = 1 + \frac{n(n+1)}{2}$

$\square$

- $A(L)$ is a planar subdivision of quadratic complexity
- Use doubly-connected edge list to store $A(L)$, with an additional bounding box that encloses all vertices of $A(L)$ $\leftarrow B(L)$
- Sweep line construction: $O(n^2 \log n)$
- Here: Incremental construction in $O(n^2)$ time

Let $A_i := A(\{l_1, \dots, l_i\})$ inside $B(L)$

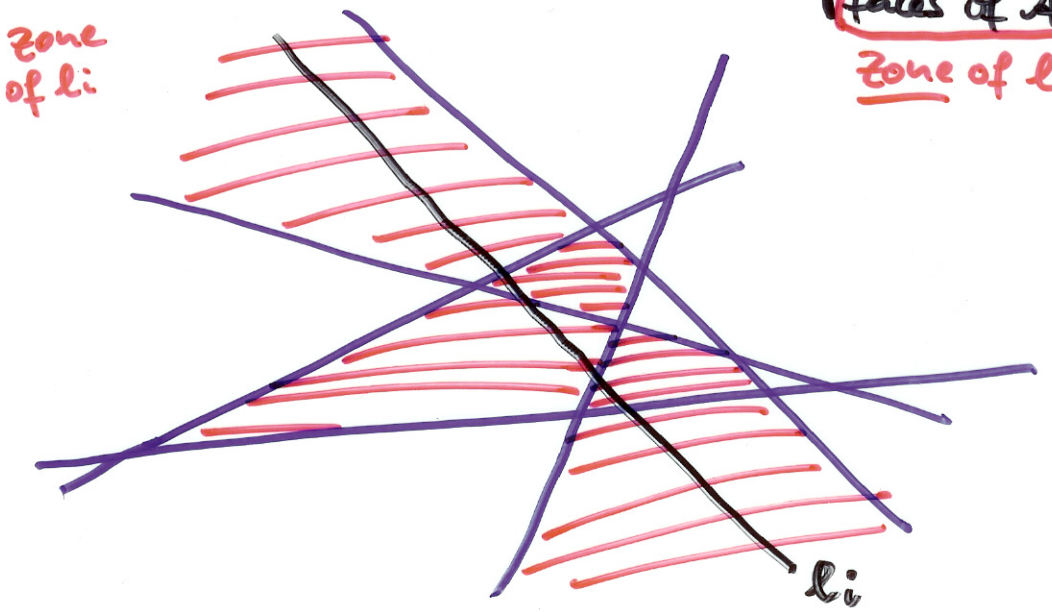
Algorithm Construct_Arrangement(L):

Input: Set of n line segments $L = \{l_1, \dots, l_n\}$

Output: dcel for $A(L)$ inside $B(L)$

- Compute bounding box $B(L)$ $\quad O(n^2)$
- Construct dcel for $B(L)$ $\quad O(1)$
- **For** $i := 1$ **to** n **do**
  - Find edge e on $B(L)$ that contains leftmost intersection point of l_i with A_{i-1} $\Big\}\ O(n)$
  - $f :=$ bounded face incident to e
  - **while** f is inside $B(L)$ // i.e., not unbounded
    do · split f $\quad O(\text{complexity of } f)$
    · set f to be next intersected face

Time to insert l_i in A_{i-1}: $O\left(\underbrace{\boxed{\text{sum of complexities of faces of } A_{i-1} \text{ intersected by } l_i}}_{\text{Zone of } l_{i-1} \text{ in arrangement } A_{i-1}}\right)$

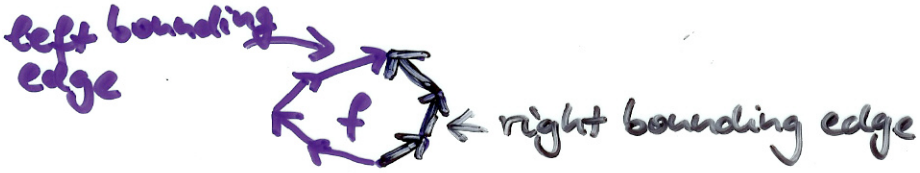


Theorem ("Zone Theorem"):

The complexity of the zone of a line in an arrangement of n lines is $O(n)$.

Proof:

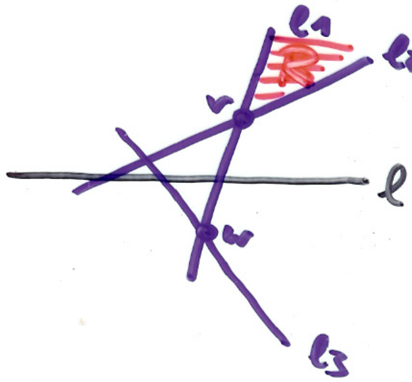
- Assume ℓ is a horizontal line (rotate)
- Assume $A(L)$ is simple and has no horizontal cuts



- Count # of left bounding edges (right is similar)
- Induction on n; Show that # left bounding edges of the zone is $\leq 3n$

$n=1$ ✓

$n>1$:



- Let ℓ_1 be the line which has the rightmost intersection with ℓ
- Let $v :=$ 1st intersection point of ℓ_1 with another line in L above ℓ ($w :=$ " below ℓ)
- Zone of $A(L \setminus \{\ell_1\})$ has $3(n-1)$ left bounding edges
- 1 new edge $\overline{vw}$, two old edges split by v or w $\rightsquigarrow 3(n-1)+3 = 3n$ left bounding edges in total
- No more new edges: Region R is not in zone(ℓ), but is the only part of ℓ_1 above v that could contibute with left bounding edges

Theorem: An arrangement of n lines in the plane can be constructed by an incremental algorithm in $O(n^2)$ time.

Proof: With zone theorem the total runtime is $\sum_{i=1}^{n} O(i) = O(n^2)$

□

Higher Dimensions:

Let $H = \{h_1, \dots, h_n\}$ be a set of $(d-1)$-dimensional hyperplanes in $\mathbb{R}^d$.

Then $\underline{A(H)}$ the arrangement of all hyperplanes in H is the subdivision of $\mathbb{R}^d$ that H induces.

- $A(H)$ consists of vertices (0-dimensional faces),
  edges (1-dimensional faces),
  ⋮
  k-dimensional faces ; $0 \le k \le d$
- The combinatorial complexity of $A(H)$ is $\Theta(n^d)$
  (# vertices $= \binom{n}{d} = O(n^d)$)
- $A(H)$ is simple iff no $d+1$ hyperplanes pass through the same point and every d hyperplanes meet in a single point
- The complexity of ~~the~~ the zone of a hyperplane in an arrangement of hyperplanes in $\mathbb{R}^d$ is $O(n^{d-1})$
- The arrangement of n hyperplanes in $\mathbb{R}^d$ can be constructed by an incremental algorithm in $O(n^d)$ time.