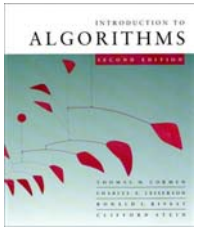
CS 5633 -- Spring 2010



Recurrences and Divide & Conquer

Carola Wenk

Slides courtesy of Charles Leiserson with small changes by Carola Wenk

Merge sort

MERGE-SORT $A[1 \ldots n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1 \ldots \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \ldots n\,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: **MERGE**

Merging two sorted arrays

20 12
13 11
7 9
2 1

Merging two sorted arrays

20 12
13 11
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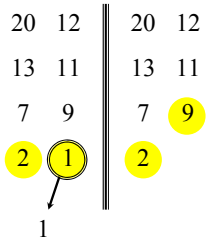
1

Merging two sorted arrays



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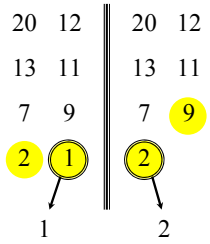

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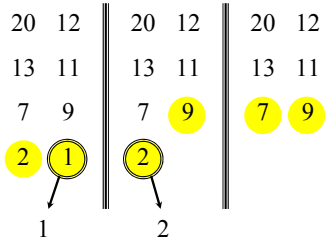

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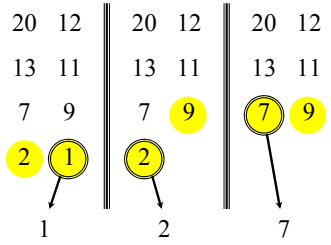

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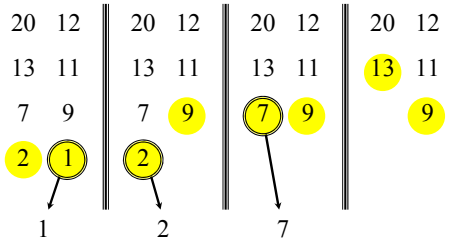

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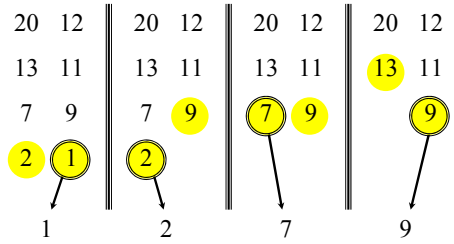

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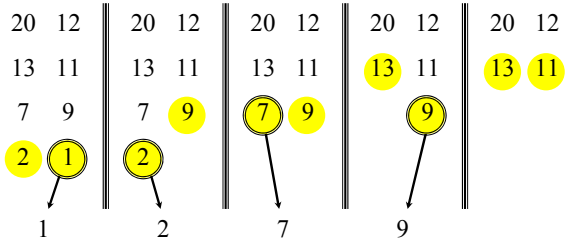

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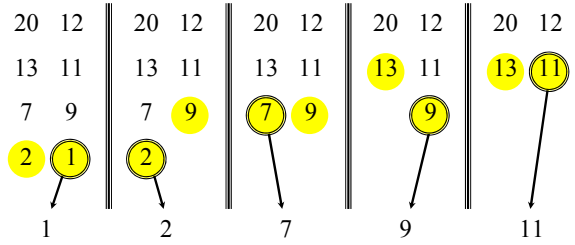

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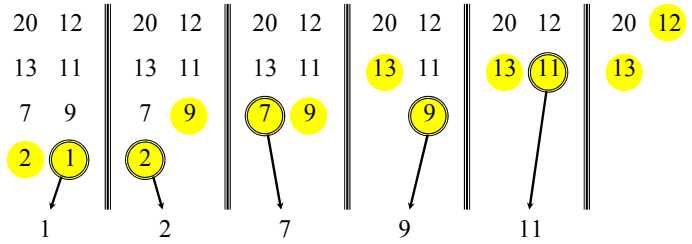

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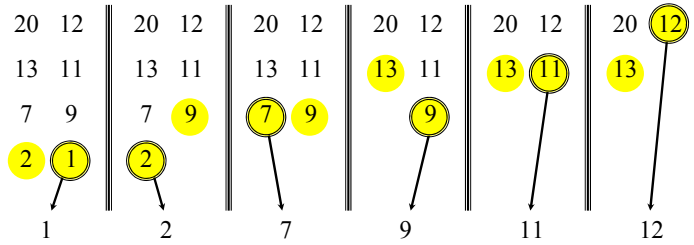

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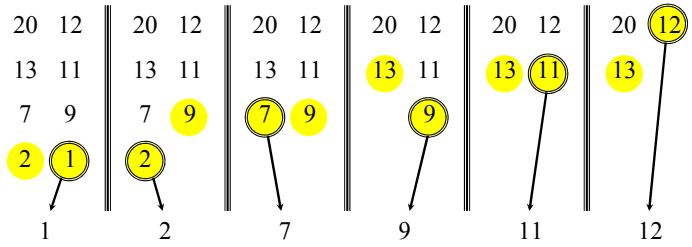

Time $dn = \Theta(n)$ to merge a total of n elements (linear time).

Analyzing merge sort

MERGE-SORT $A[1 \,..\, n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1 \,..\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$.
3. ***"Merge"*** the 2 sorted lists

Sloppiness: Should be $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$, but it turns out not to matter asymptotically.

Recurrence for merge sort

$$T(n) = \begin{cases} d_0 \text{ if } n = 1; \\ 2T(n/2) + dn \text{ if } n > 1. \end{cases}$$

- Later we shall often omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.

- But what does $T(n)$ solve to? I.e., is it $O(n)$ or $O(n^2)$ or $O(n^3)$ or …?

The divide-and-conquer design paradigm

1. ***Divide*** the problem (instance) into subproblems of sizes that are fractions of the original problem size
2. ***Conquer*** the subproblems by solving them recursively.
3. ***Combine*** subproblem solutions.

Example: merge sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.

$$T(n) = 2T(n/2) + \Theta(n)$$

 # subproblems

subproblem size

work dividing and combining

Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3	5	7	8	9	12	15

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Example: Find 9

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Recurrence for binary search

$$T(n) = 1T(n/2) + \Theta(1)$$

 # subproblems

subproblem size

work dividing and combining

Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1; \\ 2T(n/2) + \Theta(n) & \text{if } n > 1. \end{cases}$$

- How do we solve $T(n)$? I.e., how do we find out if it is $O(n)$ or $O(n^2)$ or …?

Recursion tree

Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

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$T(n)$

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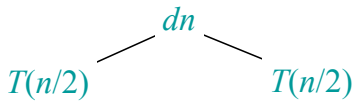

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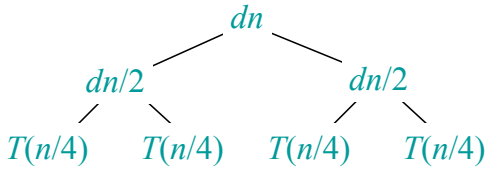

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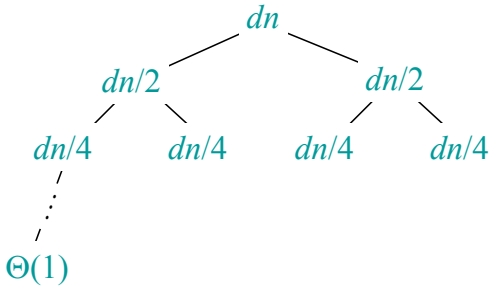

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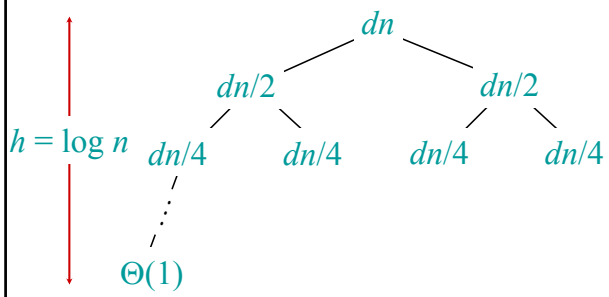

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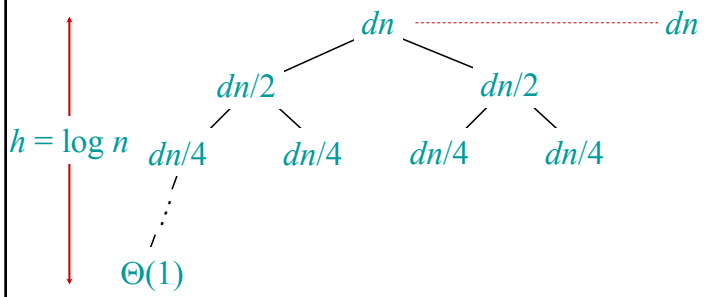

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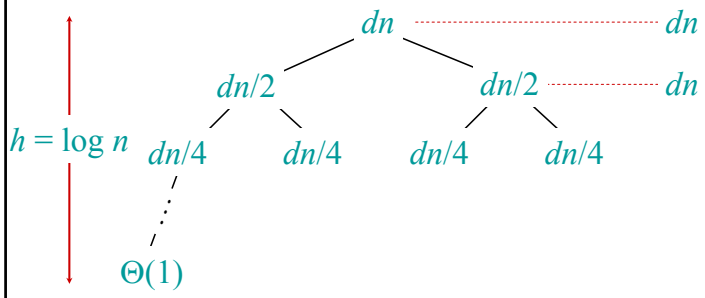

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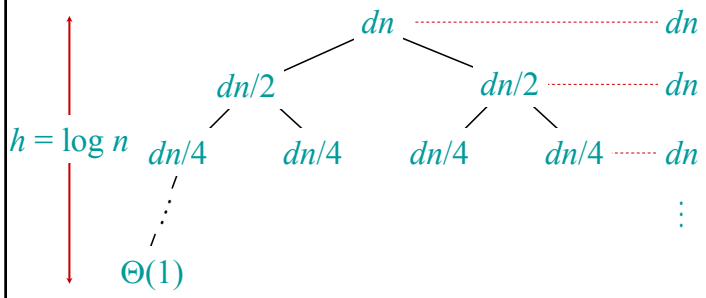

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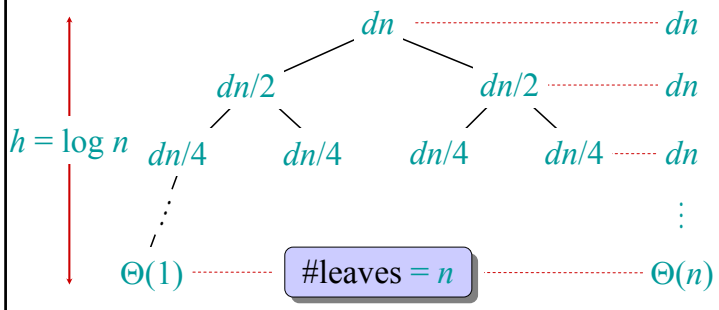

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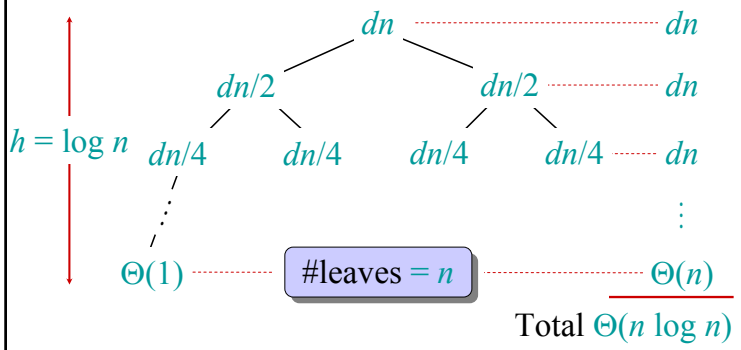

Conclusions

- Merge sort runs in $\Theta(n \log n)$ time.
- $\Theta(n \log n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so. (Why not earlier?)

Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- It is good for generating **guesses** of what the runtime could be.

But: Need to **verify** that the guess is right.
→ Induction (substitution method)

Substitution method

The most general method to solve a recurrence (prove O and Ω separately):

1. ***Guess*** the form of the solution:
   (e.g. using recursion trees, or expansion)
2. ***Verify*** by induction (inductive step).
3. ***Solve*** for O-constants n_0 and c (base case of induction)

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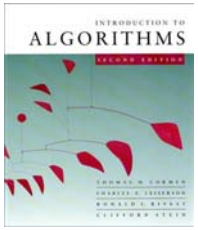


More Divide & Conquer

Carola Wenk

Slides courtesy of Charles Leiserson with small changes by Carola Wenk

The divide-and-conquer design paradigm

1. ***Divide*** the problem (instance) into subproblems.
   a subproblems, **each** of size n/b
2. ***Conquer*** the subproblems by solving them recursively.
3. ***Combine*** subproblem solutions.
   Runtime is $f(n)$

Example: merge sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort $a=2$ subarrays of size $n/2=n/b$
3. ***Combine:*** Linear-time merge, runtime $f(n) \in O(n)$

$$T(n) = 2\,T(n/2) + O(n)$$

 # subproblems — *subproblem size* — *work dividing and combining*

$$T(n) = a\,T(n/b) + f(n)$$

The master method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\ ,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.
2. $f(n) = \Theta(n^{\log_b a} \log^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

   and $f(n)$ satisfies the ***regularity condition*** that $af(n/b) \leq cf(n)$ for some constant $c < 1$.

   Solution: $T(n) = \Theta(f(n))$.

Examples

Ex. $T(n) = 4T(n/2) + \text{sqrt}(n)$
$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = \text{sqrt}(n)$.
CASE 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1.5$.
$\therefore\ T(n) = \Theta(n^2)$.

Ex. $T(n) = 4T(n/2) + n^2$
$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^2$.
CASE 2: $f(n) = \Theta(n^2 \log^0 n)$, that is, $k = 0$.
$\therefore\ T(n) = \Theta(n^2 \log n)$.

Examples

Ex. $T(n) = 4T(n/2) + n^3$
$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^3$.
CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$
and $4(n/2)^3 \leq cn^3$ (reg. cond.) for $c = 1/2$.
$\therefore\ T(n) = \Theta(n^3)$.

Ex. $T(n) = 4T(n/2) + n^2/\log n$
$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^2/\log n$.
Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $\log n \in o(n^{\varepsilon})$.

Master theorem (summary)

$$T(n) = a\,T(n/b) + f(n)$$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon})$

$\Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$

$\Rightarrow T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$ and $a f(n/b) \le c f(n)$ for some constant $c < 1$.

$\Rightarrow T(n) = \Theta(f(n))$.

Example: merge sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.

$$T(n) = 2\,T(n/2) + O(n)$$

 # subproblems — *subproblem size* — *work dividing and combining*

$n^{\log_b a} = n^{\log_2 2} = n^1 = n \Rightarrow$ CASE 2 ($k = 0$)

$\Rightarrow T(n) = \Theta(n \log n)$.

Recurrence for binary search

$$T(n) = 1\,T(n/2) + \Theta(1)$$

 # subproblems — *subproblem size* — *work dividing and combining*

$n^{\log_b a} = n^{\log_2 1} = n^0 = 1 \Rightarrow$ CASE 2 ($k = 0$)

$\Rightarrow T(n) = \Theta(\log n)$.

Powering a number

Problem: Compute a^n, where $n \in \boldsymbol{N}$.

Naive algorithm: $\Theta(n)$.

Divide-and-conquer algorithm: **(recursive squaring)**

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

$T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = \Theta(\log n)$.

Fibonacci numbers

Recursive definition:

$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

0 1 1 2 3 5 8 13 21 34 ...

Naive recursive algorithm: $\Omega(\phi^n)$ (exponential time), where $\phi = (1+\sqrt{5})/2$ is the ***golden ratio***.

Computing Fibonacci numbers

Naive recursive squaring:

$F_n = \phi^n/\sqrt{5}$ rounded to the nearest integer.

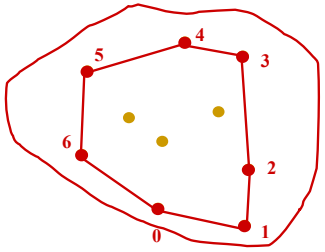
- Recursive squaring: $\Theta(\log n)$ time.
- This method is unreliable, since floating-point arithmetic is prone to round-off errors.

Bottom-up (one-dimensional dynamic programming):

- Compute $F_0, F_1, F_2, \ldots, F_n$ in order, forming each number by summing the two previous.
- Running time: $\Theta(n)$.

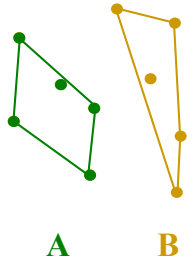
Convex Hull

- Given a set of pins on a pinboard
- And a rubber band around them
- How does the rubber band look when it snaps tight?
- We represent convex hull as the sequence of points on the convex hull polygon, in counter-clockwise order.



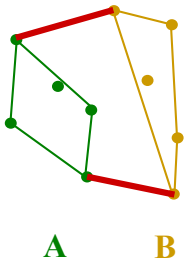
Convex Hull: Divide & Conquer

- Preprocessing: sort the points by x-coordinate
- Divide the set of points into two sets **A** and **B**:
  - **A** contains the left $\lfloor n/2 \rfloor$ points,
  - **B** contains the right $\lceil n/2 \rceil$ points
- Recursively compute the convex hull of **A**
- Recursively compute the convex hull of **B**
- Merge the two convex hulls



Merging

- **Find upper and lower tangent**
- With those tangents the convex hull of A∪B can be computed from the convex hulls of A and the convex hull of B in O(n) linear time



Finding the lower tangent

a = rightmost point of A

b = leftmost point of B

while T=ab not lower tangent to both convex hulls of A and B do{

  while T not lower tangent to convex hull of A do{

    a=a-1

  }

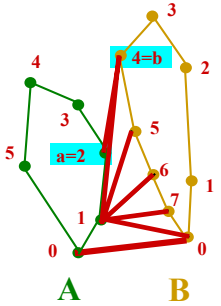
  while T not lower tangent to convex hull of B do{

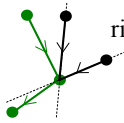
    b=b+1

  }

}

can be checked in constant time





right turn or left turn?

Convex Hull: Runtime

- Preprocessing: sort the points by x-coordinate — O(n log n) just once
- Divide the set of points into two sets **A** and **B**: — O(1)
  - **A** contains the left ⌊n/2⌋ points,
  - **B** contains the right ⌈n/2⌉ points
- Recursively compute the convex hull of **A** — T(n/2)
- Recursively compute the convex hull of **B** — T(n/2)
- Merge the two convex hulls — O(n)

Convex Hull: Runtime

- Runtime Recurrence:

  $T(n) = 2\,T(n/2) + cn$

- Solves to $T(n) = \Theta(n \log n)$

Matrix multiplication

Input: $A = [a_{ij}]$, $B = [b_{ij}]$. } $i, j = 1, 2, \ldots, n$.
Output: $C = [c_{ij}] = A \cdot B$.

$$\begin{bmatrix} c_{11} & c_{12} & \Lambda & c_{1n} \\ c_{21} & c_{22} & \Lambda & c_{2n} \\ M & M & O & M \\ c_{n1} & c_{n2} & \Lambda & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \Lambda & a_{1n} \\ a_{21} & a_{22} & \Lambda & a_{2n} \\ M & M & O & M \\ a_{n1} & a_{n2} & \Lambda & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \Lambda & b_{1n} \\ b_{21} & b_{22} & \Lambda & b_{2n} \\ M & M & O & M \\ b_{n1} & b_{n2} & \Lambda & b_{nn} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Standard algorithm

```
for i ← 1 to n
    do for j ← 1 to n
           do c_ij ← 0
              for k ← 1 to n
                  do c_ij ← c_ij + a_ik · b_kj
```

Running time = $\Theta(n^3)$

Divide-and-conquer algorithm

IDEA:
$n \times n$ matrix = 2×2 matrix of $(n/2) \times (n/2)$ submatrices:

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = A \cdot B$$

$$\begin{aligned} r &= a \cdot e + b \cdot g \\ s &= a \cdot f + b \cdot h \\ t &= c \cdot e + d \cdot g \\ u &= c \cdot f + d \cdot h \end{aligned}$$

8 recursive mults of $(n/2) \times (n/2)$ submatrices
4 adds of $(n/2) \times (n/2)$ submatrices

Analysis of D&C algorithm

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

 # submatrices
submatrix size
work adding submatrices

$n^{\log_b a} = n^{\log_2 8} = n^3 \Rightarrow$ CASE 1 $\Rightarrow T(n) = \Theta(n^3)$.

No better than the ordinary algorithm.

Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$
\begin{aligned}
P_1 &= a \cdot (f - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (g - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}
$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
s &= P_1 + P_2 \\
t &= P_3 + P_4 \\
u &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

7 mults, 18 adds/subs.
Note: No reliance on commutativity of mult!

Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$
\begin{aligned}
P_1 &= a \cdot (f - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (g - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}
$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
&= (a + d)(e + h) \\
&\quad + d(g - e) - (a + b)h \\
&\quad + (b - d)(g + h) \\
&= ae + ah + de + dh \\
&\quad + dg - de - ah - bh \\
&\quad + bg + bh - dg - dh \\
&= ae + bg
\end{aligned}
$$

Strassen's algorithm

1. ***Divide:*** Partition A and B into $(n/2)\times(n/2)$ submatrices. Form P-terms to be multiplied using + and – .
2. ***Conquer:*** Perform 7 multiplications of $(n/2)\times(n/2)$ submatrices recursively.
3. ***Combine:*** Form C using + and – on $(n/2)\times(n/2)$ submatrices.

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

Analysis of Strassen

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \Rightarrow$ CASE 1 $\Rightarrow T(n) = \Theta(n^{\log 7})$.

The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 30$ or so.

Best to date (of theoretical interest only): $\Theta(n^{2.376\Lambda})$.

Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- Can lead to more efficient algorithms