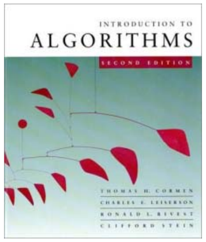
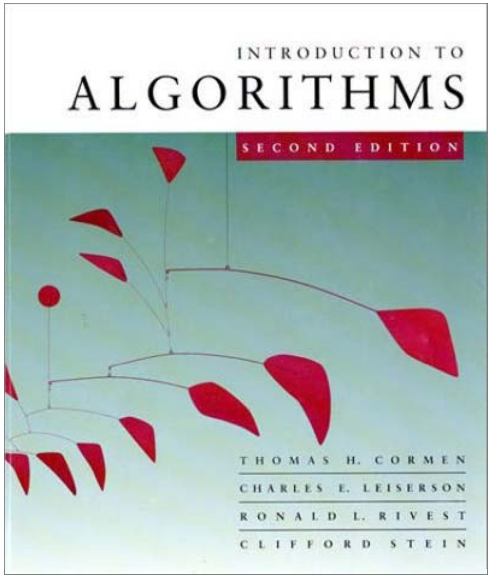


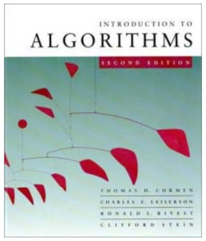
CS 5633 – Spring 2012



Graphs

Carola Wenk

Slides courtesy of Charles Leiserson with changes and additions by Carola Wenk



Graphs (review)

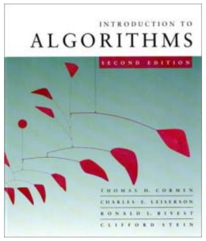
Definition. A ***directed graph (digraph)*** $G = (V, E)$ is an ordered pair consisting of

- a set V of ***vertices*** (singular: ***vertex***),
- a set $E \subseteq V \times V$ of ***edges***.

In an ***undirected graph*** $G = (V, E)$, the edge set E consists of *unordered* pairs of vertices.

In either case, we have $|E| = O(|V|^2)$. Moreover, if G is connected, then $|E| \geq |V| - 1$.

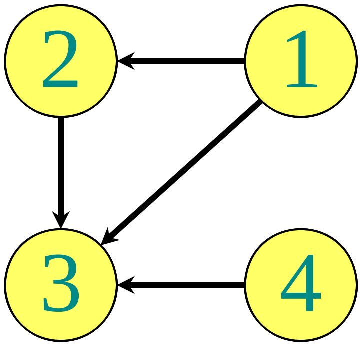
(Review CLRS, Appendix B.4 and B.5.)



Adjacency-matrix representation

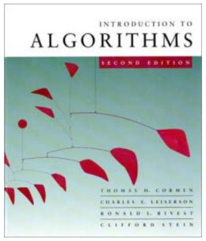
The ***adjacency matrix*** of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1 \,.\,.\, n, 1 \,.\,.\, n]$ given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E. \end{cases}$$



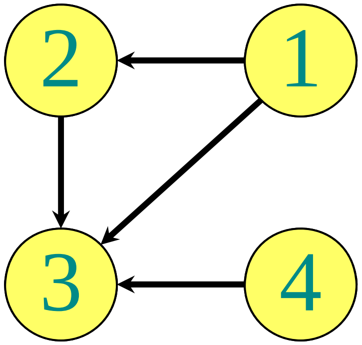
$$\begin{array}{c|cccc} A & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

$\Theta(|V|^2)$ storage $\Rightarrow$ ***dense*** representation.



Adjacency-list representation

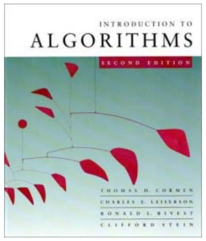
An ***adjacency list*** of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to v.



$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3\}$

For undirected graphs, $|Adj[v]| = degree(v)$.

For digraphs, $|\,Adj[v]\,| = out\text{-}degree(v)$.



Adjacency-list representation

Handshaking Lemma:

Every edge is counted twice

- For undirected graphs:

  $\sum_{v \in V} degree(v) = 2|E|$

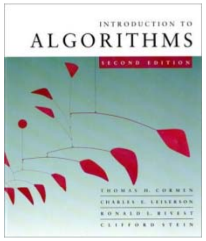
- For digraphs:

  $\sum_{v \in V} in\text{-}degree(v) + \sum_{v \in V} out\text{-}degree(v) = 2 \mid E \mid$

⇒ adjacency lists use $\Theta(|V| + |E|)$ storage

⇒ a ***sparse*** representation

⇒ We usually use this representation,
unless stated otherwise



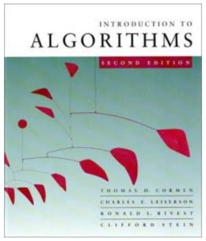
Graph Traversal

Let $G=(V,E)$ be a (directed or undirected) graph, given in adjacency list representation.

$|V| = n$, $|E| = m$

A graph traversal visits every vertex:

- Breadth-first search (BFS)
- Depth-first search (DFS)

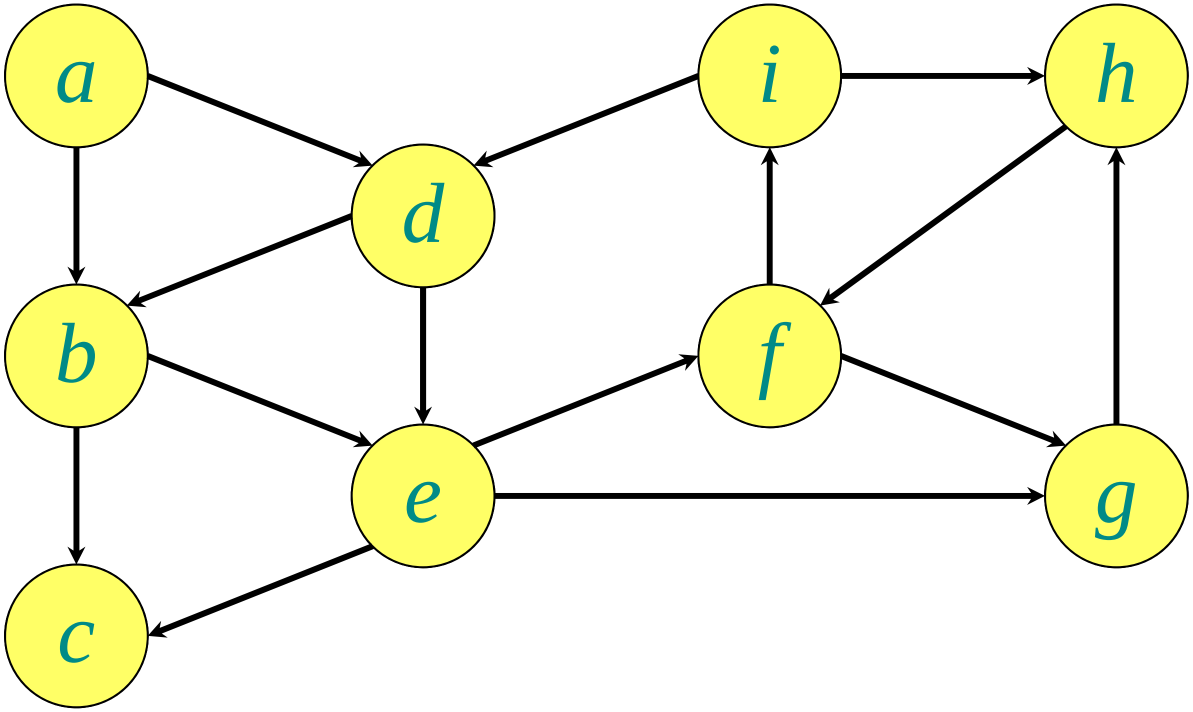


Breadth-First Search (BFS)

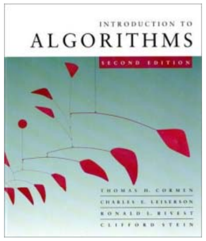
```
BFS($G=(V,E)$)
    Mark all vertices in $G$ as “unvisited” // time=0
    Initialize empty queue $Q$
    for each vertex $v \in V$ do
        if $v$ is unvisited
            visit $v$ // time++
            $Q$.enqueue($v$)
            BFS_iter($G$)
```

```
BFS_iter($G$)
    while $Q$ is non-empty do
        $v$ = $Q$.dequeue()
        for each $w$ adjacent to $v$ do
            if $w$ is unvisited
                visit $w$   // time++
                Add edge ($v$,$w$) to $T$
                $Q$.enqueue($w$)
```

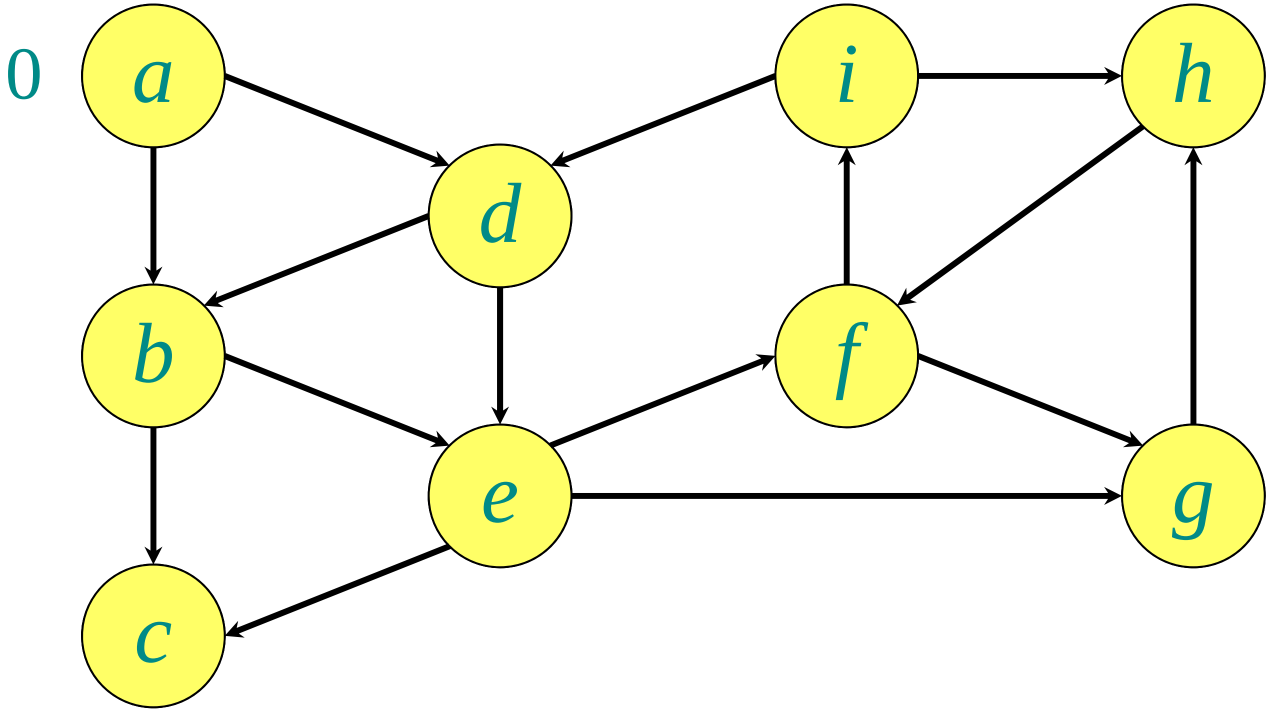
Example of breadth-first search



Q:

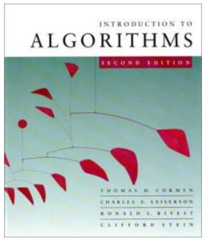


Example of breadth-first search

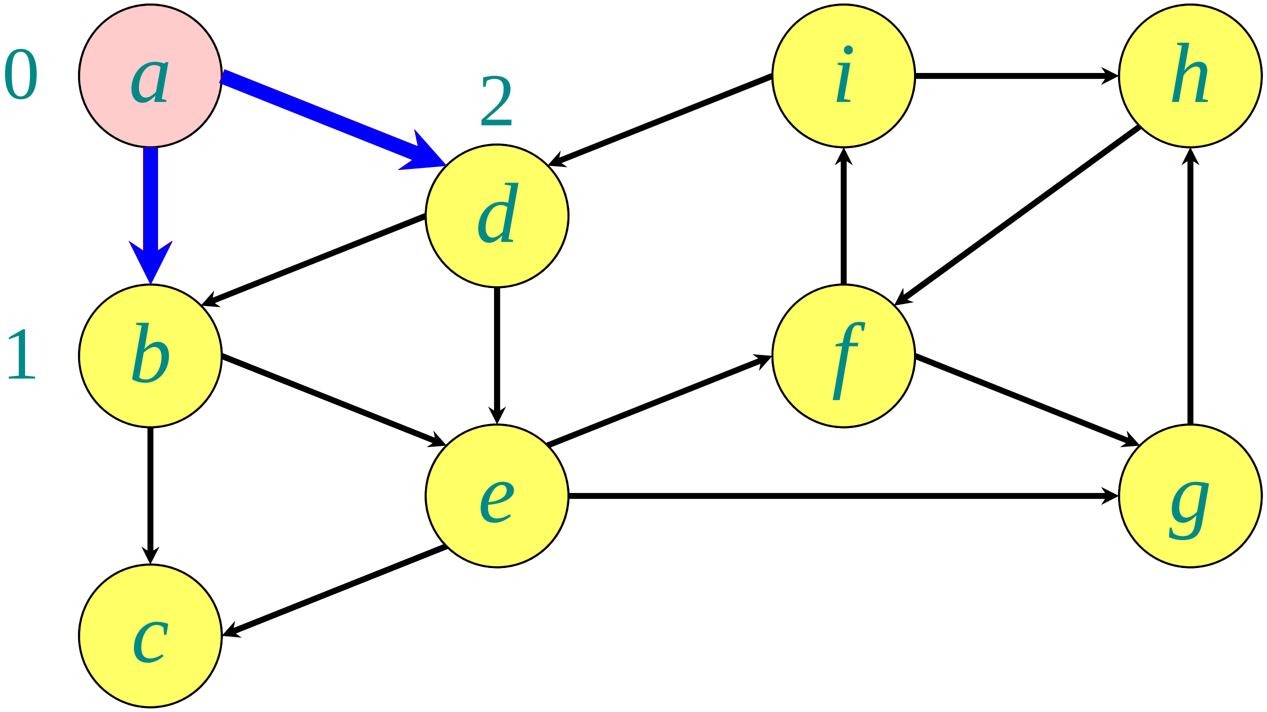


0

Q: *a*

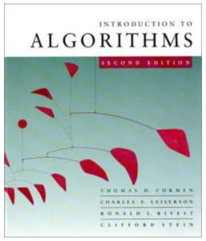


Example of breadth-first search

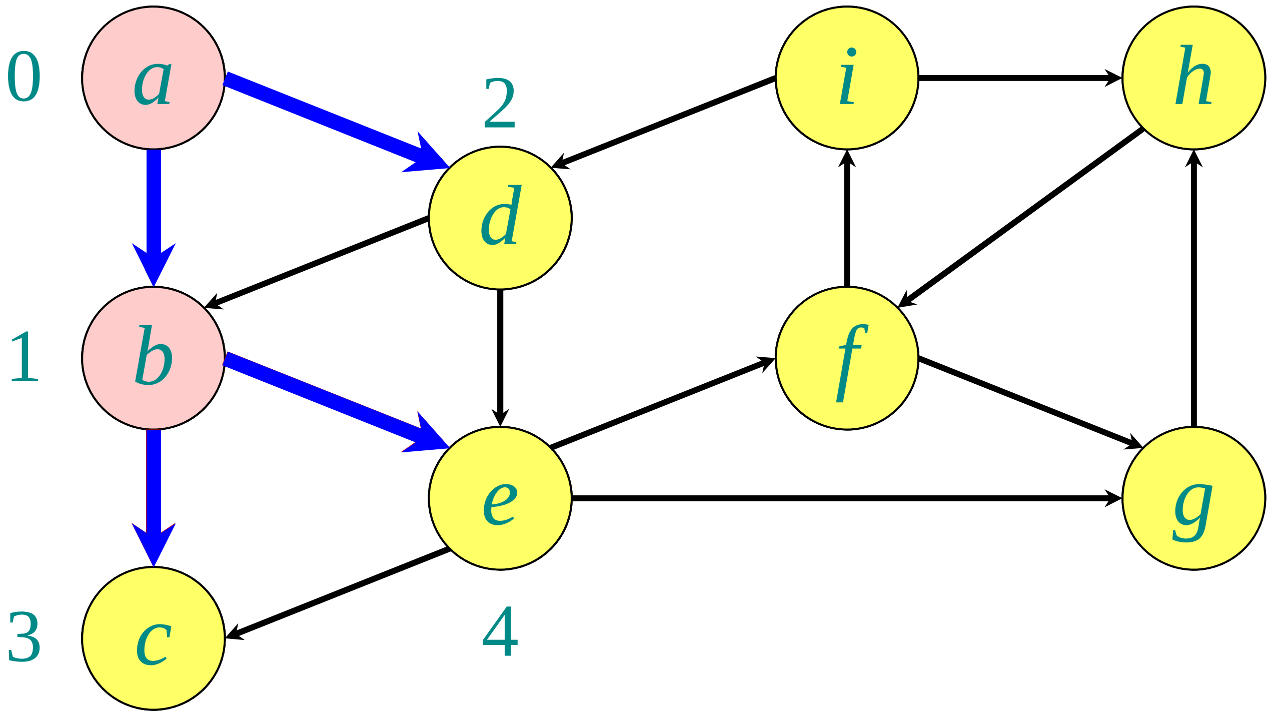


1 2

Q: a b d

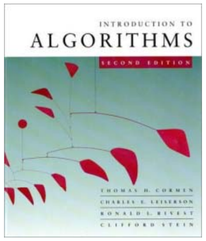


Example of breadth-first search

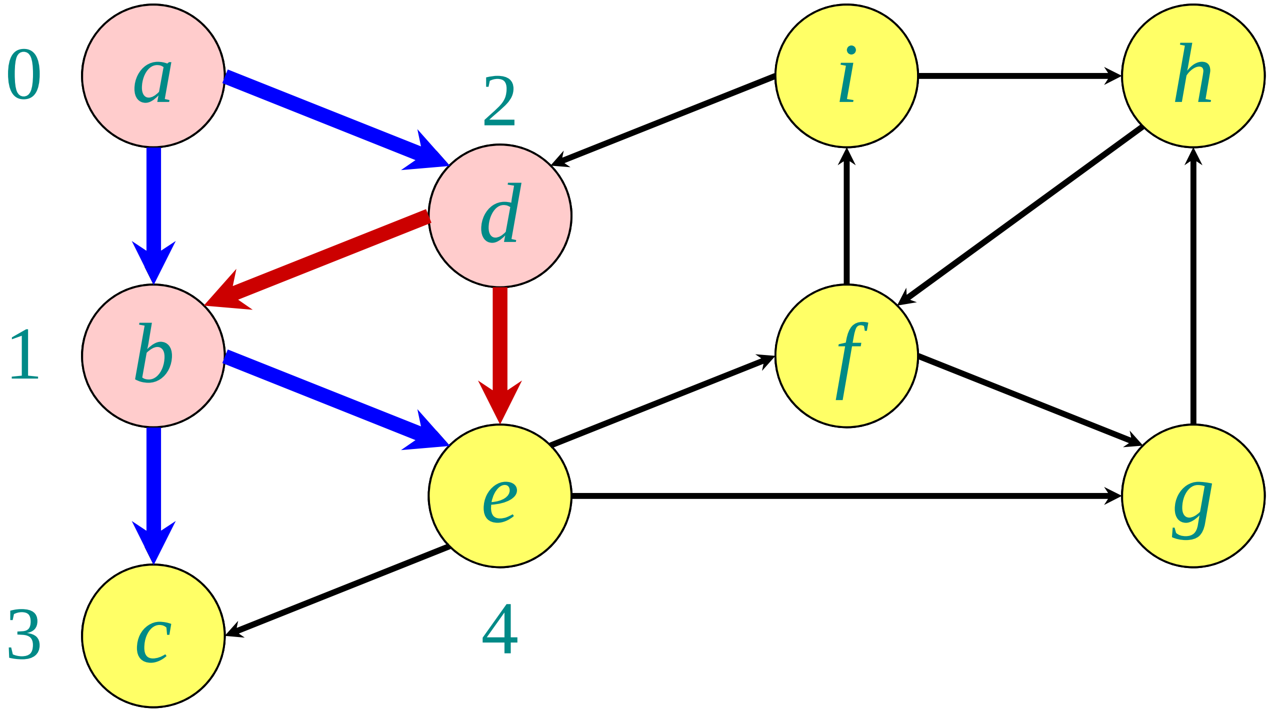


2 3 4

Q: *a* *b* *d* *c* *e*

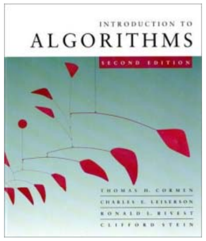


Example of breadth-first search

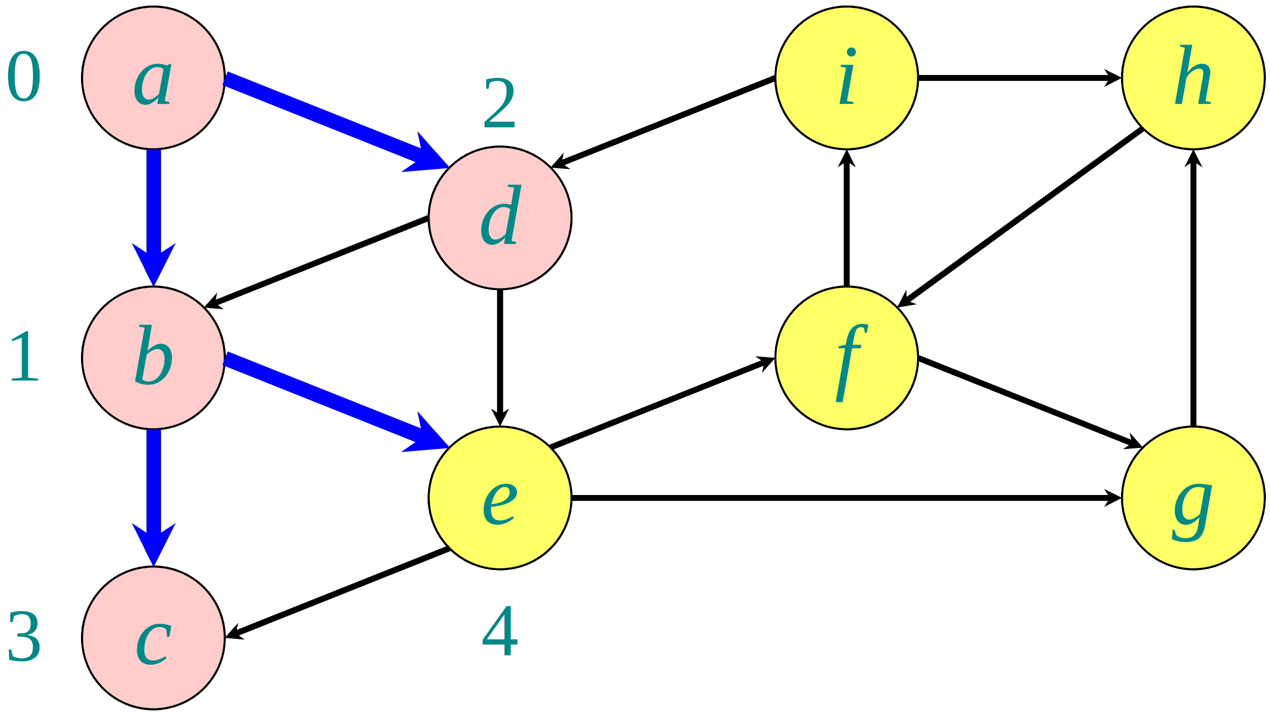


3 4

Q: a b d c e

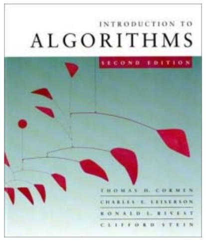


Example of breadth-first search

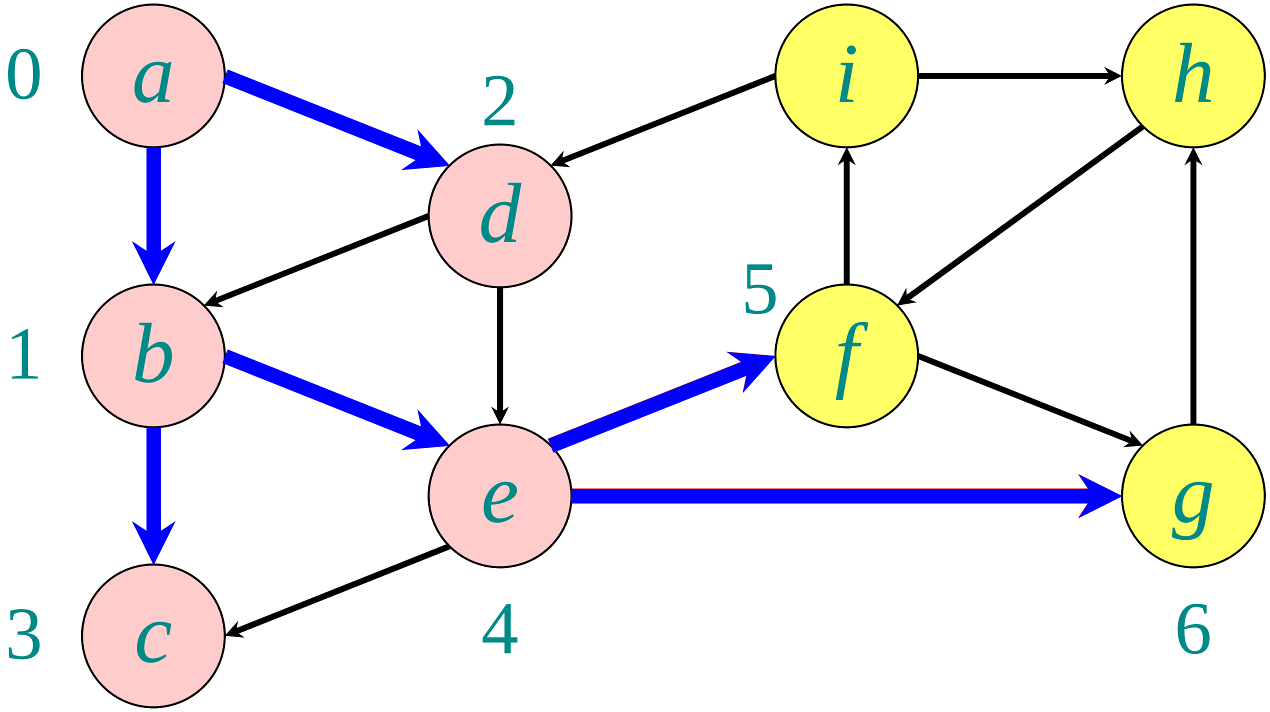


4

Q: a b d c e

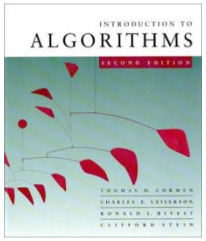


Example of breadth-first search

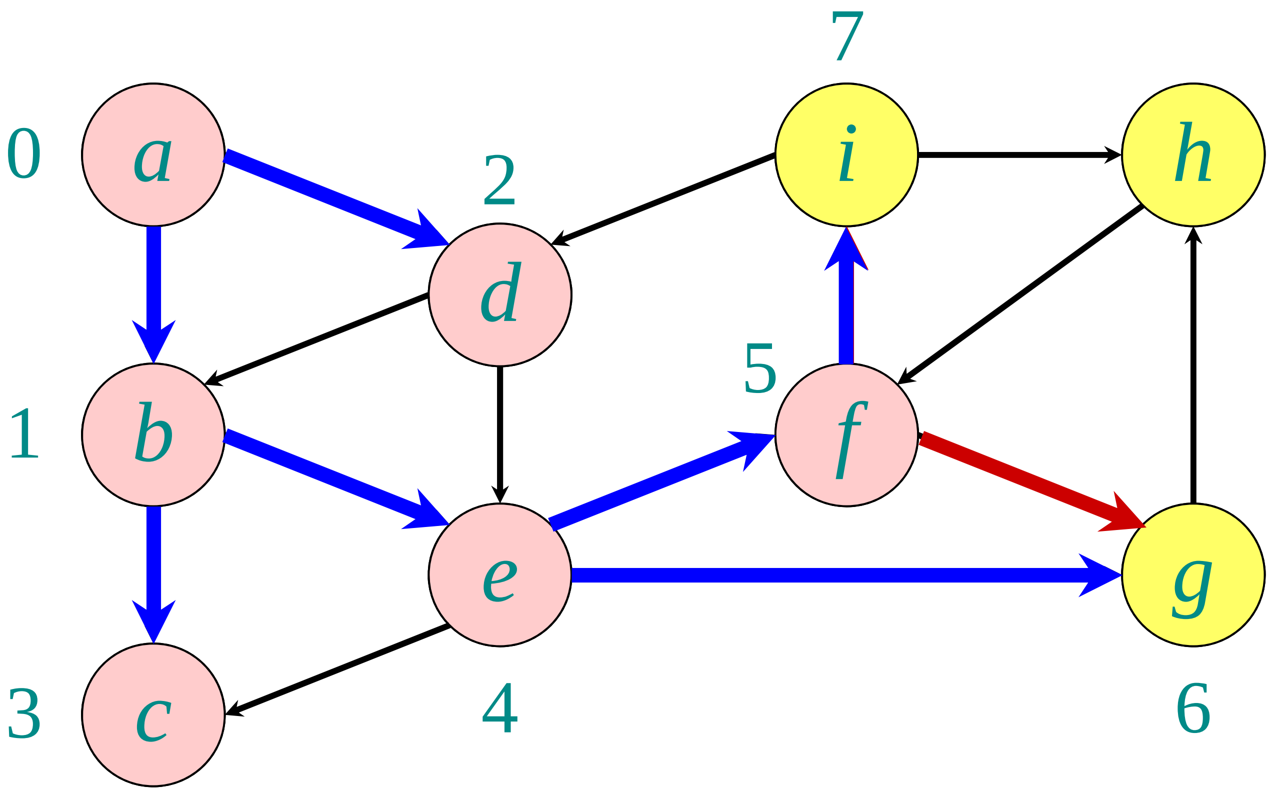


5 6

Q: *a b d c e f g*

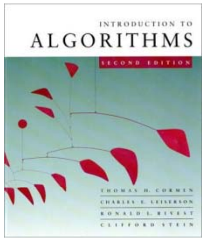


Example of breadth-first search

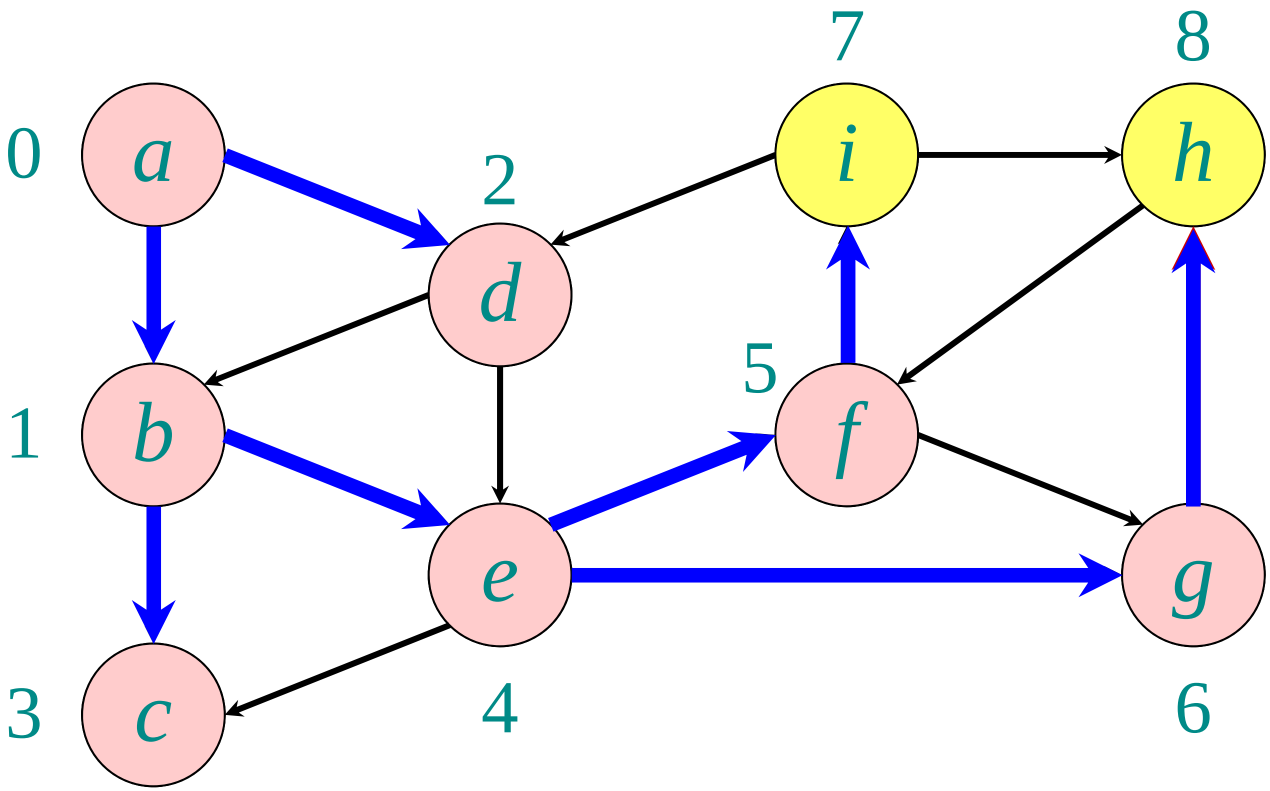


6 7

Q: a b d c e f g i



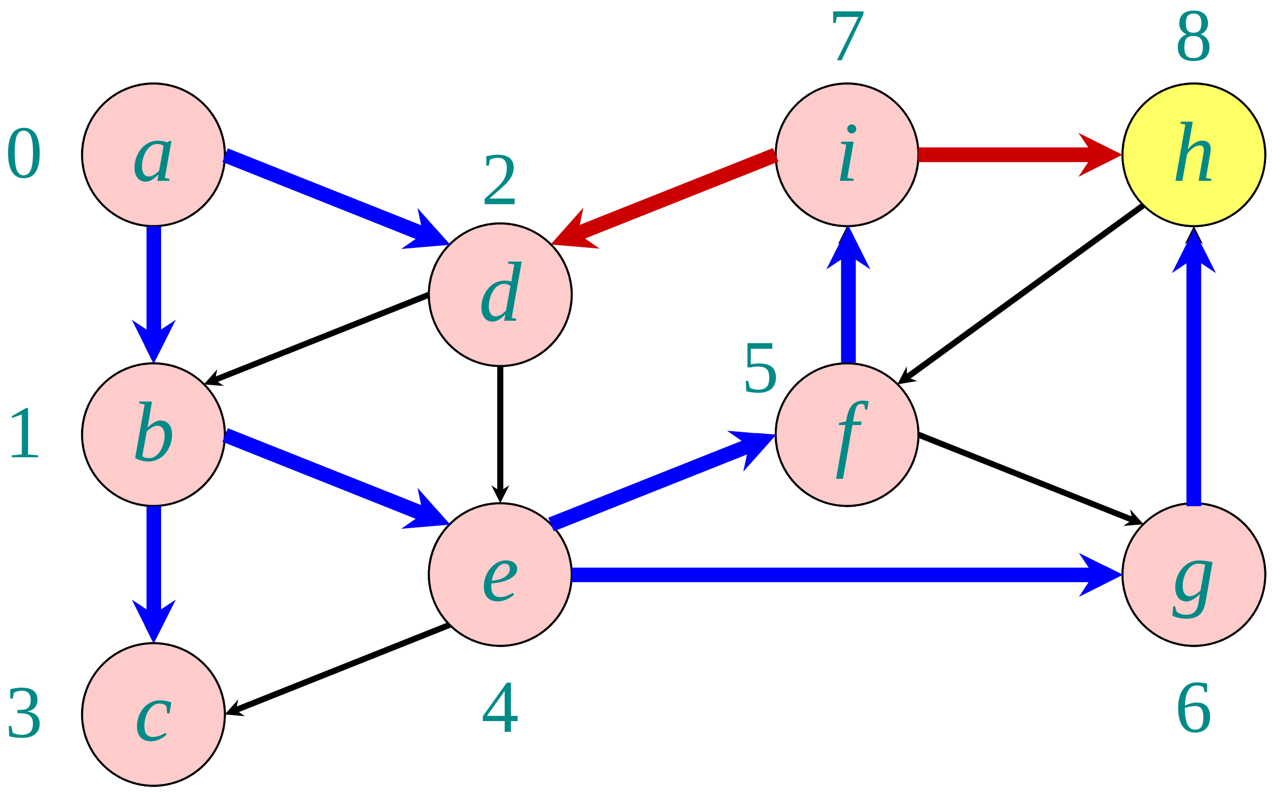
Example of breadth-first search



7 8

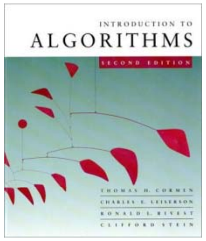
Q: a b d c e f g i h

Example of breadth-first search

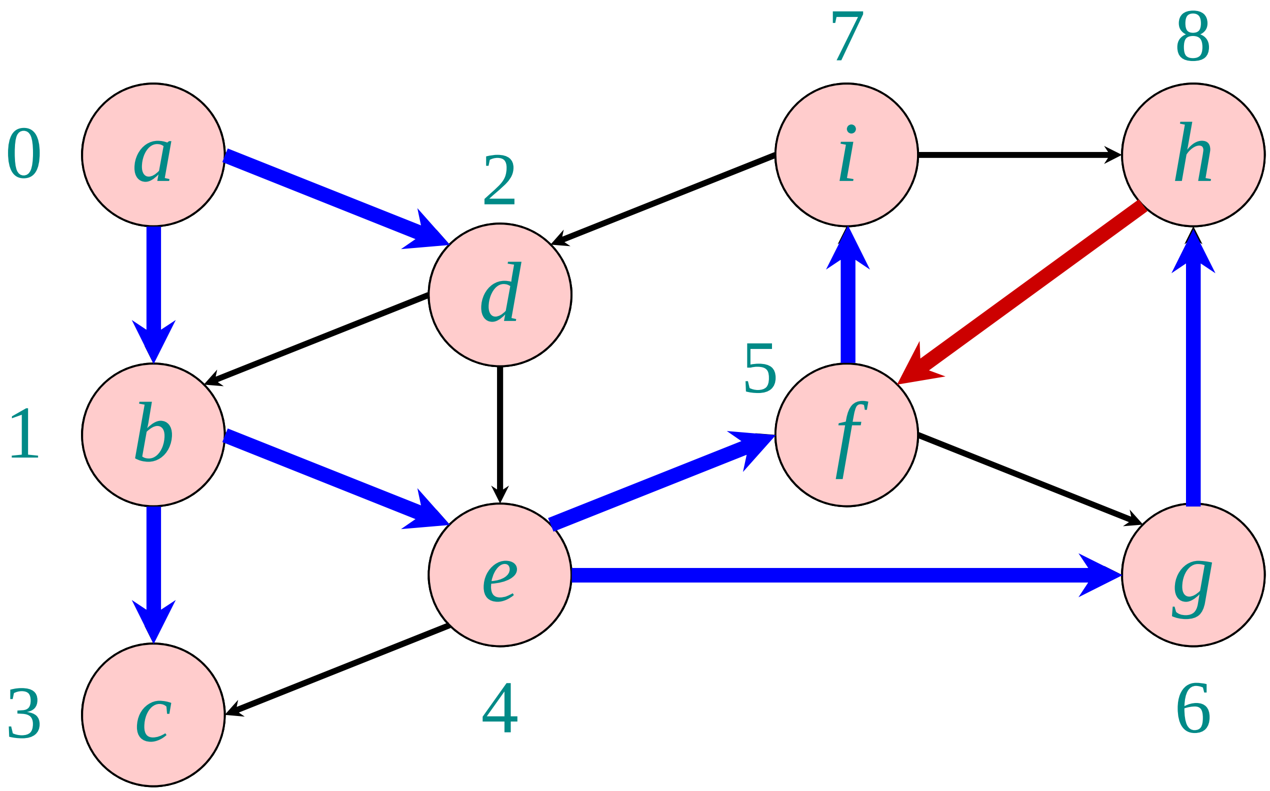


8

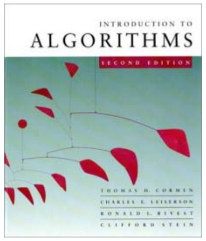
Q: *a b d c e f g i h*



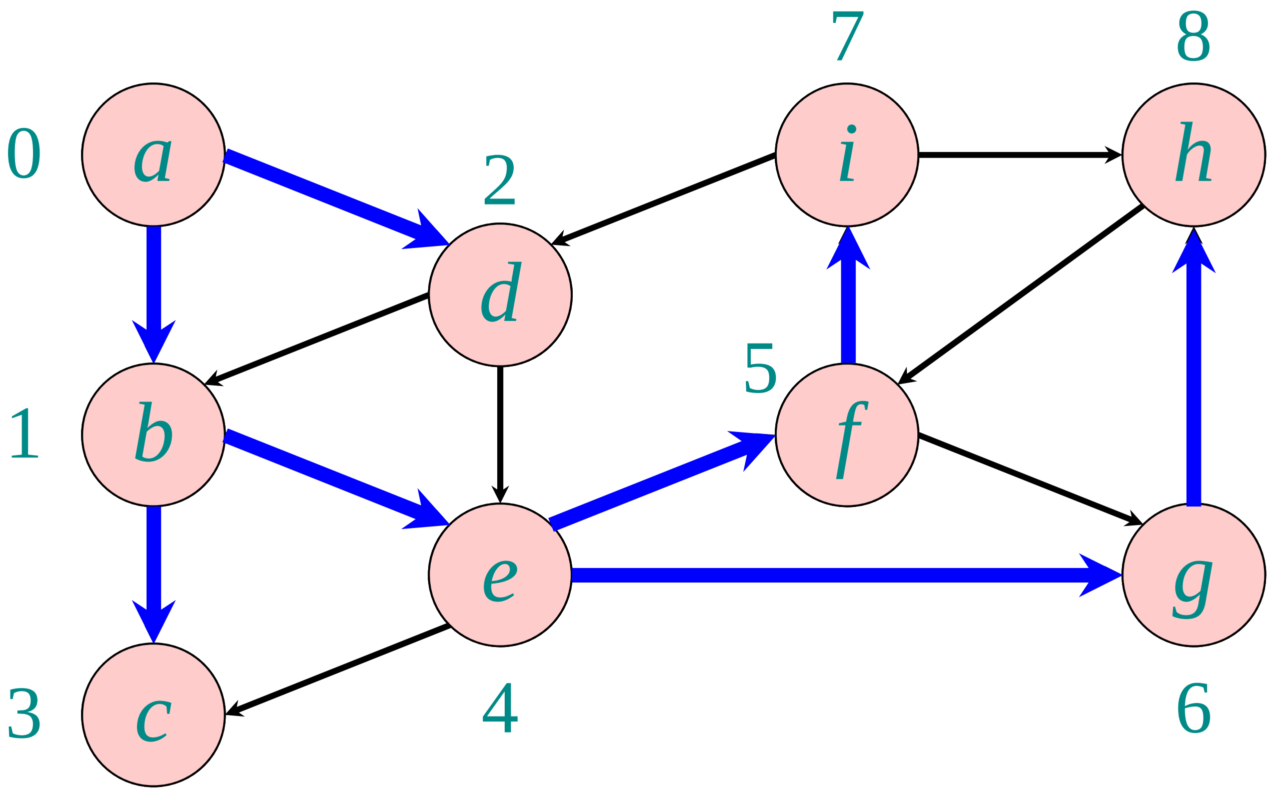
Example of breadth-first search



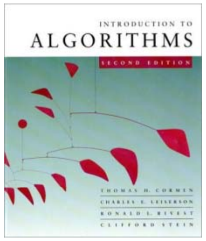
Q: *a b d c e f g i h*



Example of breadth-first search



Q: *a b d c e f g i h*



Breadth-First Search (BFS)

```
BFS(G=(V,E))
    Mark all vertices in G as "unvisited" // time=0
    Initialize empty queue Q
    for each vertex $v \in V$ do
        if v is unvisited
            visit v // time++
            Q.enqueue(v)
            BFS_iter(G)
```

O(*n*)

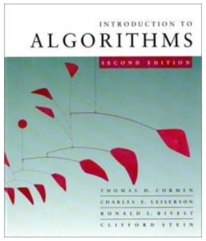
O(1)

O(*n*)
without
BFS_iter

```
BFS_iter(G)
    while Q is non-empty do
        v = Q.dequeue()
        for each w adjacent to v do
            if w is unvisited
                visit w   // time++
                Add edge (v,w) to T
                Q.enqueue(w)
```

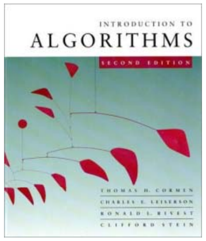
O(*deg*(*v*))

$\sum_{v \in V} deg(v) =$ O(*m*)



BFS runtime

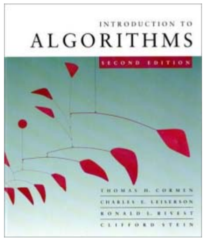
- Each vertex is marked as unvisited in the beginning ⇒ O(n) time
- Each vertex is marked at most once, enqueued at most once, and therefore dequeued at most once
- The time to process a vertex is proportional to the size of its adjacency list (its degree), since the graph is given in adjacency list representation
 ⇒ O(m) time
- Total runtime is O($n+m$) = O(|V| + |E|)



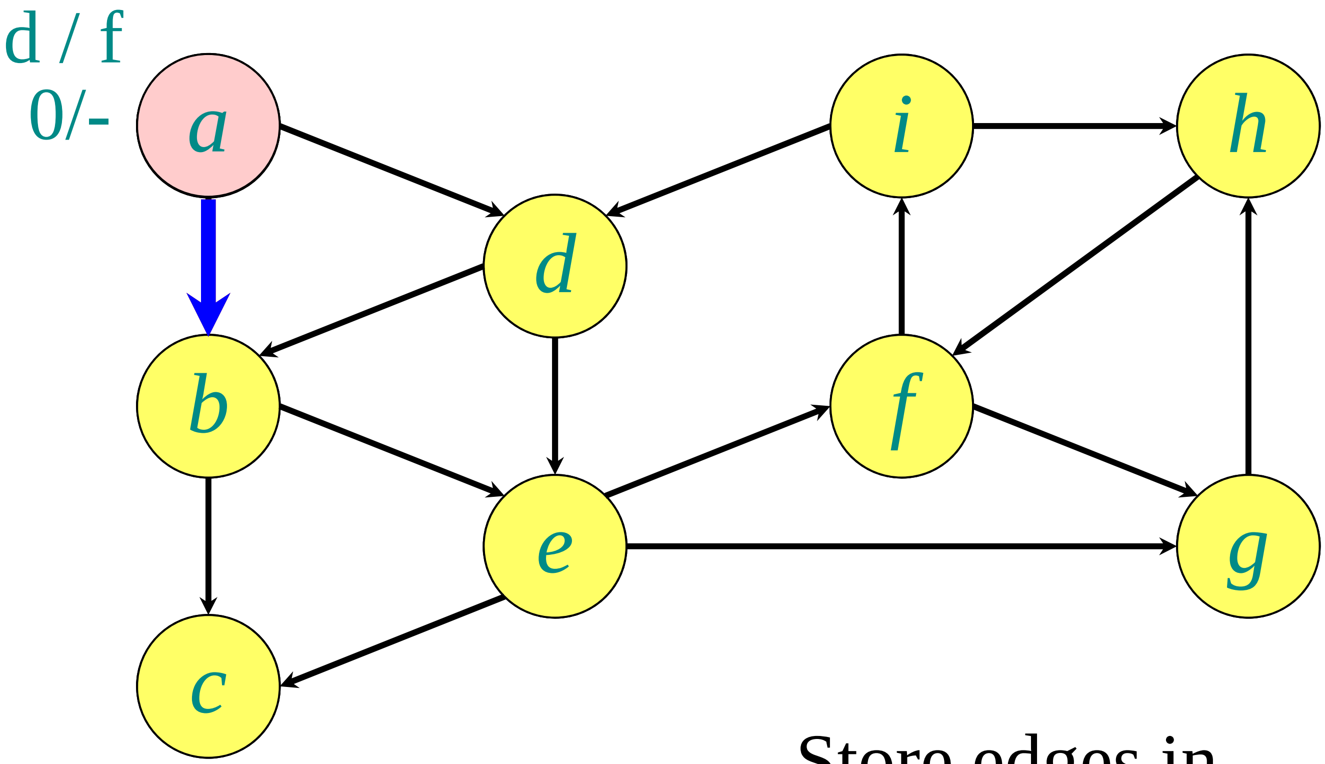
Depth-First Search (DFS)

```
DFS($G=(V,E)$)
    Mark all vertices in $G$ as “unvisited”  // time=0
    **for** each vertex $v \in V$ **do**
        **if** $v$ is unvisited
            DFS_rec($G$,$v$)
```

```
DFS_rec($G$, $v$)
    mark $v$ as “visited” // $d[v]$=++time
    **for** each $w$ adjacent to $v$ **do**
        **if** $w$ is unvisited
            Add edge $(v,w)$ to tree $T$
            DFS_rec($G$,$w$)
    mark $v$ as “finished” // $f[v]$=++time
```

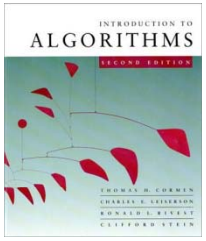


Example of depth-first search

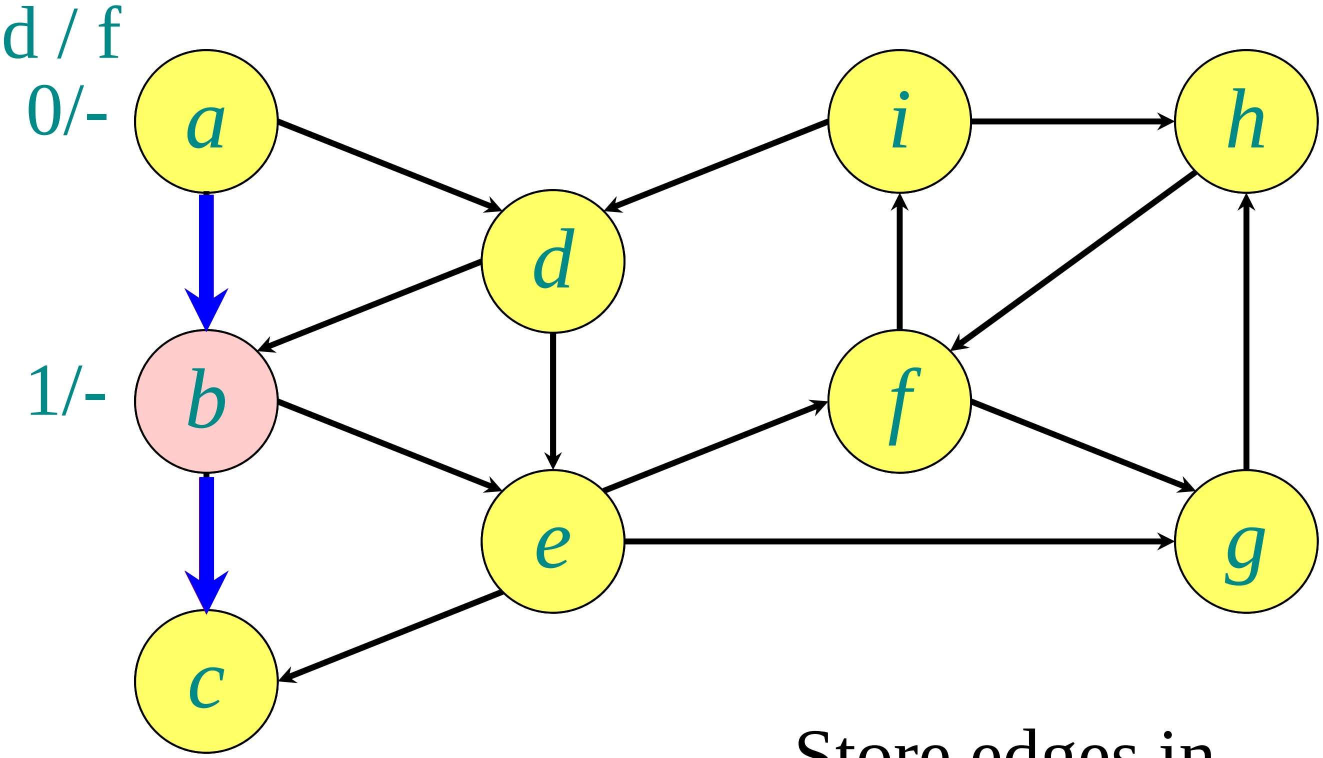


Store edges in
predecessor array

π:	a	b	c	d	e	f	g	h	i
	-	a							

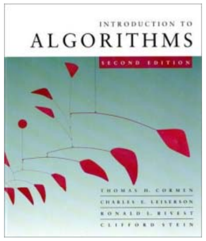


Example of depth-first search

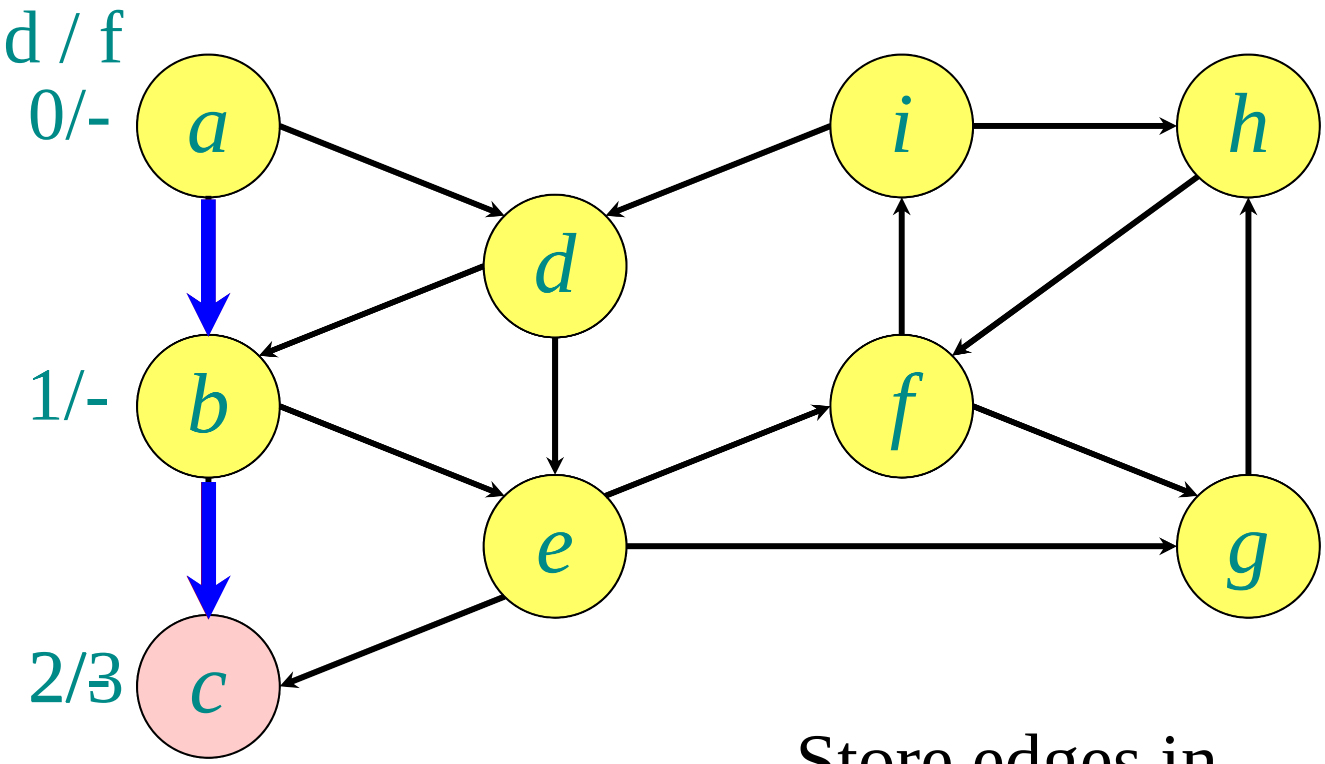


Store edges in
predecessor array

π:	a	b	c	d	e	f	g	h	i
	-	a	b						

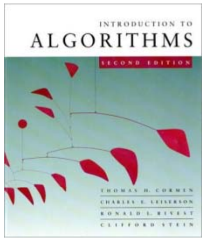


Example of depth-first search

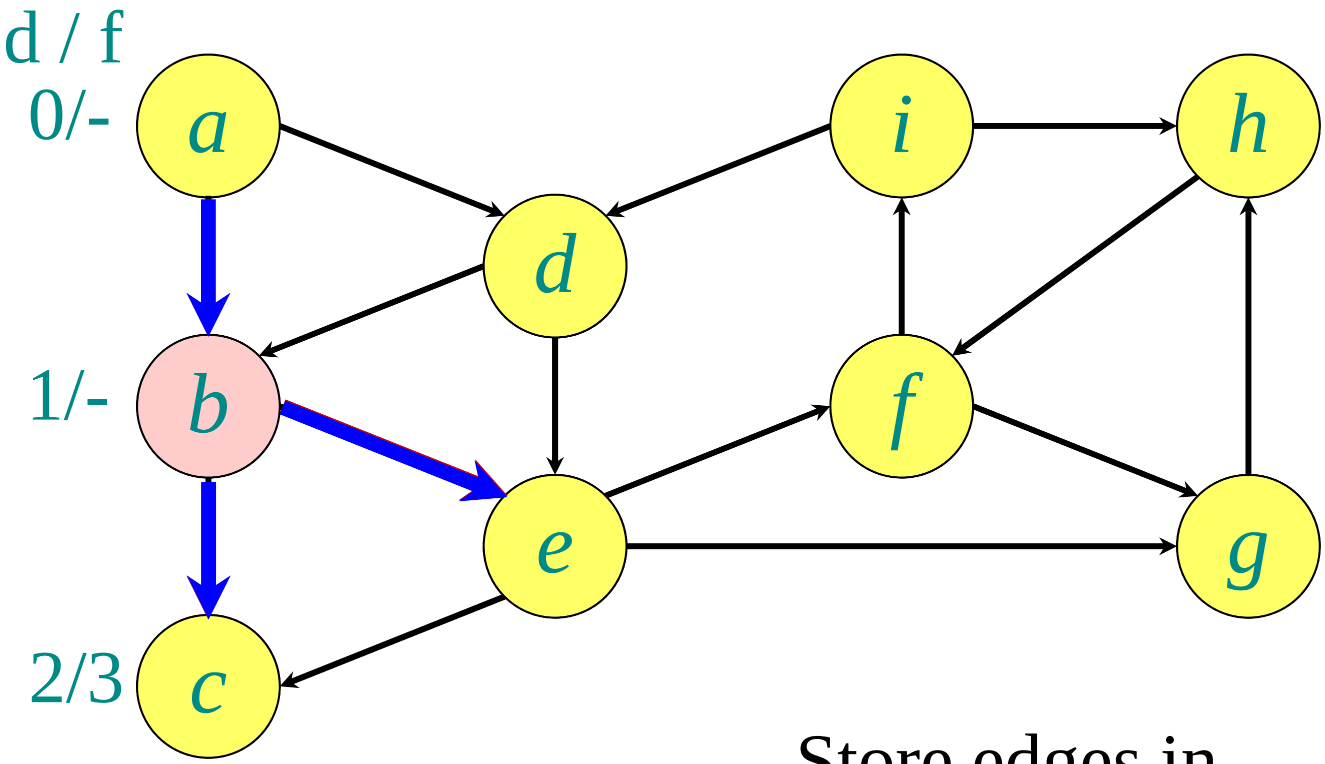


Store edges in
predecessor array

π:	a	b	c	d	e	f	g	h	i
	-	a	b						

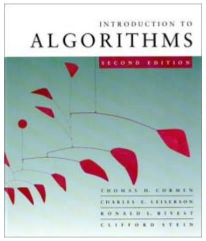


Example of depth-first search

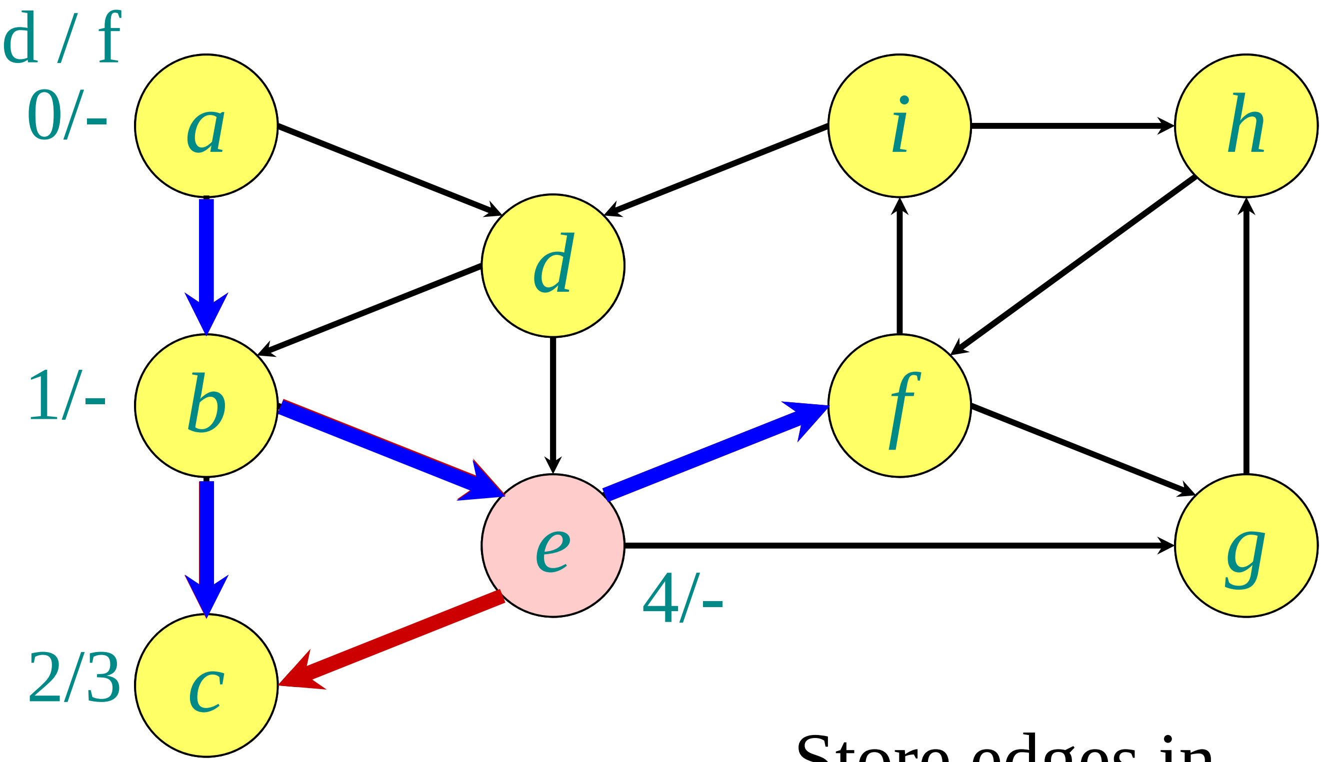


π:	a	b	c	d	e	f	g	h	i
	-	a	b		b				

Store edges in
predecessor array

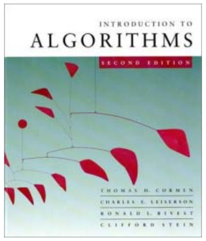


Example of depth-first search

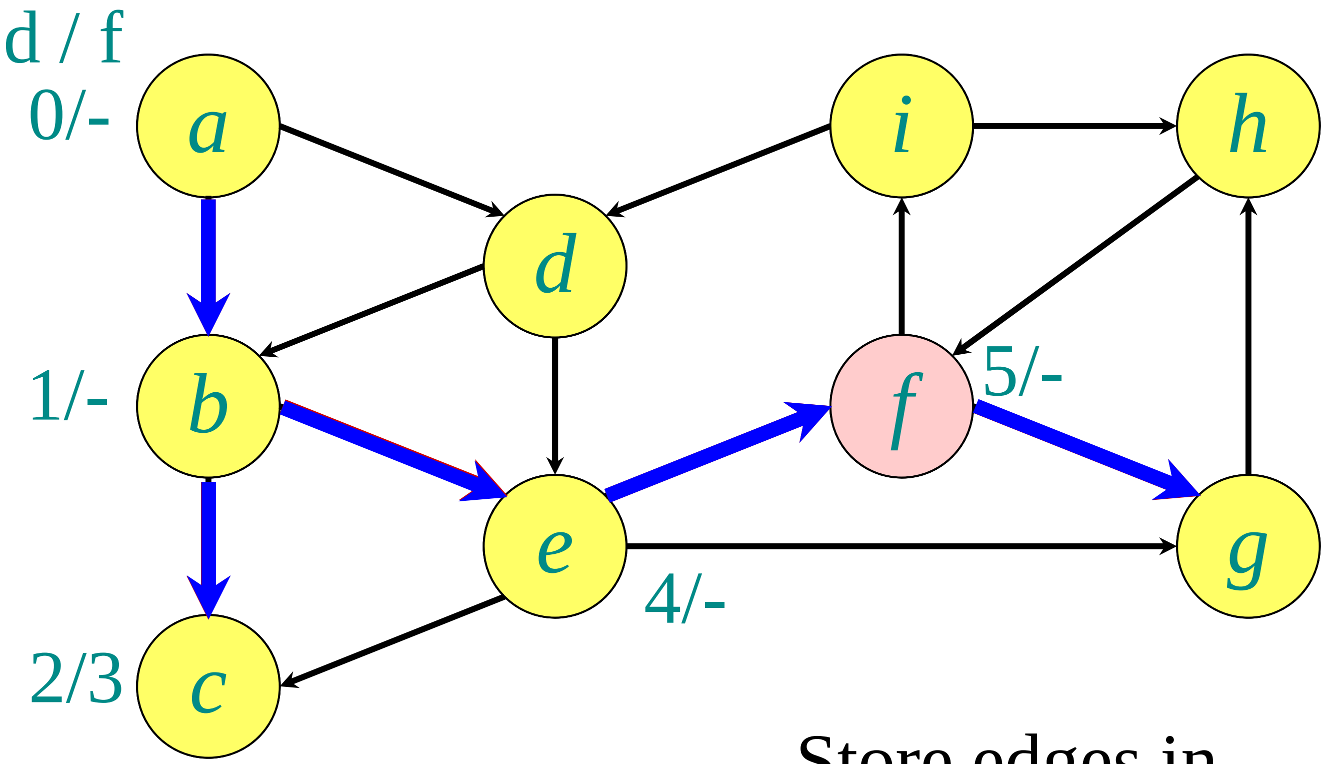


π:	a	b	c	d	e	f	g	h	i
	-	a	b		b	e			

Store edges in
predecessor array

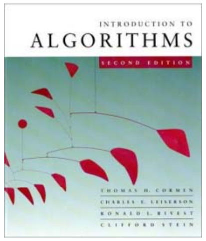


Example of depth-first search

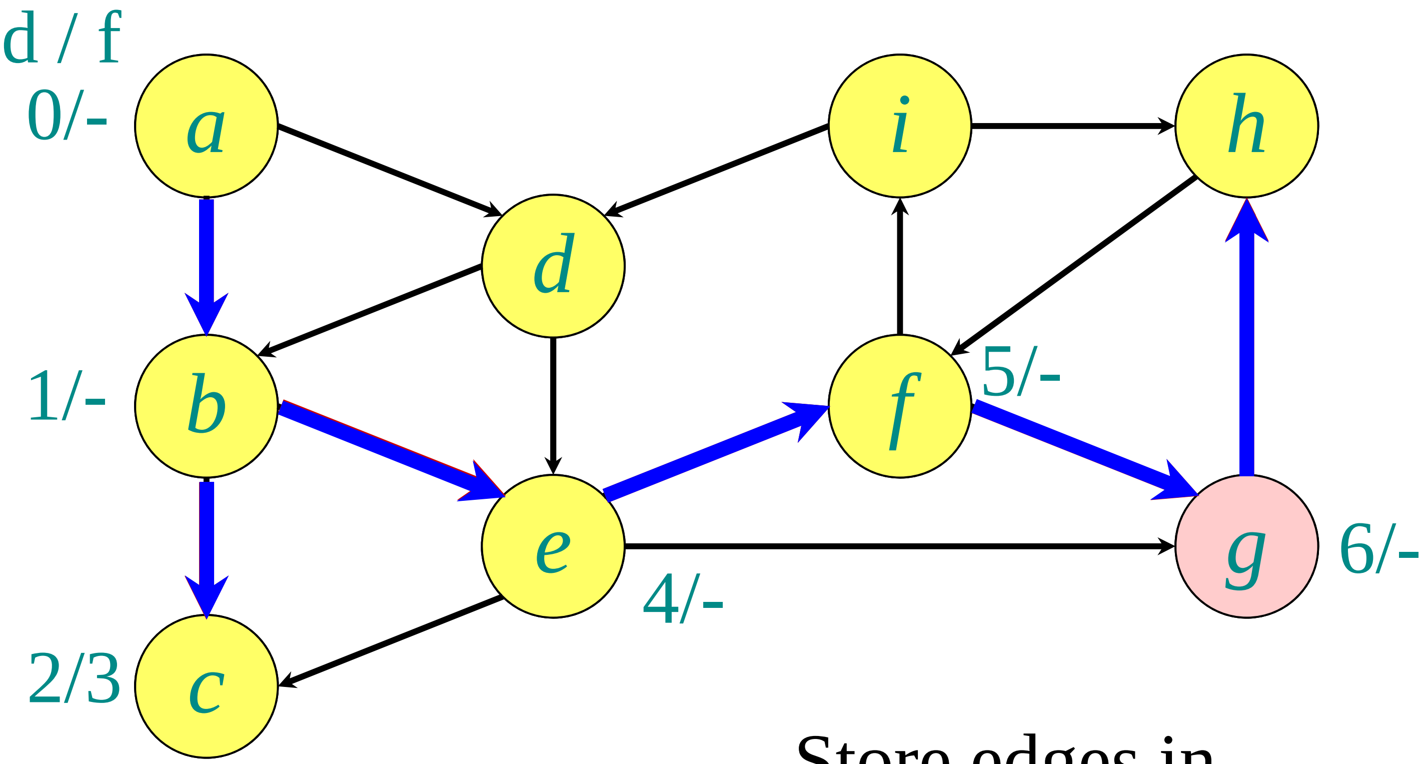


π:	a	b	c	d	e	f	g	h	i
	-	a	b		b	e	f		

Store edges in predecessor array

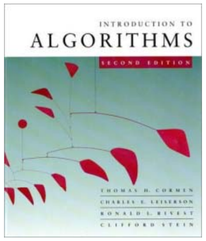


Example of depth-first search

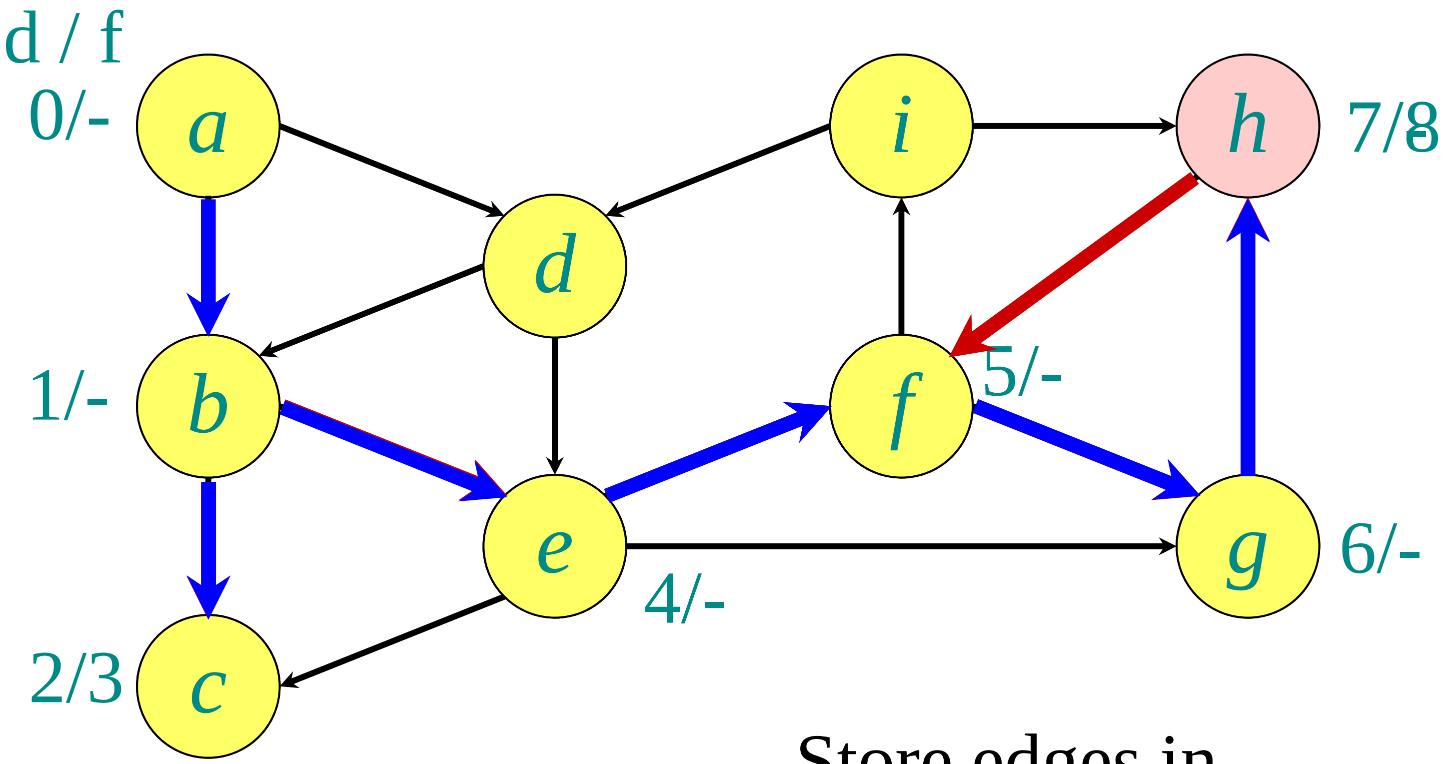


π:	a	b	c	d	e	f	g	h	i
	-	a	b		b	e	f	g	

Store edges in predecessor array

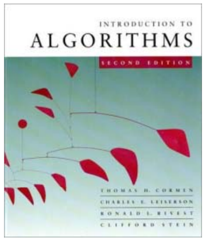


Example of depth-first search

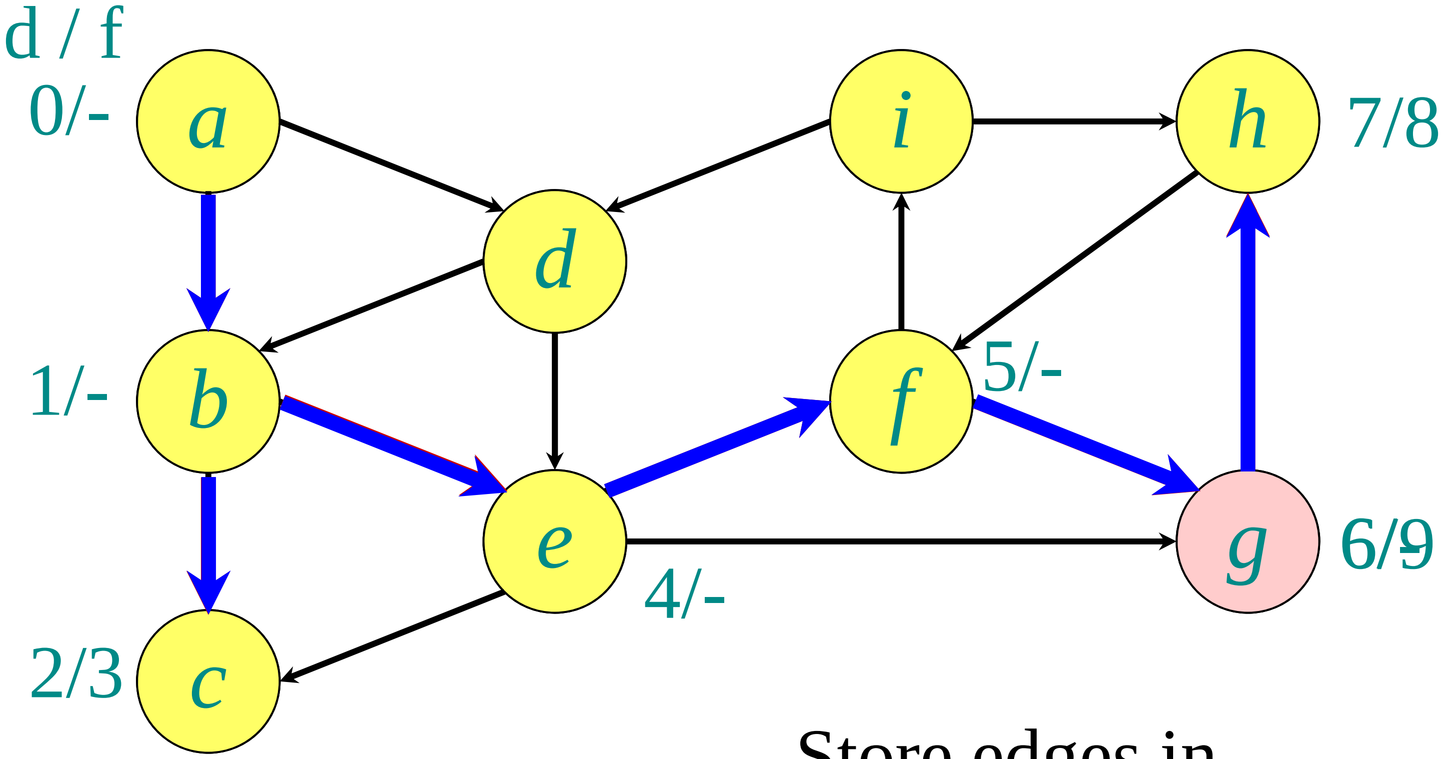


π:	a	b	c	d	e	f	g	h	i
	-	a	b		b	e	f	g	

Store edges in predecessor array

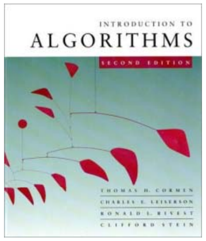


Example of depth-first search

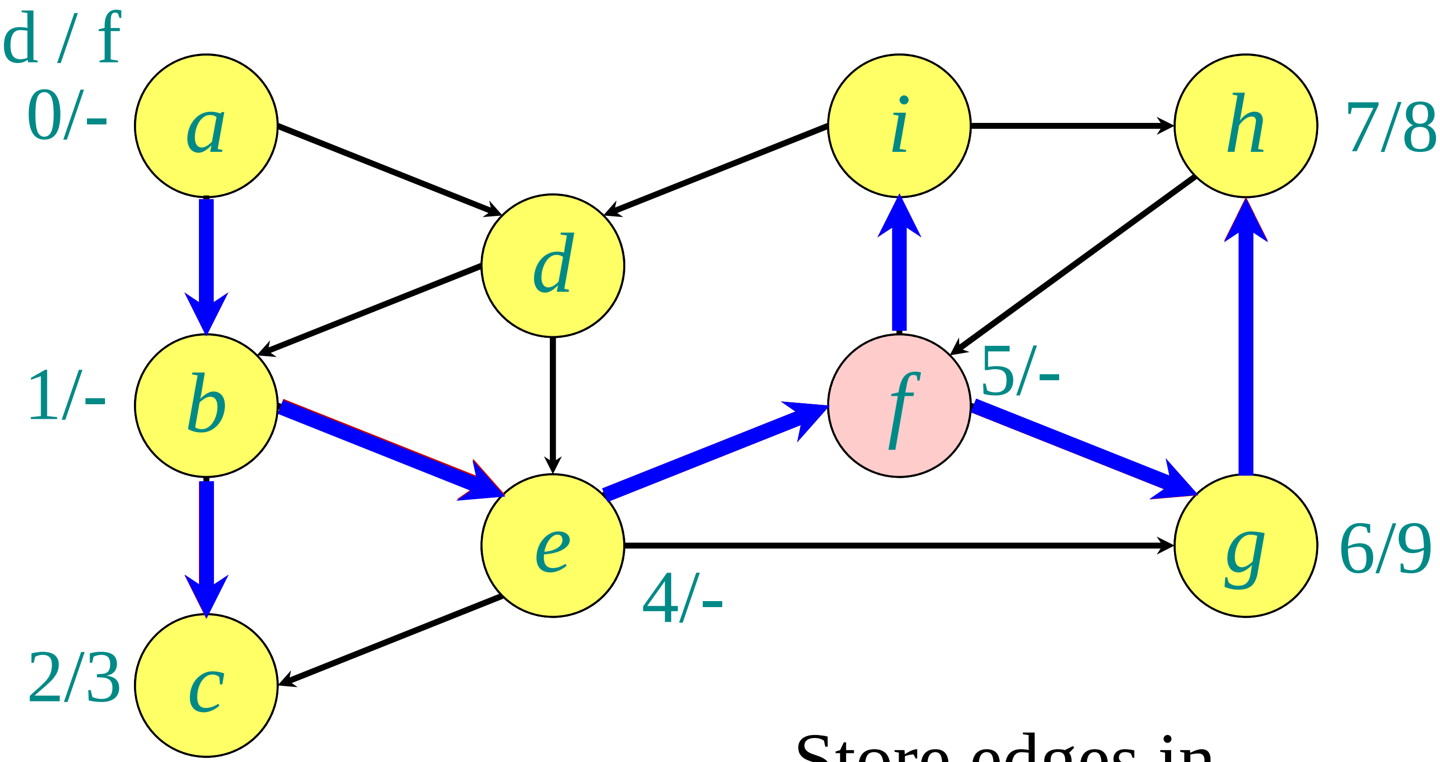


Store edges in predecessor array

π:	a	b	c	d	e	f	g	h	i
	-	a	b		b	e	f	g	

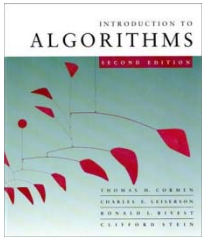


Example of depth-first search

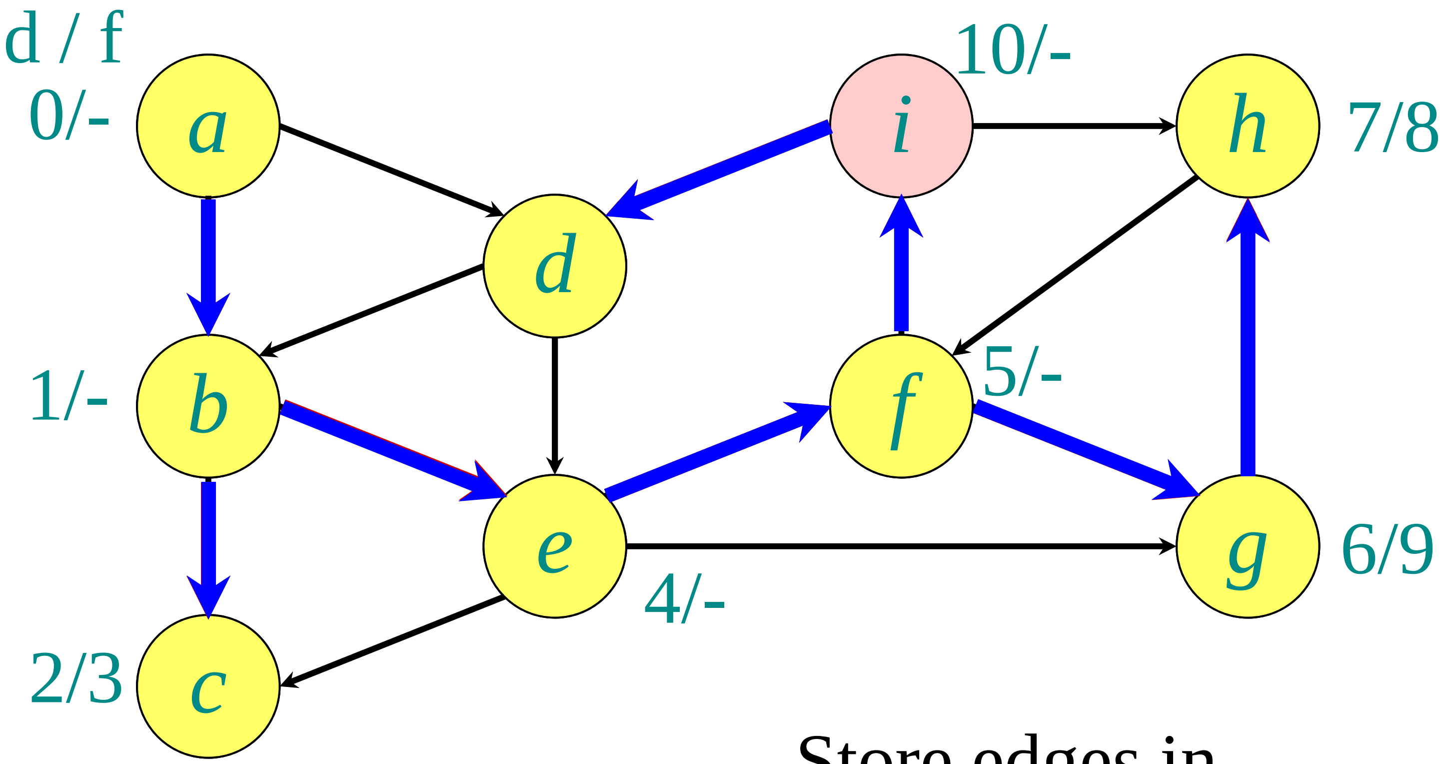


Store edges in
predecessor array

π:	a	b	c	d	e	f	g	h	i
	-	a	b		b	e	f	g	f

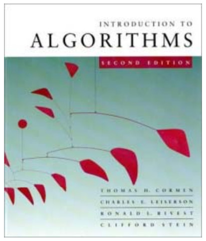


Example of depth-first search



π:	a	b	c	d	e	f	g	h	i
	-	a	b	i	b	e	f	g	f

Store edges in predecessor array



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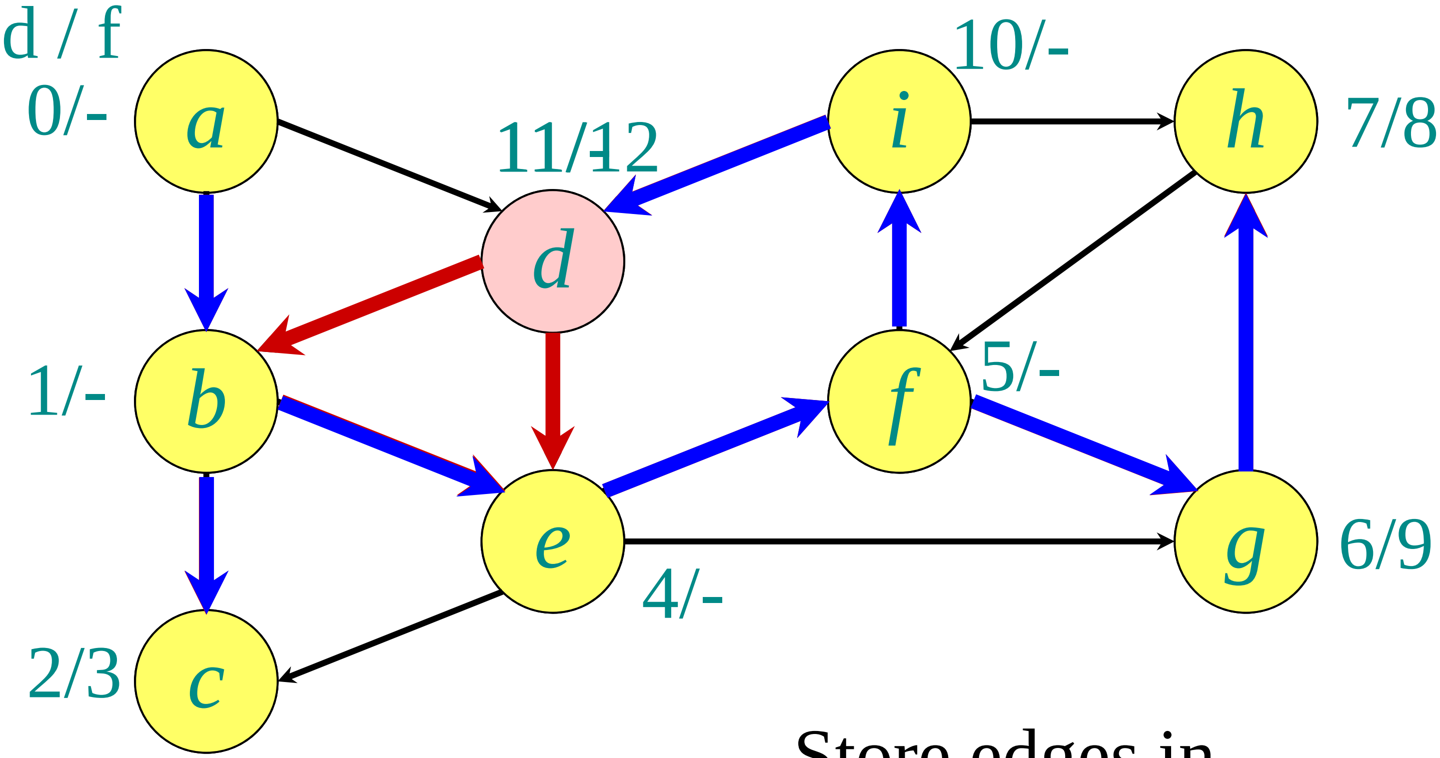
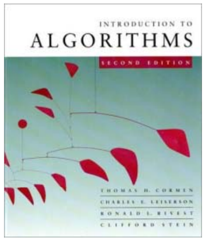


π:	a	b	c	d	e	f	g	h	i
	-	a	b	i	b	e	f	g	f

Store edges in
predecessor array



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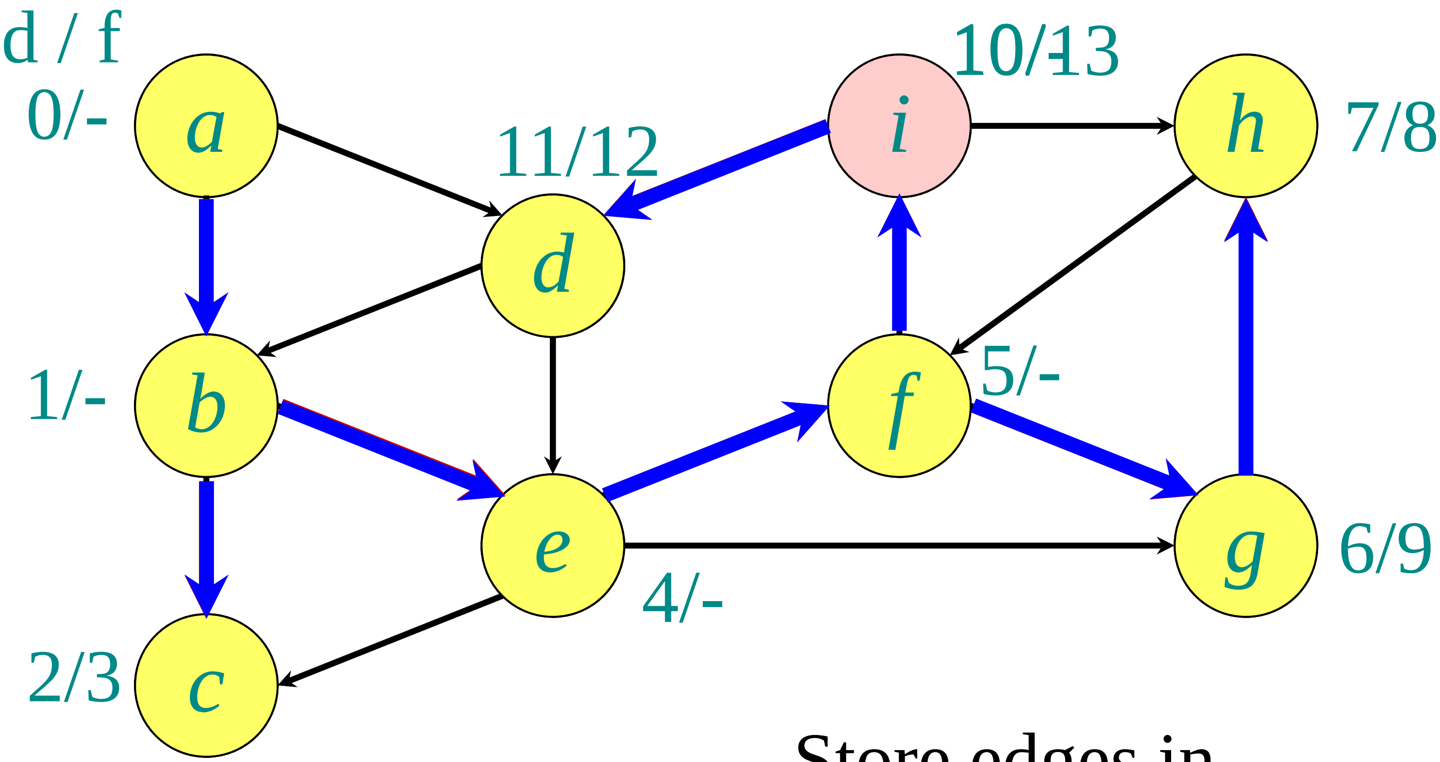

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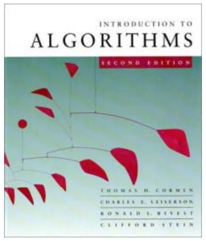

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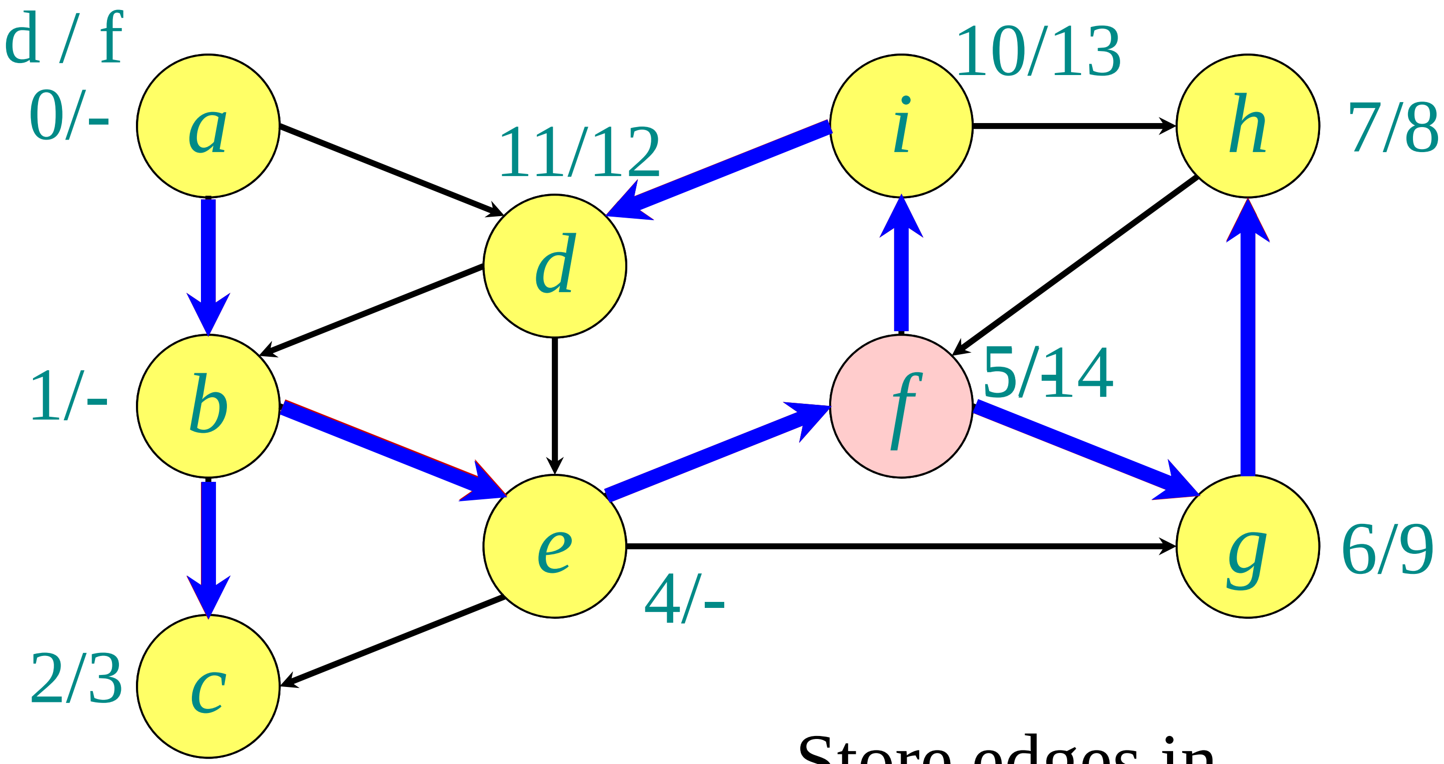

π:	a	b	c	d	e	f	g	h	i
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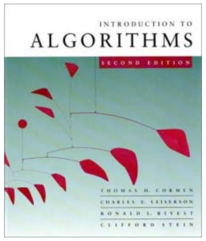
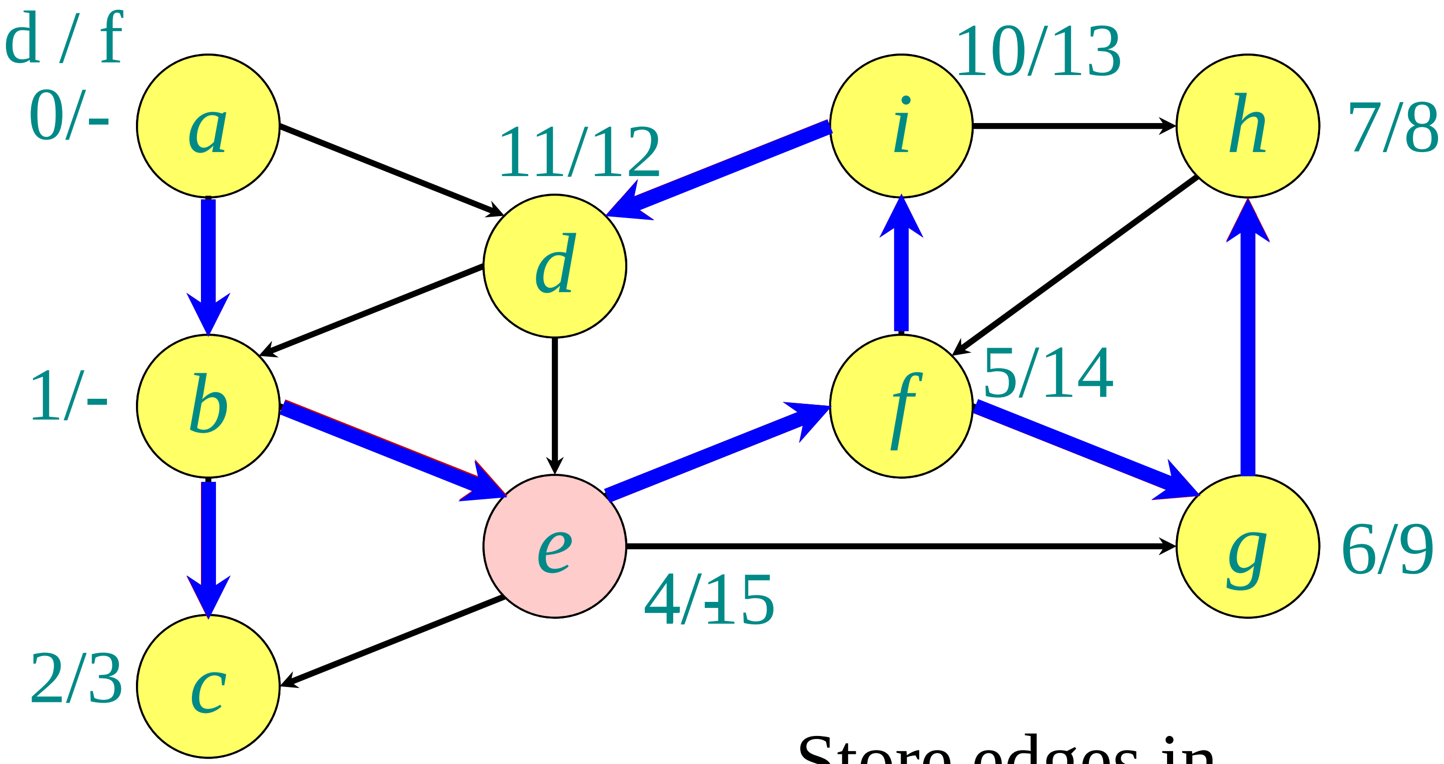
predecessor array



Example of depth-first search



π:	a	b	c	d	e	f	g	h	i
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Store edges in
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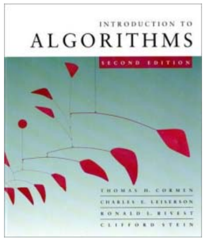
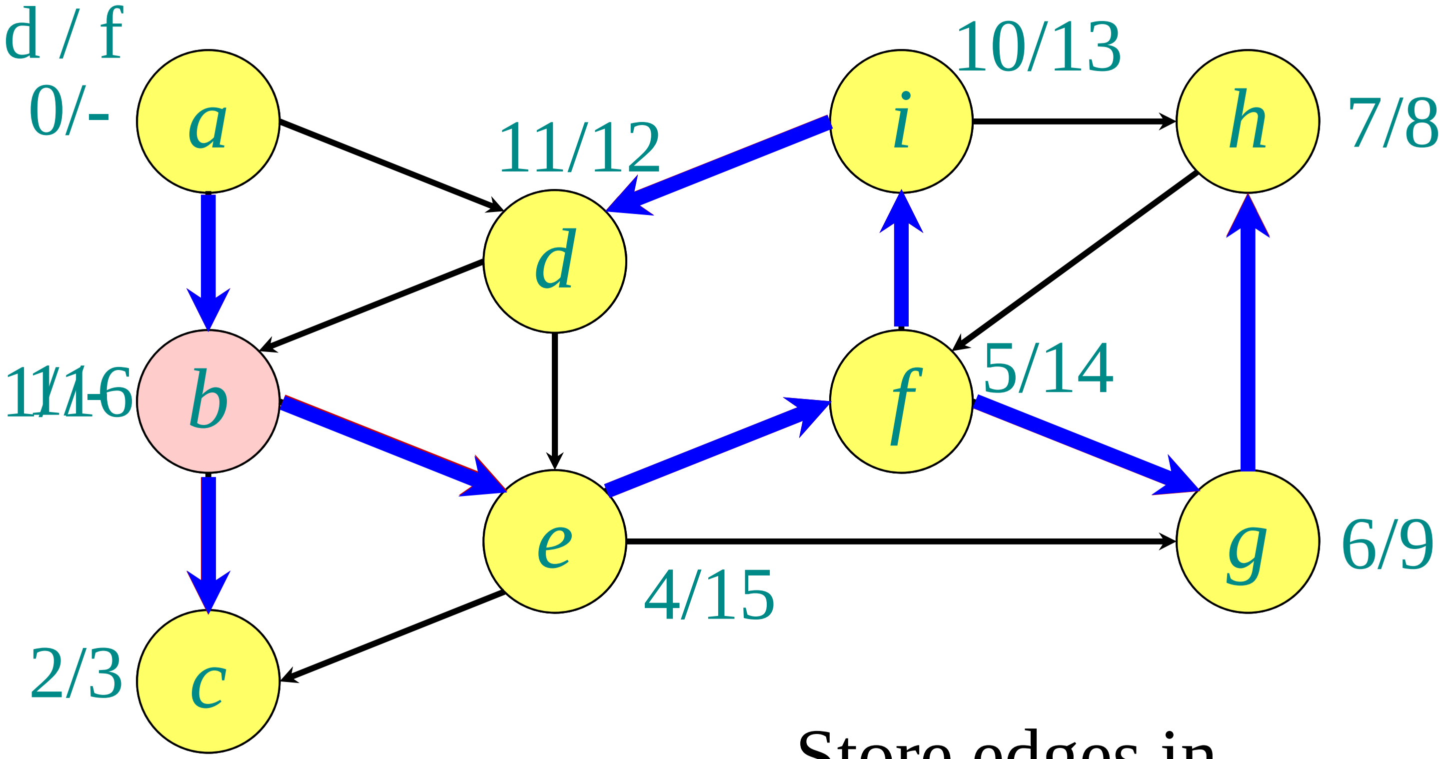
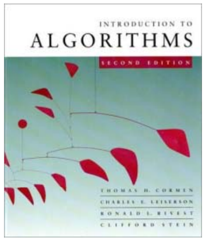


Example of depth-first search

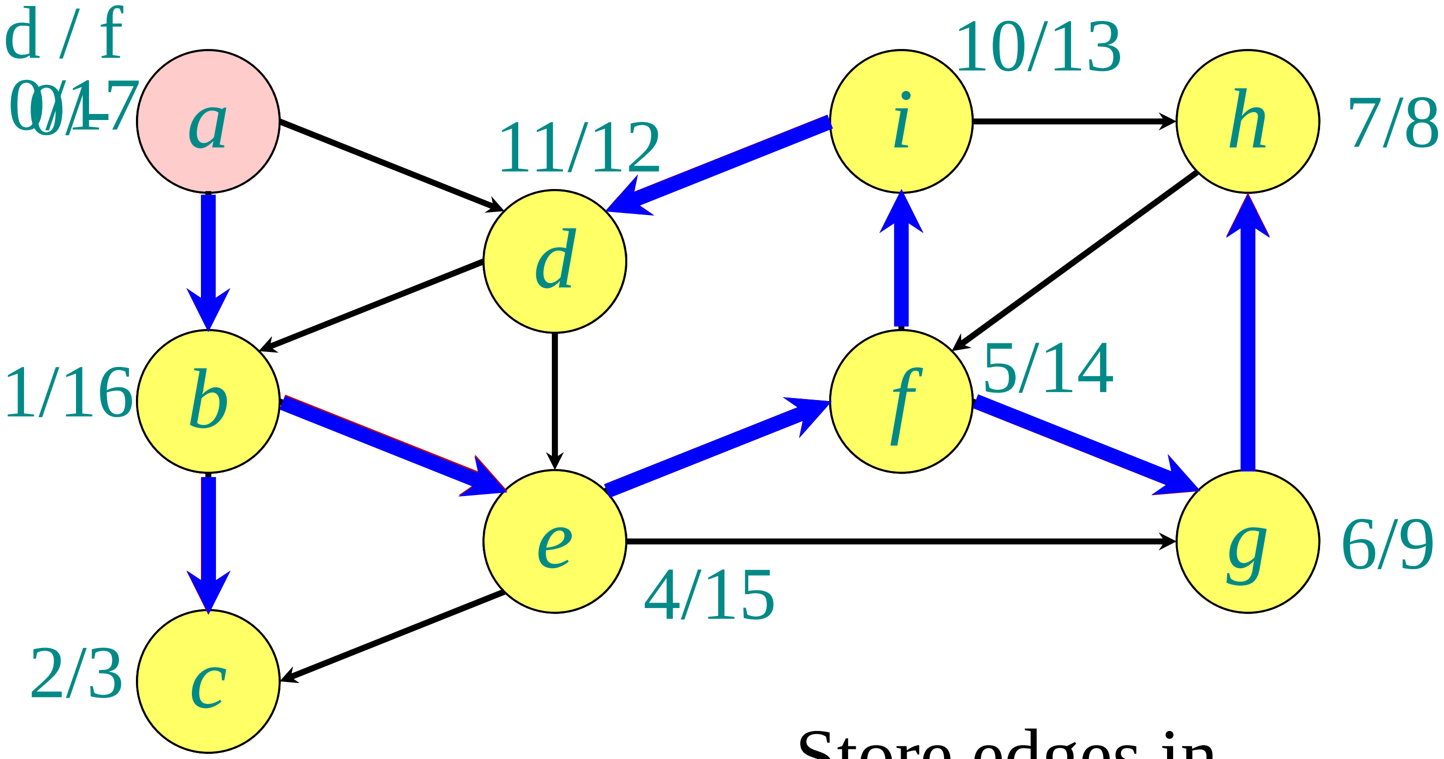


Store edges in predecessor array

π:	a	b	c	d	e	f	g	h	i
	-	a	b	i	b	e	f	g	f



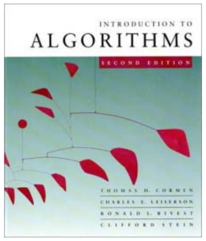
Example of depth-first search



Store edges in predecessor array

π: a b c d e f g h i

\- a b i b e f g f



Depth-First Search (DFS)

$O(n)$

$O(n)$ without DFS_rec

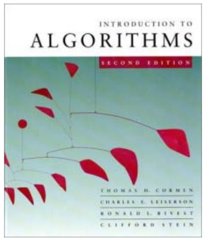
```
DFS($G=(V,E)$)
    Mark all vertices in $G$ as "unvisited"  // time=0
    for each vertex $v \in V$ do
        if $v$ is unvisited
            DFS_rec($G,v$)
```

$O(1)$

$O(deg(v))$ without recursive call

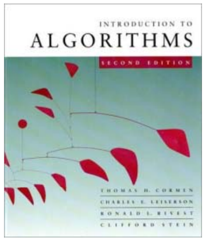
```
DFS_rec($G, v$)
    mark $v$ as "visited" // $d[v]$=++time
    for each $w$ adjacent to $v$ do
        if $w$ is unvisited
            Add edge $(v,w)$ to tree $T$
            DFS_rec($G,w$)
    mark $v$ as "finished" // $f[v]$=++time
```

⇒ With Handshaking Lemma, all recursive calls are $O(m)$, for a total of $O(n + m)$ runtime

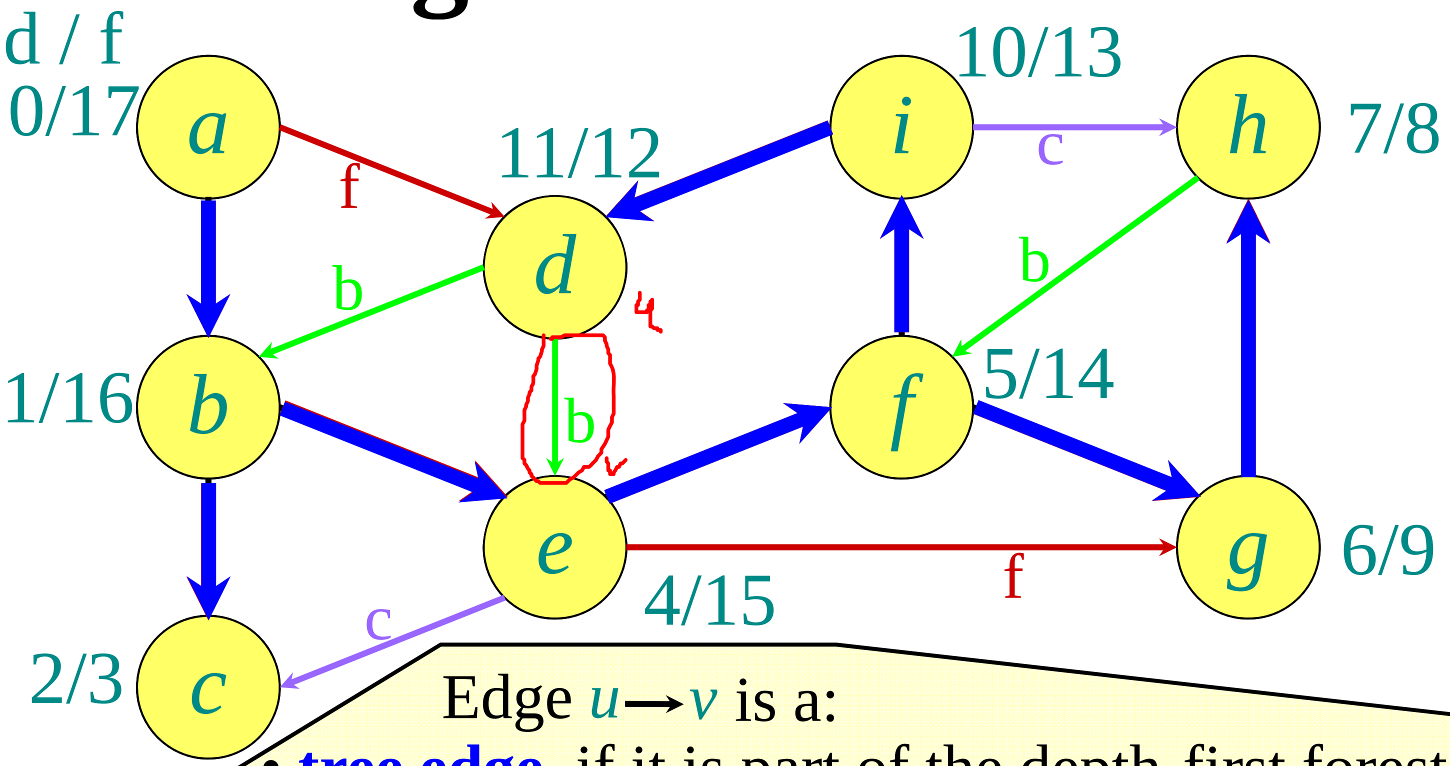


DFS runtime

- Each vertex is visited at most once ⇒ O(n) time
- The body of the **for** loops (except the recursive call) take constant time per graph edge
- All **for** loops take O(m) time
- Total runtime is O(n+m) = O(|V| + |E|)

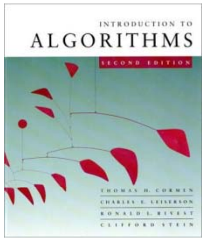


DFS edge classification

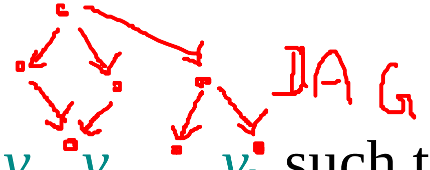


Edge $u \rightarrow v$ is a:

- **tree edge**, if it is part of the depth-first forest.
- back edge, if *u* connects to an ancestor *v* in a depth-first tree. It holds $d(u)>d(v)$ and $f(u)<f(v)$.
- forward edge, if it connects *u* to a descendant *v* in a depth-first tree. It holds $d(u)<d(v)$.
- cross edge, if it is any other edge. It holds $d(u)>d(v)$ and $f(u)>f(v)$.

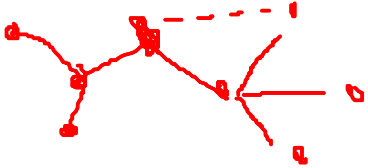


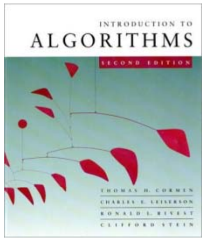
Paths, Cycles, Connectivity



Let $G=(V,E)$ be a directed (or undirected) graph

- A **path** from v_1 to v_k in G is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{\{i+1\}}) \in E$ (or $\{v_i, v_{\{i+1\}}\} \in E$ if G is undirected) for all $i \in \{1, \ldots, k-1\}$.
- A path is **simple** if all vertices in the path are distinct.
- A path $v_1, v_2, \ldots, v_k$ forms a **cycle** if $v_1 = v_k$.
- A graph with no cycles is **acyclic**.
  - An undirected acyclic graph is called a **tree**. (Trees do not have to have a root vertex specified.)
  - A directed acyclic graph is a **DAG**. (A DAG can have undirected cycles if the direction of the edges is not considered.)
- An undirected graph is **connected** if every pair of vertices is connected by a path. A directed graph is **strongly connected** if for every pair $u, v \in V$ there is a path from u to v and there is a path from v to u.
- The **(strongly) connected components** of a graph are the equivalence classes of vertices under this reachability relation.





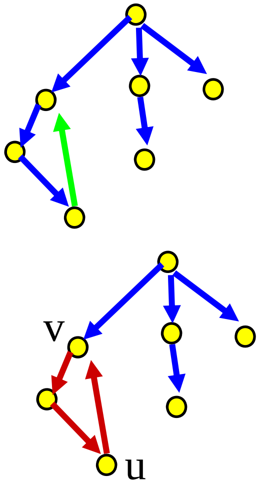
DAG Theorem

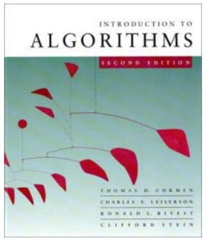
Theorem: A directed graph G is acyclic
⇔ a depth-first search of G yields no back edges.

Proof:

"⇒": Suppose there is a back edge (u,v). Then by definition of a back edge there would be a cycle.

"⇐": Suppose G contains a cycle c. Let v be the first vertex to be discovered in c, and let u be the preceding vertex in c. v is an ancestor of u in the depth-first forest, hence (u,v) is a back edge.

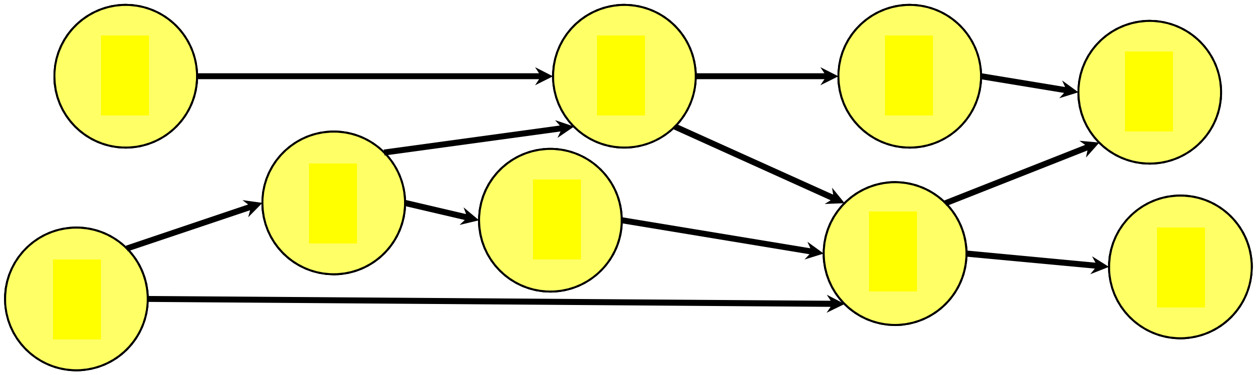


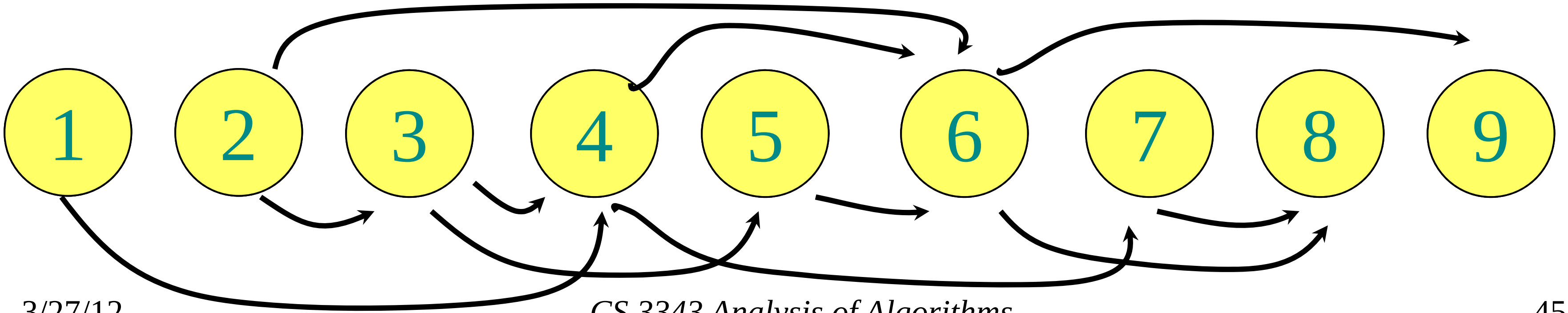


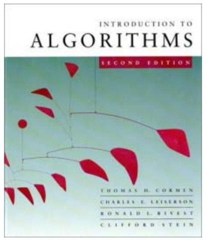
Topological Sort

Topologically sort the vertices of a ***directed acyclic graph*** (***DAG***):

- Determine $f : V \to \{1, 2, \ldots, |V|\}$ such that $(u, v) \in E$ $\Rightarrow f(u) < f(v)$.

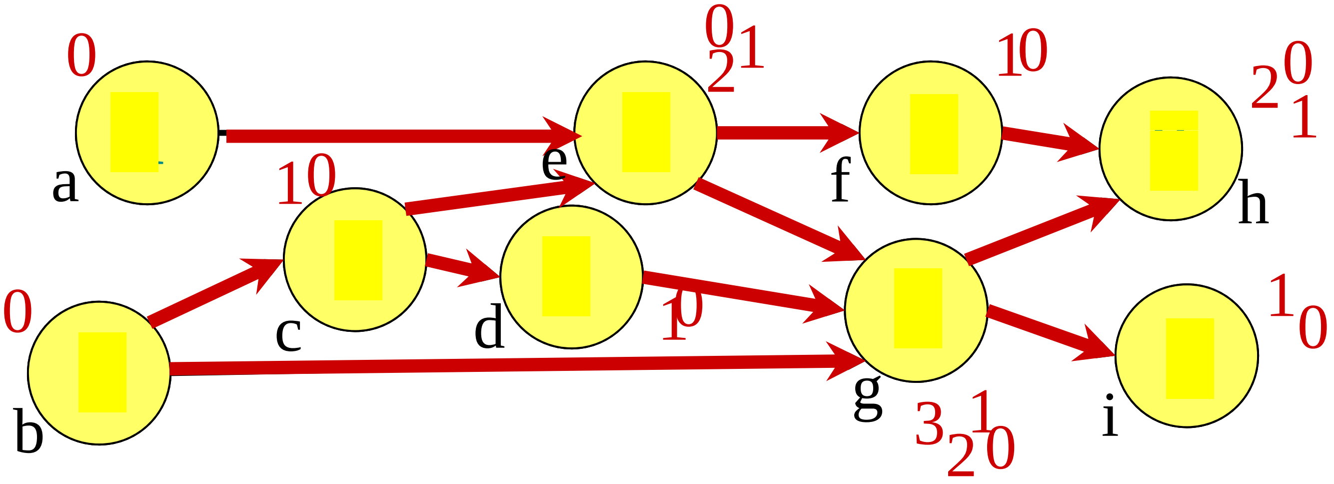




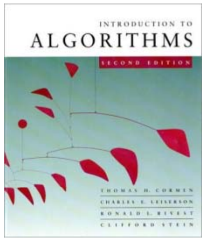


Topological Sort Algorithm

- Store vertices with in-degree 0 in a queue Q.
- While Q is not empty
  - Dequeue vertex v, and give it the next number
  - Decrease in-degree of all adjacent vertices by 1
  - Enqueue all vertices with in-degree 0



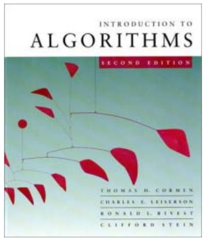
Q: a , b , c, e, d, f, g, i , h



Topological Sort Runtime

Runtime:

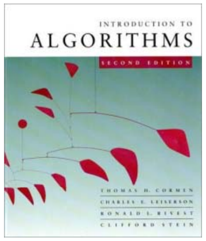
- O(|V|+|E|) because every edge is touched once, and every vertex is enqueued and dequeued exactly once



DFS-Based Topological Sort Algorithm

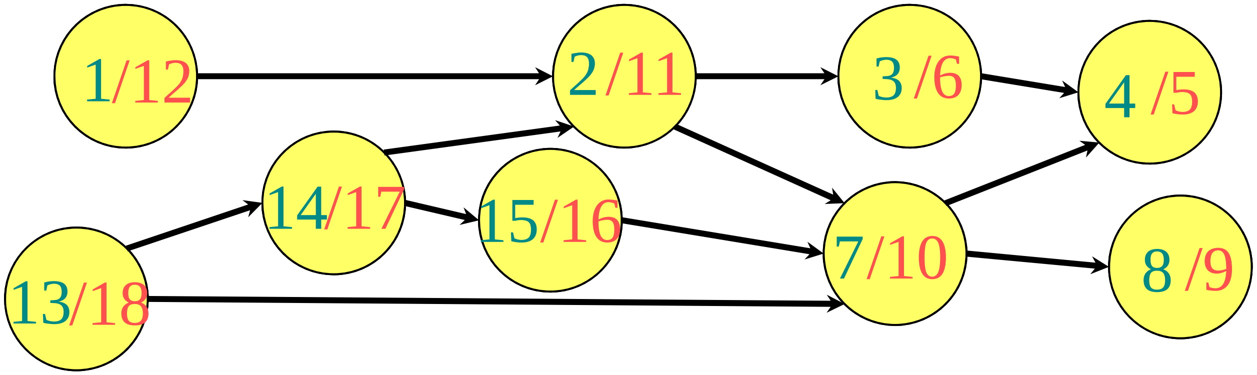
- Call DFS on the directed acyclic graph $G=(V,E)$
  $\Rightarrow$ Finish time for every vertex
- Reverse the finish times (highest finish time becomes the lowest finish time,…)
  $\Rightarrow$ Valid function $f\,'\colon V \rightarrow \{1, 2, \ldots, |\,V\,|\}$ such that
  $(u, v) \in E \Rightarrow f\,'(u) < f\,'(v)$

Runtime: $O(|V|+|E|)$

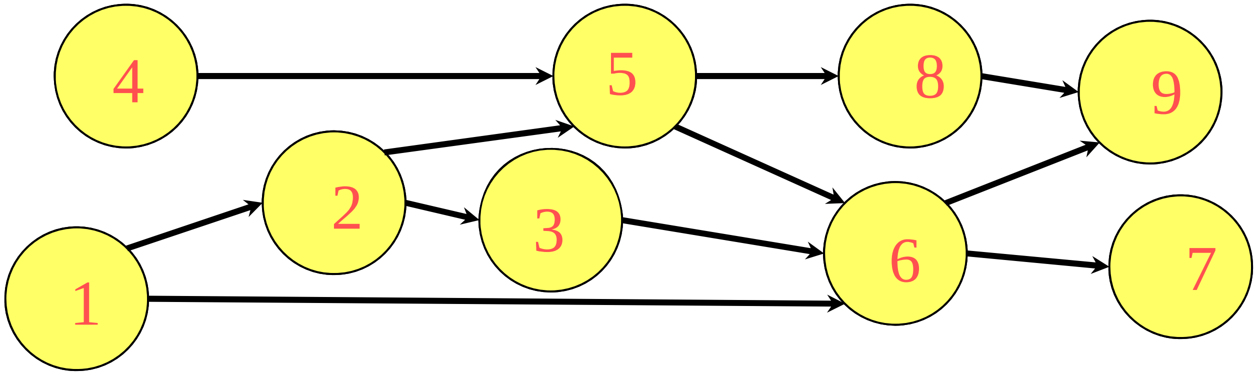


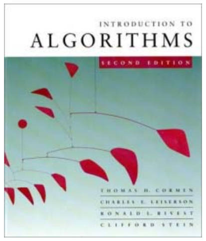
DFS-Based Topological Sort

- Run DFS:



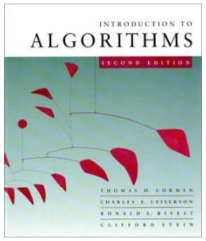
- Reverse finish times:





DFS-Based Top. Sort Correctness

- Need to show that for any $(u, v) \in E$ holds $f(v) < f(u)$. (since we consider reversed finish times)
- Consider exploring edge (u, v) in DFS:
  - v cannot be visited and unfinished (and hence an ancestor in the depth first tree), since then (u,v) would be a back edge (which by the DAG lemma cannot happen).
  - If v has not been visited yet, it becomes a descendant of u, and hence $f(v)<f(u)$. (tree edge)
  - If v has been finished, $f(v)$ has been set, and u is still being explored, hence $f(u)>f(v)$ (forward edge, cross edge) .



Topological Sort Runtime

Runtime:

- O(|V|+|E|) because every edge is touched once, and every vertex is enqueued and dequeued exactly once

- DFS-based algorithm: O(|V| + |E|)