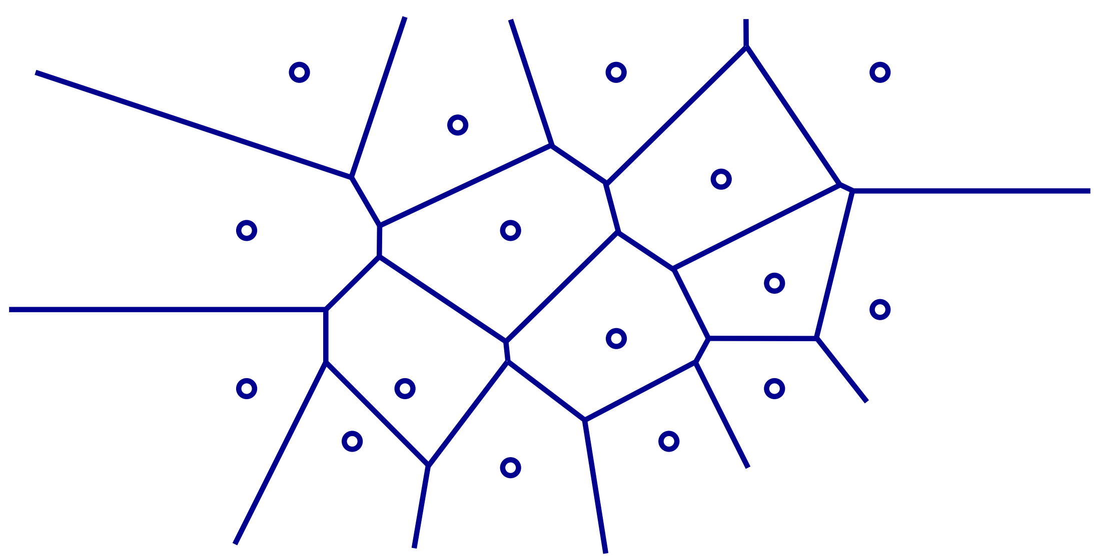
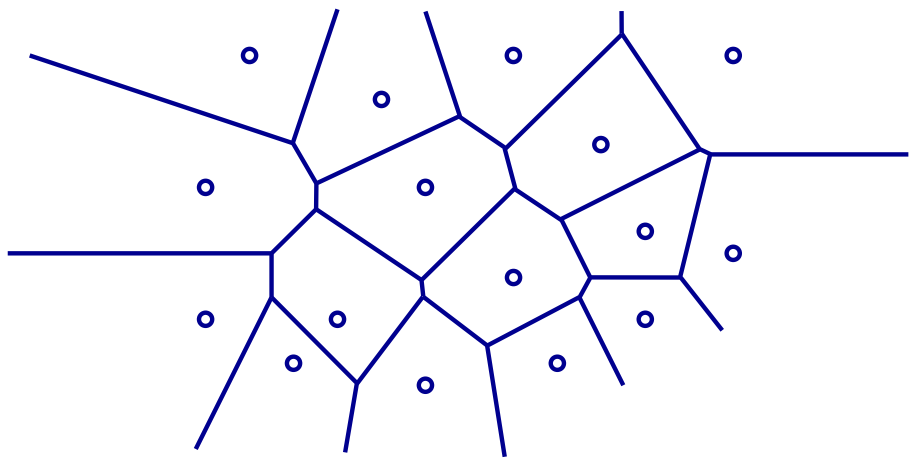
Voronoi Diagrams

- **A set $P = \{p_1, p_2, \ldots, p_n\}$ of n points, called sites.**
- **Suppose each point of the plane is attracted to the site closest to it.**
- **This induces a partition of the plane: each site owns the part attracted to it.**
- **What does this partition look like? This is called the Voronoi Diagram of P.**



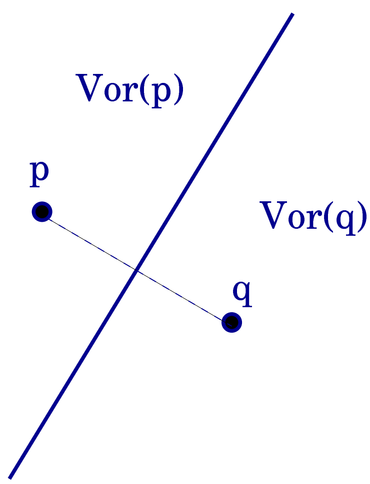
Applications

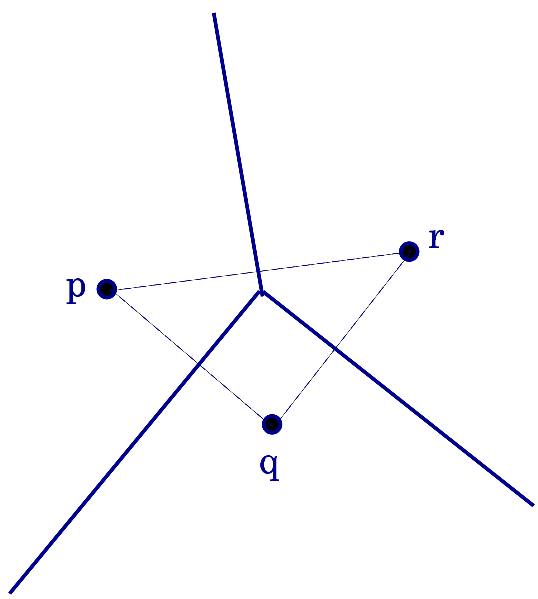


- Voronoi diagram is a promixity diagram. Think of the post office analogy.
- Many applications are proximity driven: location of retail stores, emergency services, facilities etc.
- Strategy questions like "where to locate a new store".
- In science, forces are inversely proportional to distances.
- Phenomena like crystal growth, cluster formation give rise to Voronoi diagrams.

Preliminaries

- **Begin with the simplest possible setting: two sites p and q.**
- **What does their Voronoi diagram look like?**

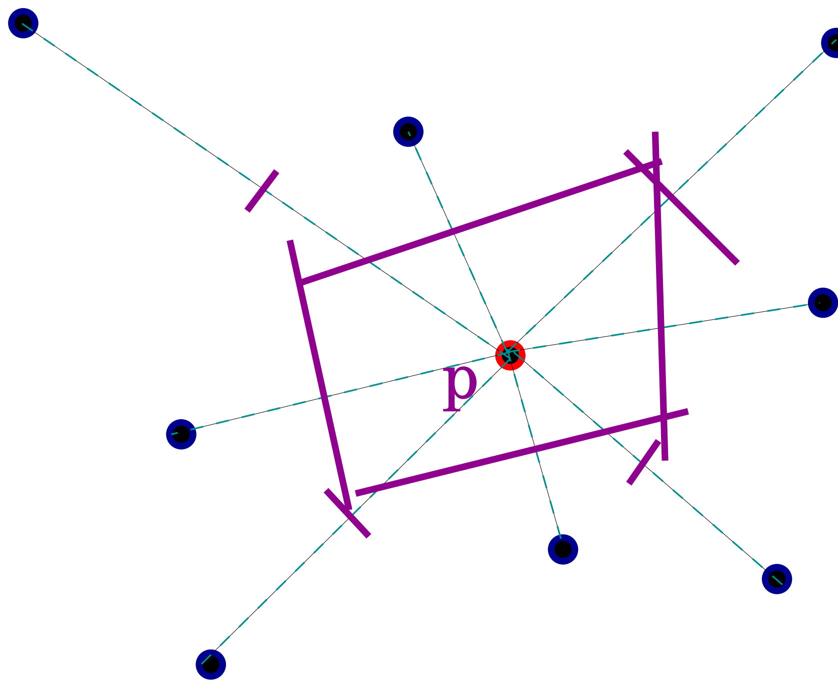




- **What happens with three sites: p, q, r?**

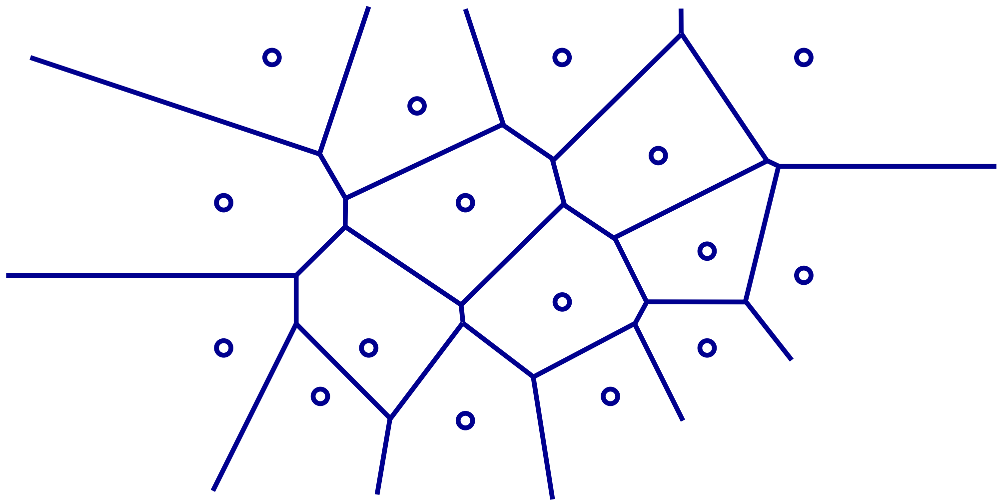
A Voronoi Cell

- **Generalizing, we realize that a Voronoi cell is the intersection of $n-1$ halfplanes.**



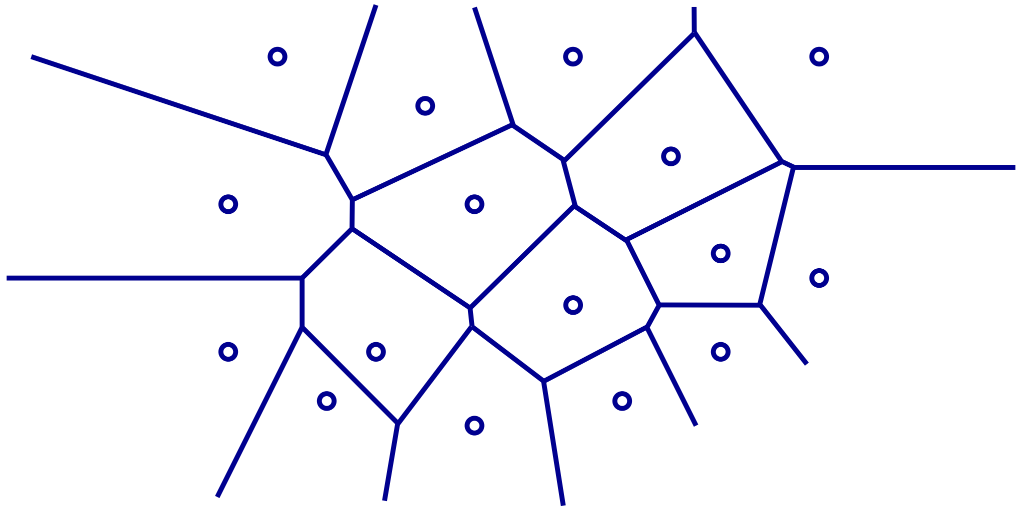
- **Each halfplane defined by the bisector line of pair (p, p_i):**
  $h(p, p_i) = \{x \mid dist(x, p) \leq dist(x, p_i\}$**.**
- **Bisectors are the building blocks of the Voronoi diagram.**

Voronoi Properties



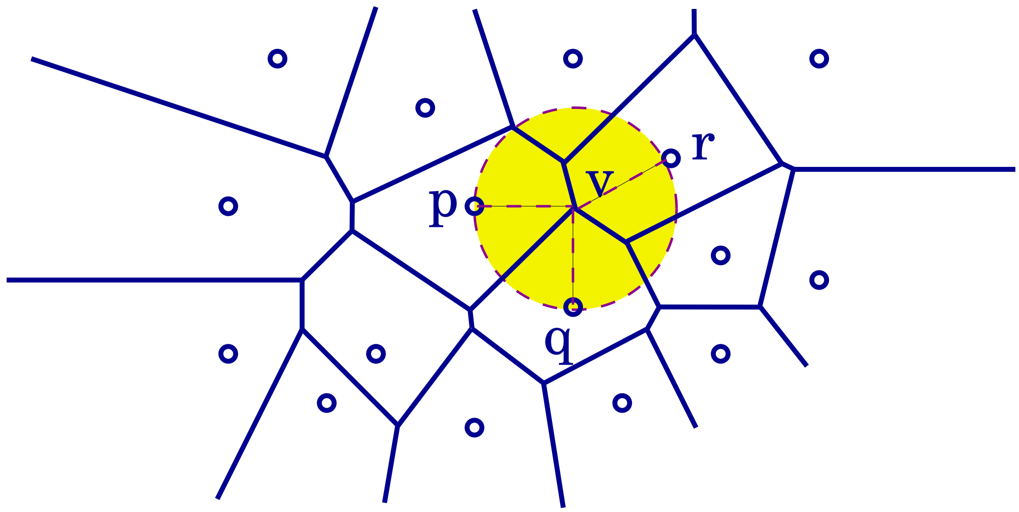
- **A site's Voronoi cell, $V(p)$, is a convex polygon.**
- **$V(p)$ has at most $n - 1$ sides.**
- **For every point x in the interior of $V(p)$, the closest site of x is p.**
- **Every point x on the boundary of $V(p)$ is equidistant from at least two sites, one of which is p.**
- **The Voronoi Diagram $V(P)$ is the union of voronoi cells of all sites in P.**

Properties



- **How do various cells inter-relate?**
- $int\ V(p) \cap \ int\ V(q) = \emptyset$; **interior of the cell belongs to exactly one site.**
- **The union of all cells covers the plane: each point x must have some closest site.**
- **Thus, $V(P)$ is a planar subdivision of R^2.**

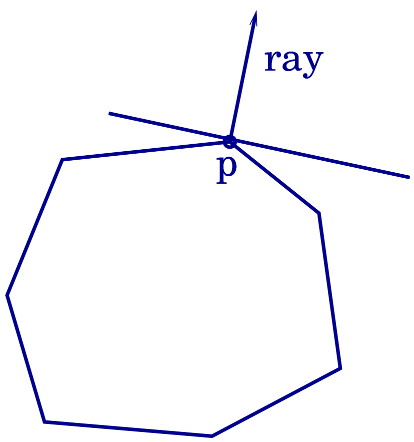
General Position

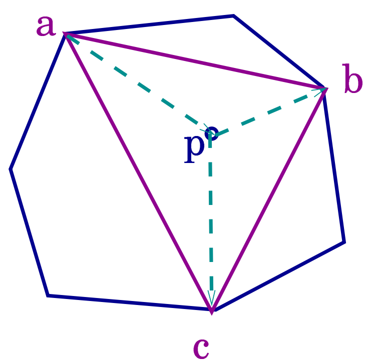


- **The degree of a vertex v in $V(P)$ is the number of edges incident to v.**
- **The degree also equals the number of cells that have v on their boundary.**
- **If $deg(v) = k$, then k sites are equi-distant from v.**
- **The equi-distant sites are co-circular.**
- **We make the general position assumption that no more than 3 sites lie on a circle, so all vertices of $V(P)$ have degree three.**

Boundedness

- **A cell $V(p)$ is unbounded if and only if p is on the convex hull of the sites.**
- **If p on $CH(P)$, consider ray normal to the support line of p.**
- **All x on this ray have p as their nearest site. Thus, $V(p)$ is unbounded.**

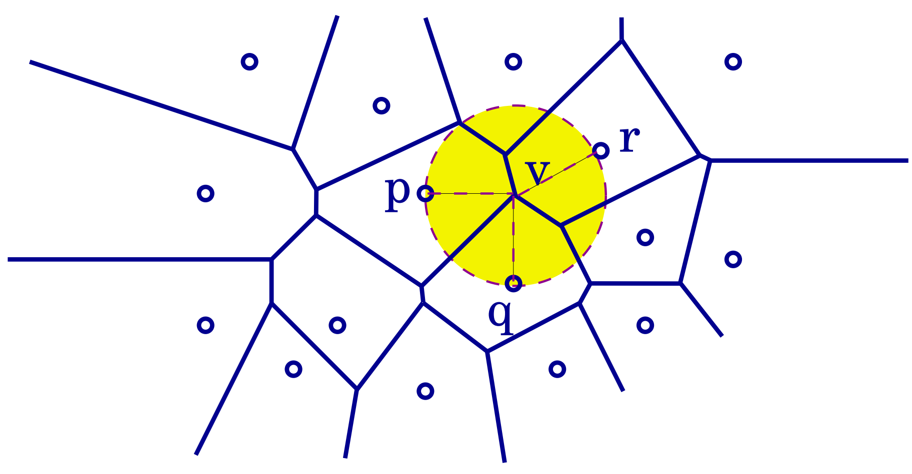




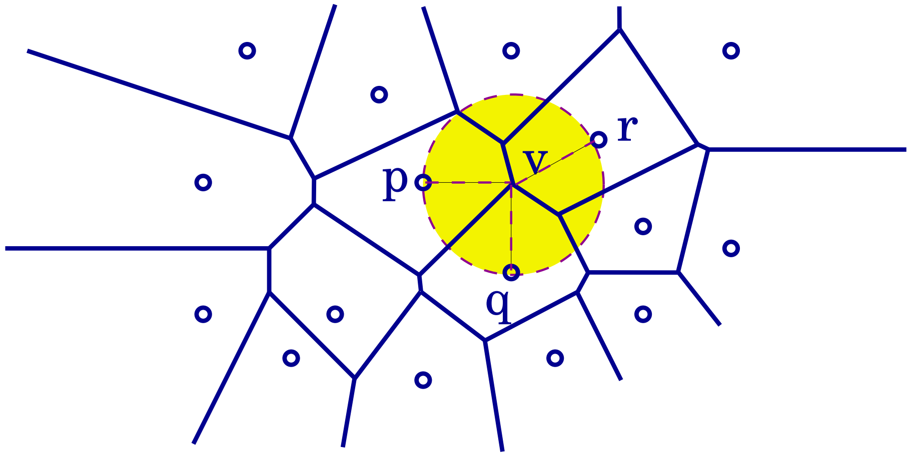
- **If $p \notin CH(P)$, then p lies inside some triangle $\triangle abc$ where a, b, c on the hull.**
- **Any point sufficient far out, has a, b or c as their closest site, not p.**
- **So, $V(p)$ must be bounded.**

Properties

- v **is a Voronoi vertex iff it is the center of an empty circle that passes through 3 sites.**
- **If** v **is center of such a circle, it must be that** $v \in V(a) \cap V(b) \cap V(c)$**, and so it's a Voronoi vertex.**
- **If** v **is a Voronoi vertex, let** $v = V(a) \cap V(b) \cap V(c)$**. No other site can be closer to** v **than** a, b, c**, and so the circle with center** v **and radius** va **is empty.**



Properties



- Choose an arbitrary point $x \in R^2$. Grow a circle from x until it hits some sites.
  - If circle hits exactly one point p, then $x \in int\ V(p)$.
  - If circle hits two points p, q simultaneously, then x is on the voronoi edge between $V(p)$ and $V(q)$.
  - If circle hits three points p, q, r, then x is the voronoi vertex.