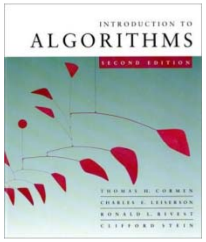
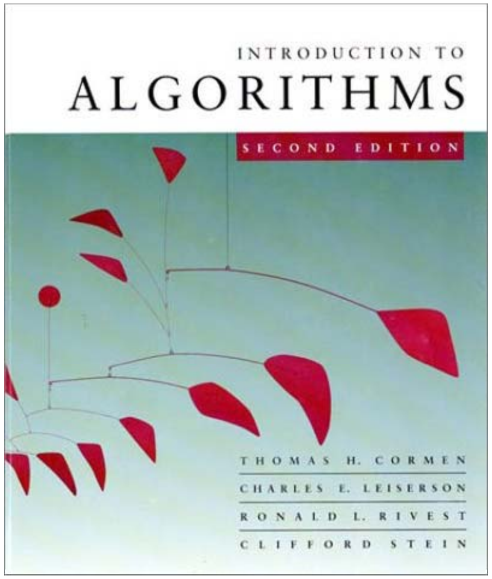


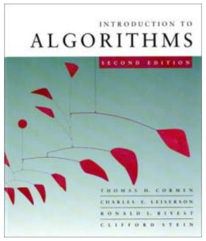
CS 5633 – Spring 2012



More on Shortest Paths

Carola Wenk

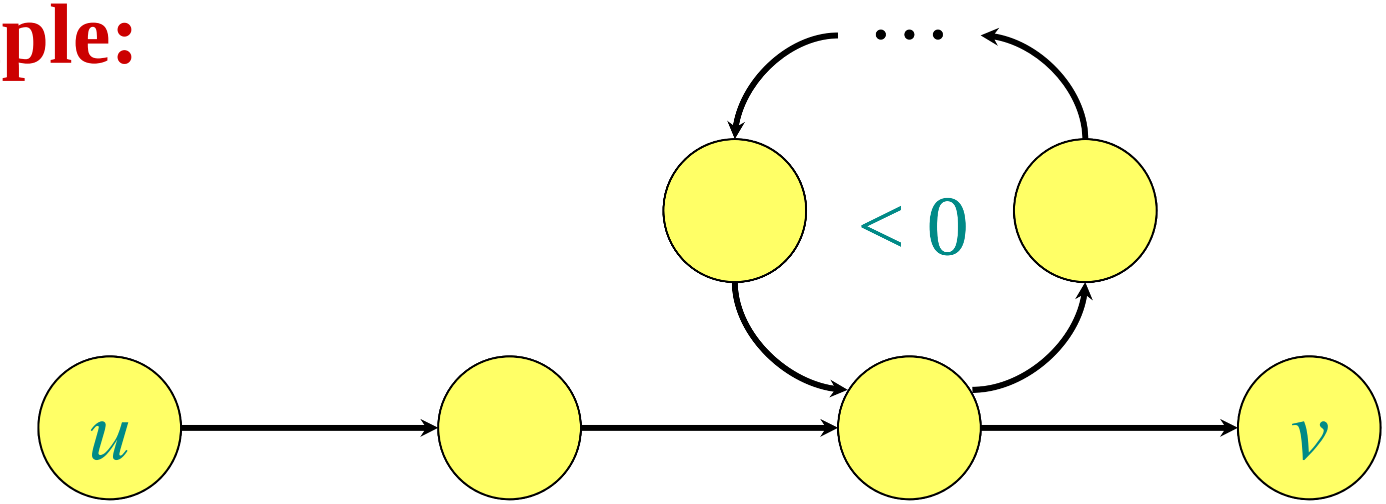
Slides courtesy of Charles Leiserson with small changes by Carola Wenk



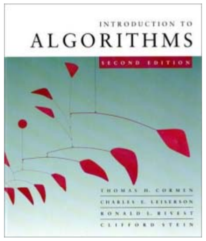
Negative-weight cycles

Recall: If a graph $G = (V, E)$ contains a negative-weight cycle, then some shortest paths may not exist.

Example:



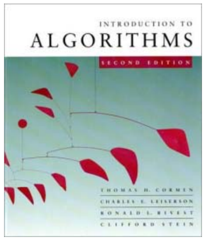
Bellman-Ford algorithm: Finds all shortest-path weights from a ***source*** $s \in V$ to all $v \in V$ or determines that a negative-weight cycle exists.



Bellman-Ford algorithm

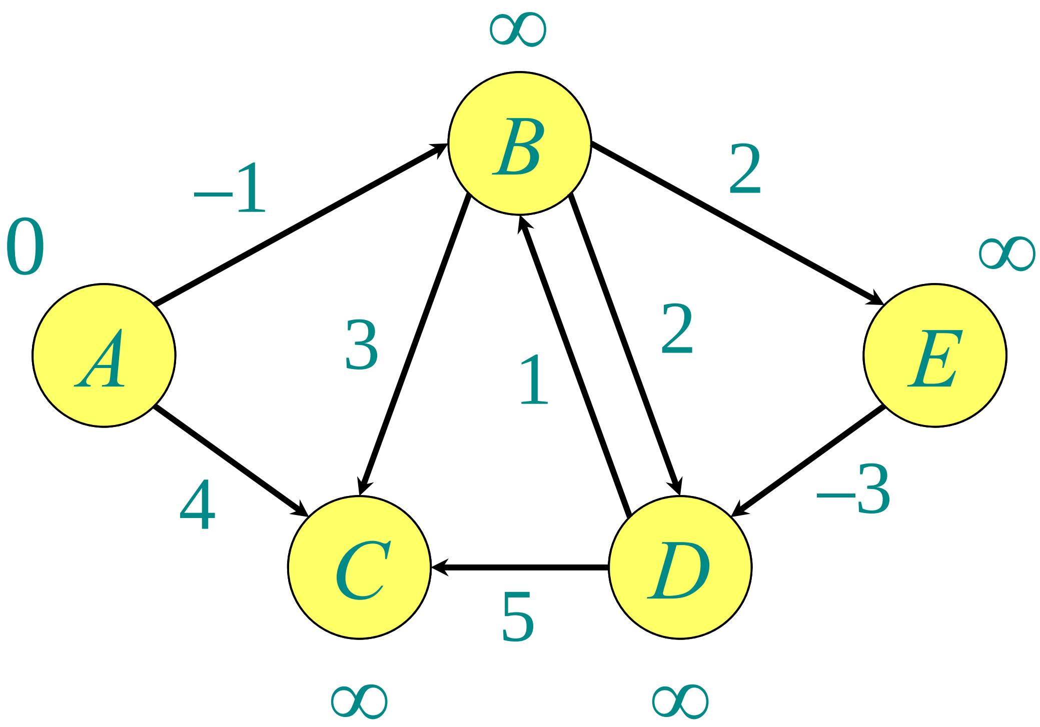
```
$d[s] \leftarrow 0$
for each $v \in V - \{s\}$                          } initialization
    do $d[v] \leftarrow \infty$

for $i \leftarrow 1$ to $|V| - 1$ do
  for each edge $(u, v) \in E$ do
    if $d[v] > d[u] + w(u, v)$ then
          $d[v] \leftarrow d[u] + w(u, v)$          } relaxation step
          $\pi[v] \leftarrow u$
for each edge $(u, v) \in E$
    do if $d[v] > d[u] + w(u, v)$
          then report that a negative-weight cycle exists
```

At the end, $d[v] = \delta(s, v)$. Time = $O(|V||E|)$.

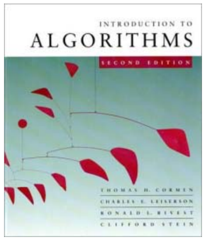


Example of Bellman-Ford

Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*

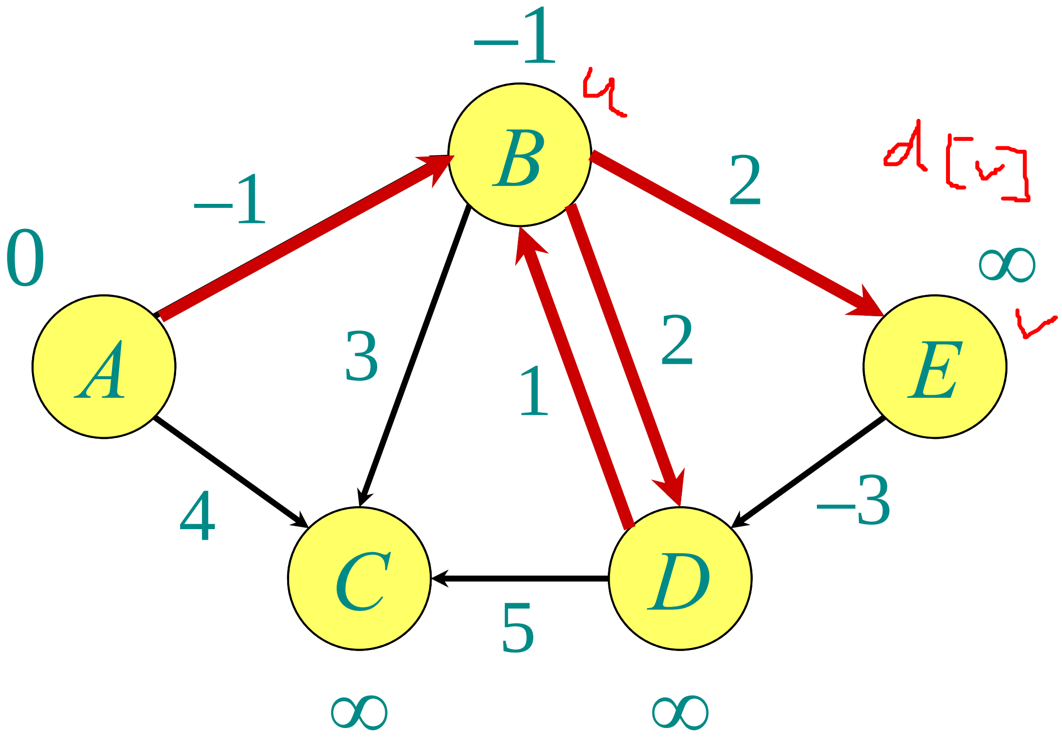


A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞

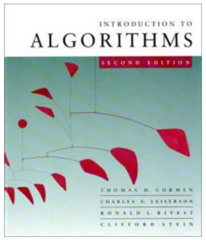


Example of Bellman-Ford

Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*

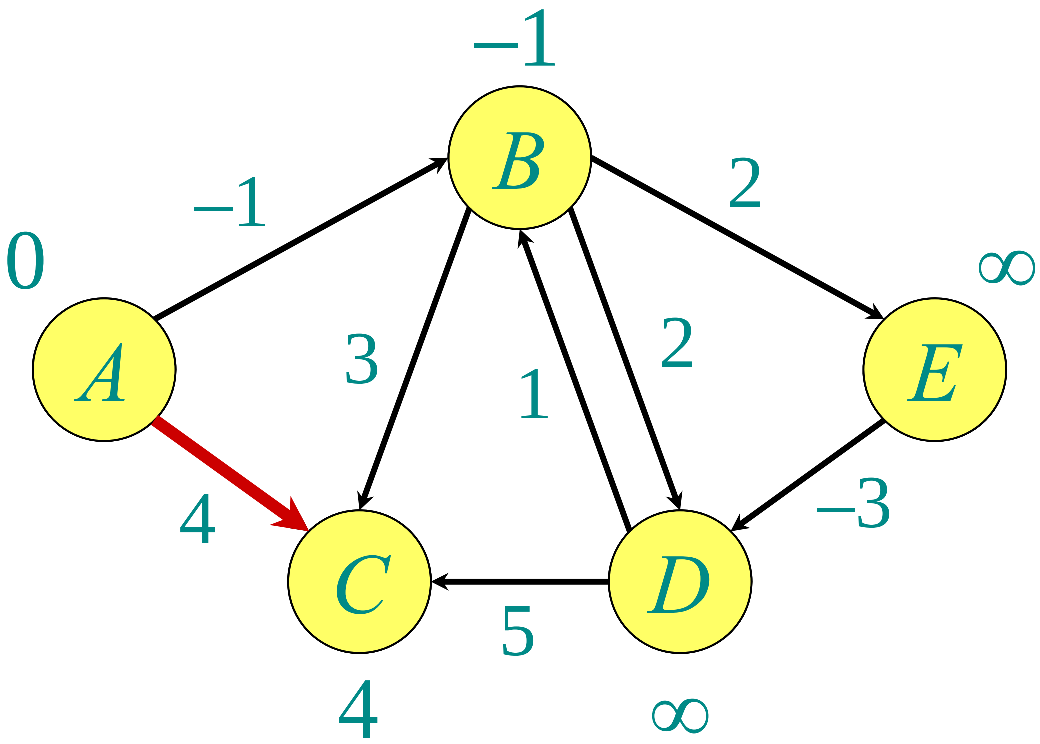


A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞

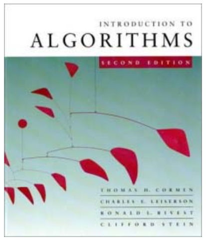


Example of Bellman-Ford

Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*

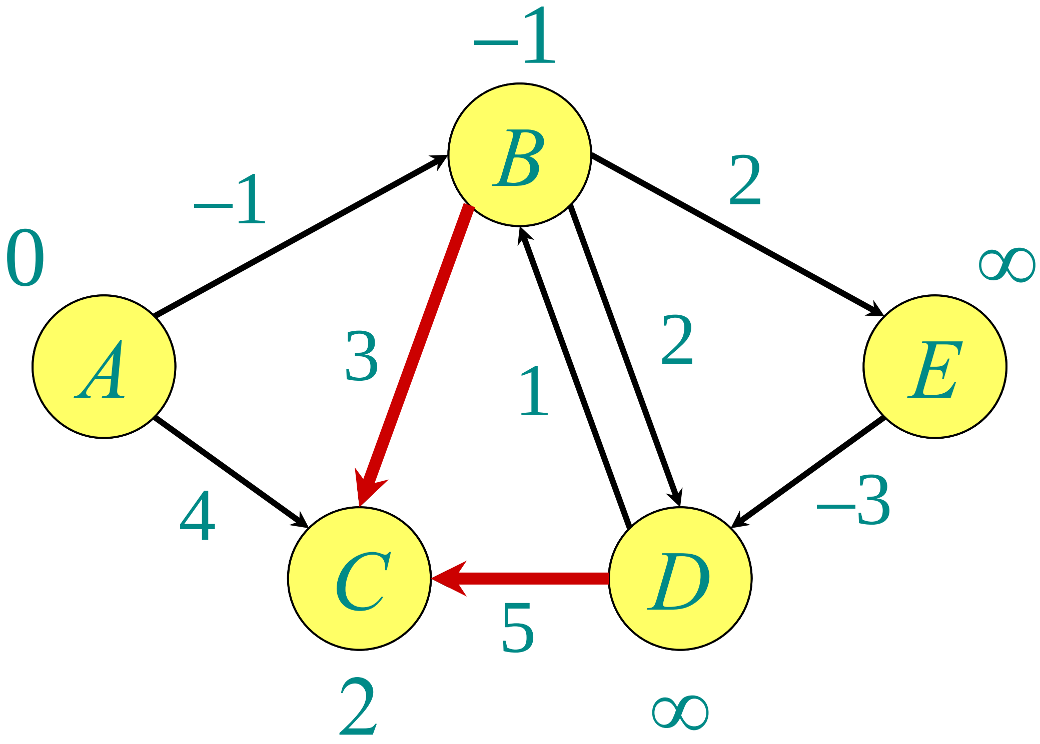


A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞

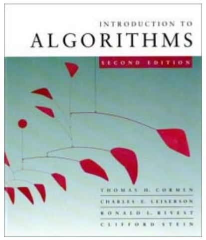


Example of Bellman-Ford

Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*

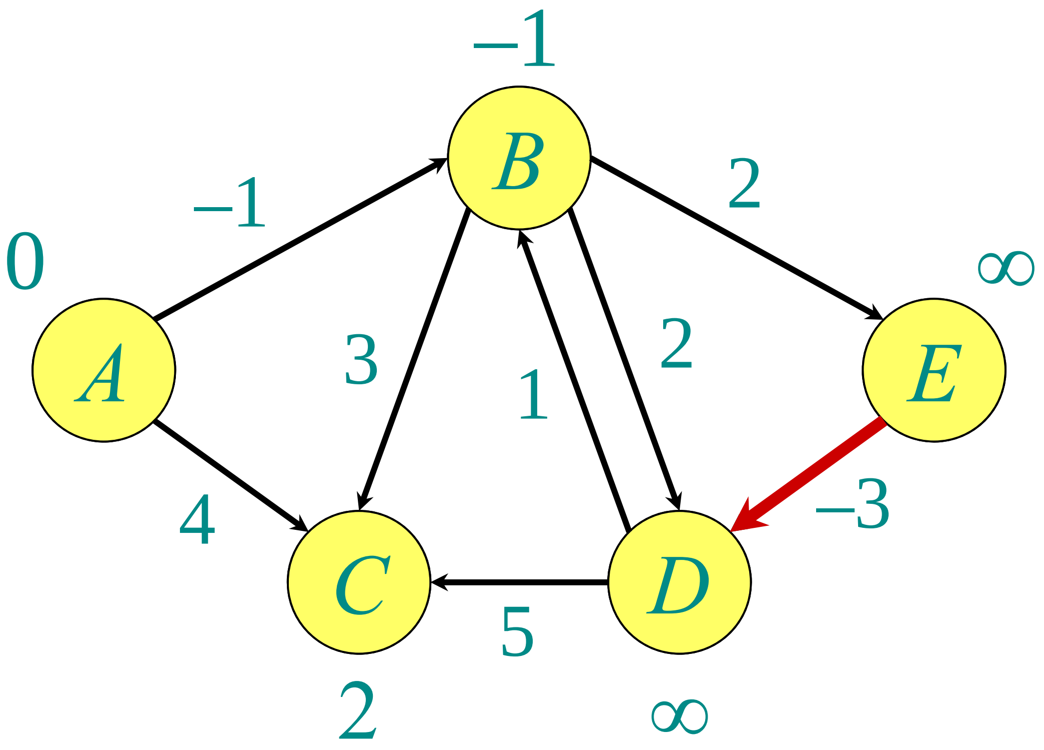


A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞

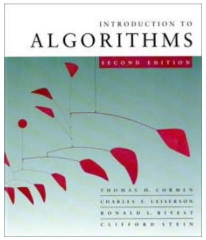


Example of Bellman-Ford

Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*

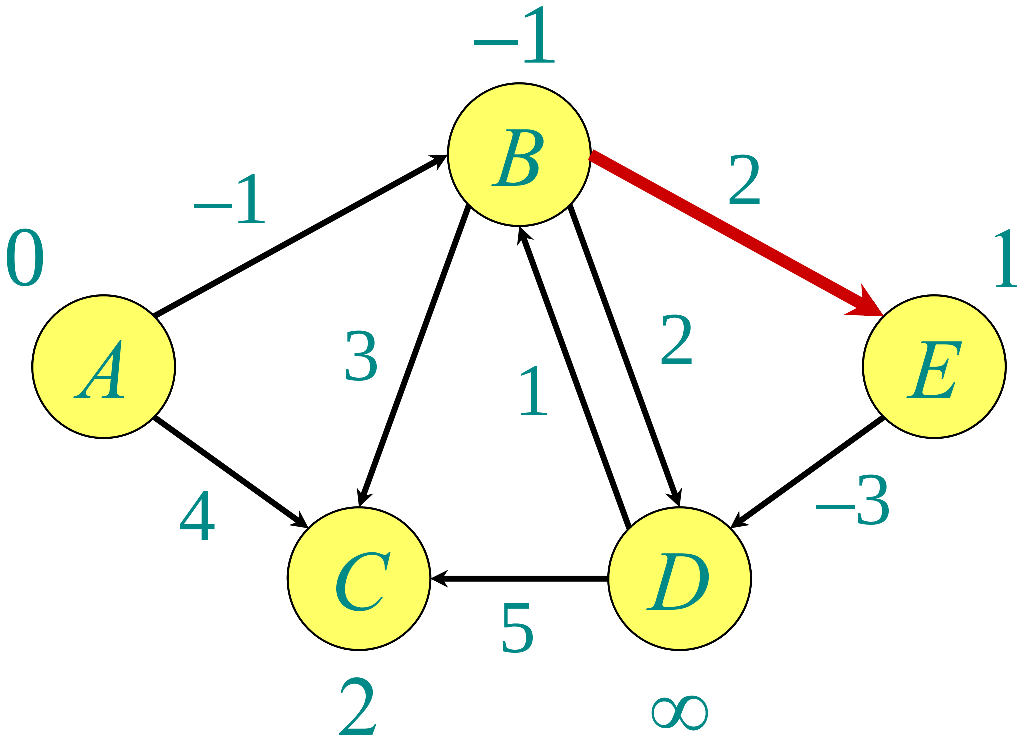


A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞

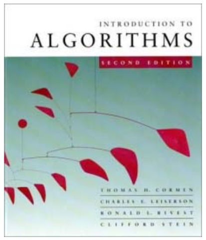


Example of Bellman-Ford

Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*

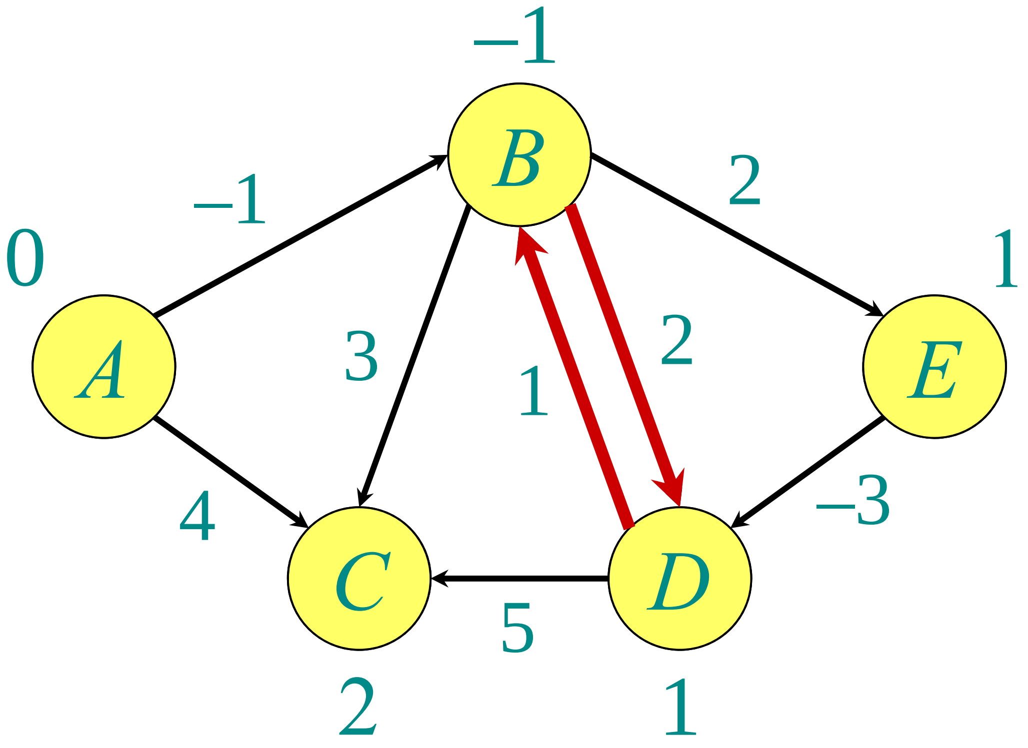


A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞
0	–1	2	∞	1

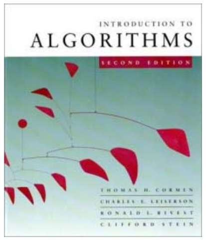


Example of Bellman-Ford

Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*

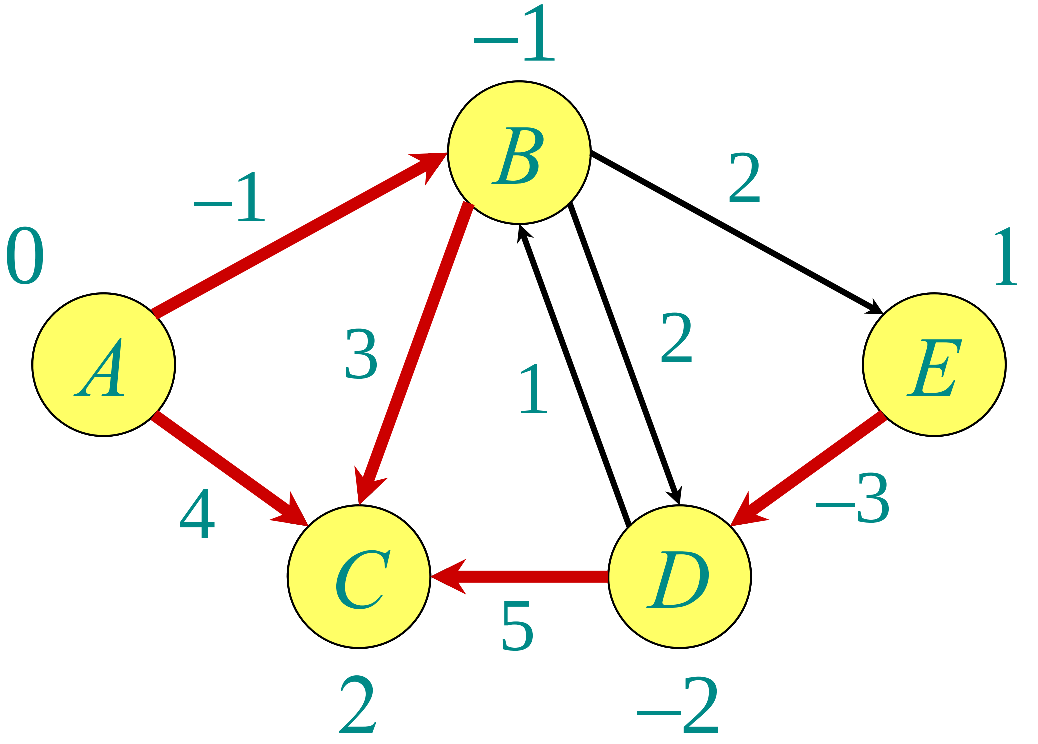


A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞
0	–1	2	∞	1
0	–1	2	1	1

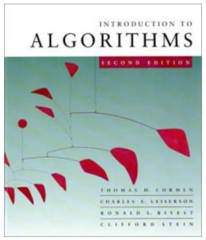


Example of Bellman-Ford

Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*

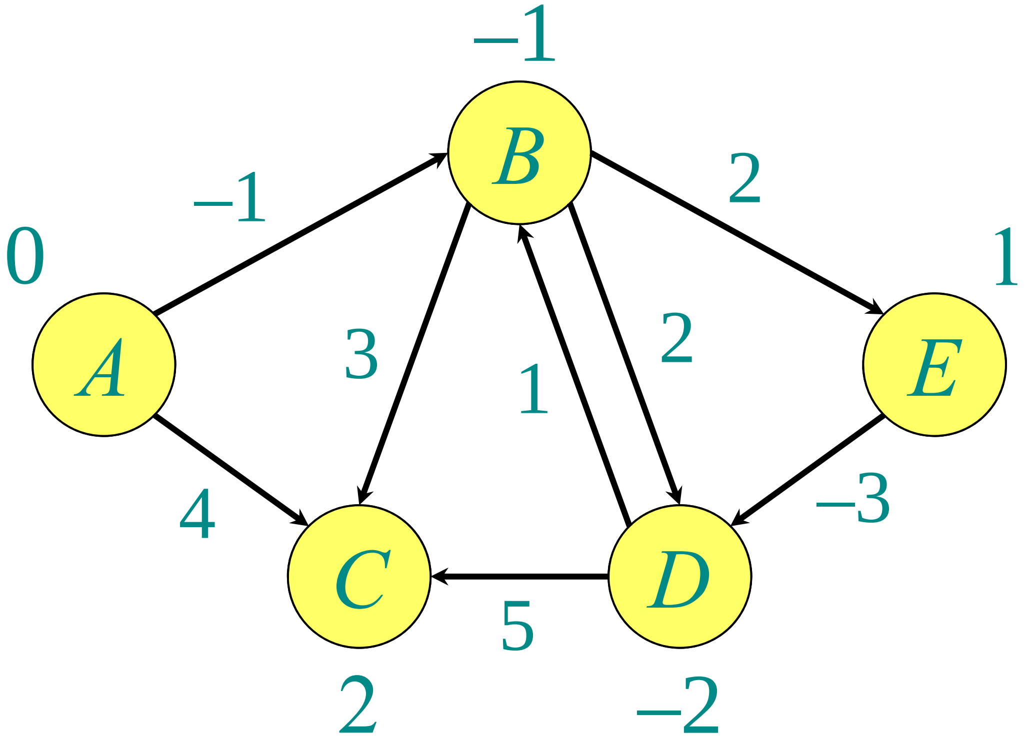


A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞
0	–1	2	∞	1
0	–1	2	1	1
0	–1	2	–2	1



Example of Bellman-Ford

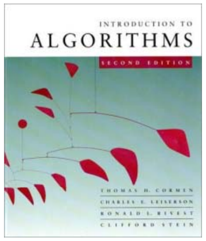
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*



A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞
0	–1	2	∞	1
0	–1	2	1	1
0	–1	2	–2	1

Note: Values decrease monotonically.

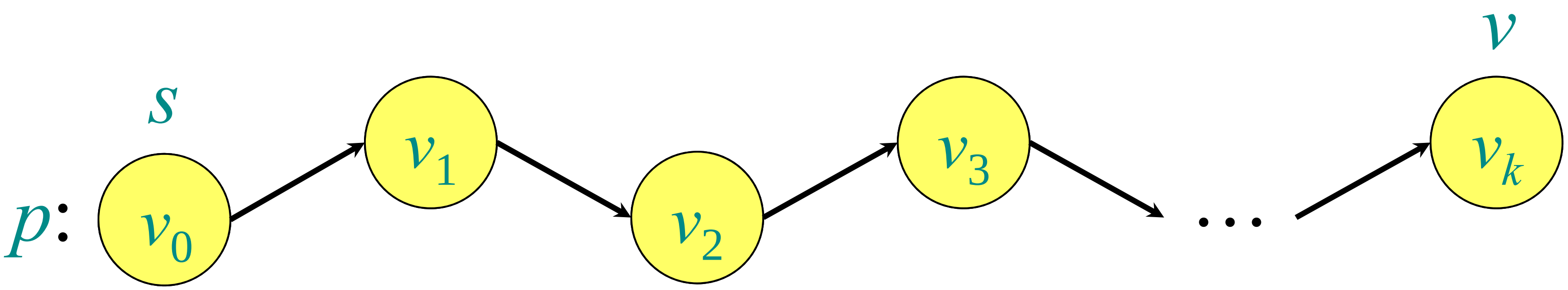
… and 2 more iterations



Correctness

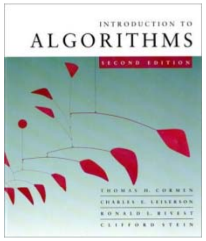
Theorem. If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.

Proof. Let $v \in V$ be any vertex, and consider a shortest path p from s to v with the minimum number of edges.

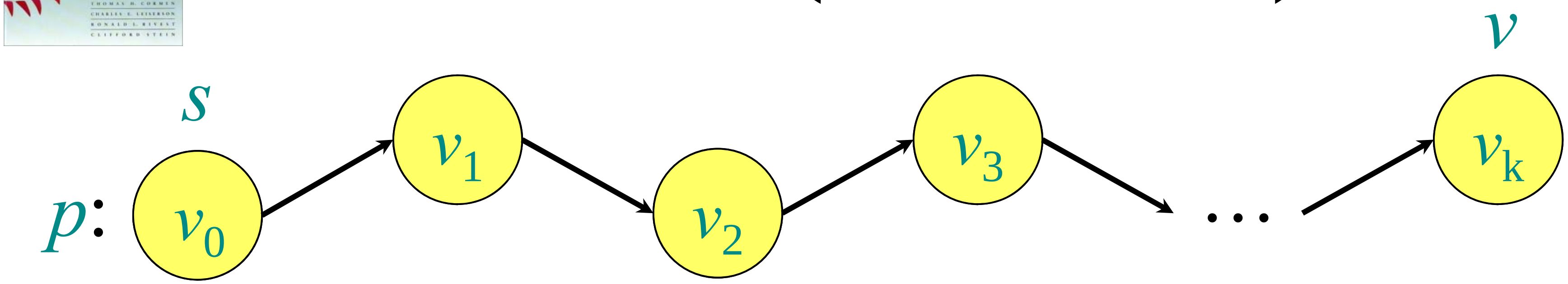


Since p is a shortest path, we have

$$\delta(s, v_i) = \delta(s, v_{i-1}) + w(v_{i-1}, v_i) \ .$$



Correctness (continued)

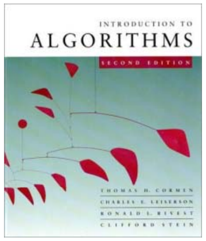


Initially, $d[v_0] = 0 = \delta(s, v_0)$, and $d[s]$ is unchanged by subsequent relaxations.

- After 1 pass through E, we have $d[v_1] = \delta(s, v_1)$.
- After 2 passes through E, we have $d[v_2] = \delta(s, v_2)$.

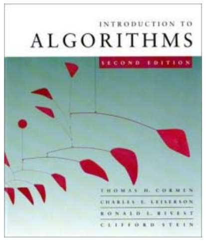
  …
- After k passes through E, we have $d[v_k] = \delta(s, v_k)$.

Since G contains no negative-weight cycles, p is simple. Longest simple path has $\leq |V| - 1$ edges. □



Detection of negative-weight cycles

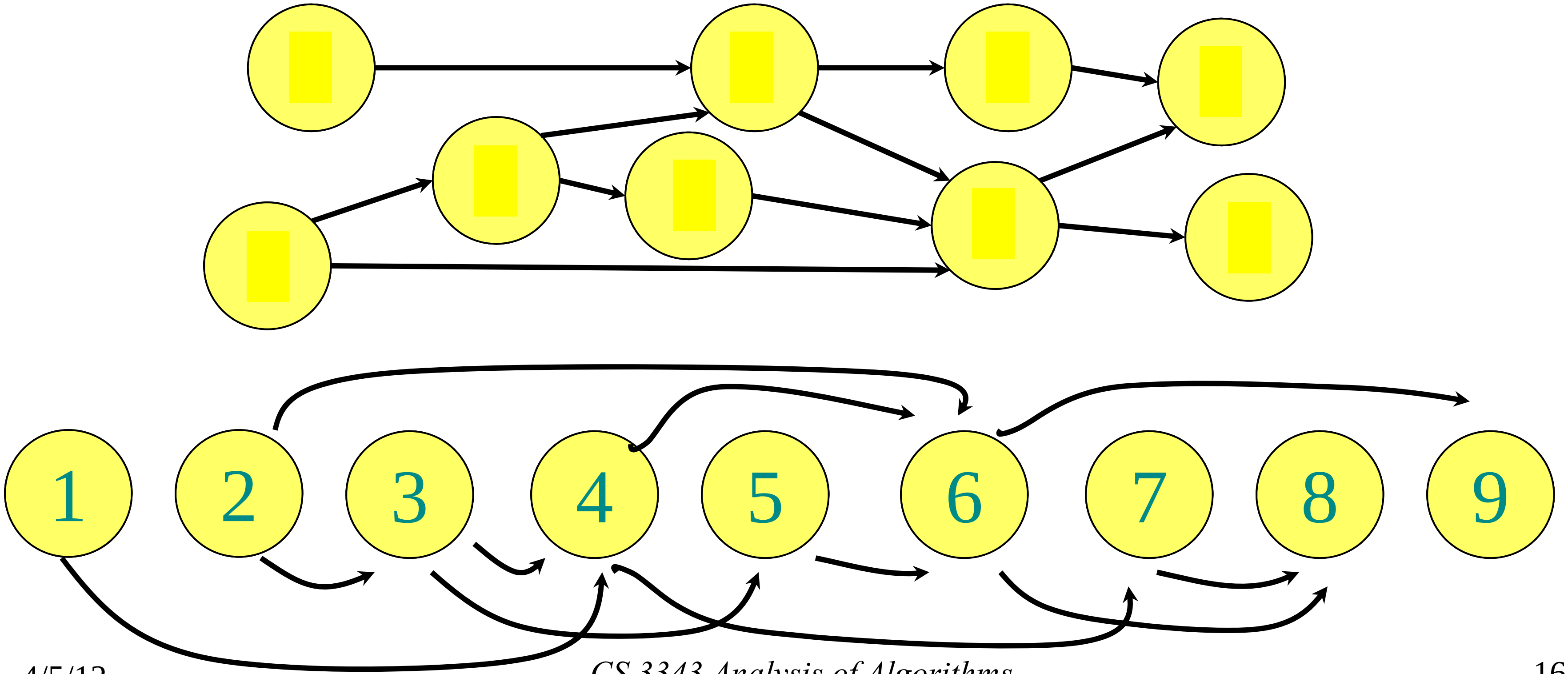
Corollary. If a value $d[v]$ fails to converge after $|V| - 1$ passes, there exists a negative-weight cycle in G reachable from s. □

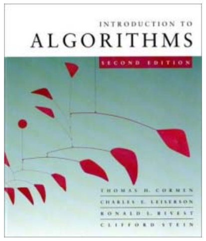


DAG shortest paths

If the graph is a ***directed acyclic graph*** (***DAG***), we first ***topologically sort*** the vertices.

- Determine $f : V \rightarrow \{1, 2, \ldots, |V|\}$ such that $(u, v) \in E \Rightarrow f(u) < f(v)$.

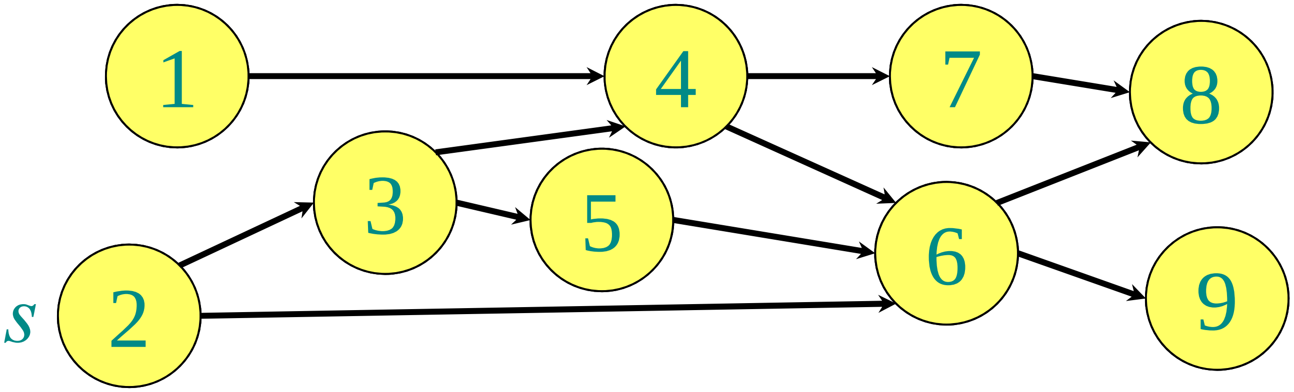




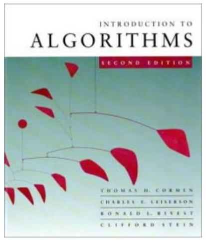
DAG shortest paths

If the graph is a ***directed acyclic graph*** (***DAG***), we first ***topologically sort*** the vertices.

- Determine $f : V \to \{1, 2, \ldots, |V|\}$ such that $(u, v) \in E$ $\Rightarrow f(u) < f(v)$.
- $O(|V| + |E|)$ time



- Walk through the vertices $u \in V$ in this order, relaxing the edges in $Adj[u]$, thereby obtaining the shortest paths from s in a total of $O(|V| + |E|)$ time.



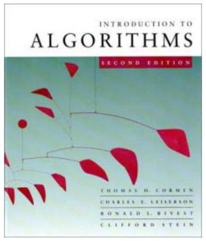
Shortest paths

Single-source shortest paths

- Nonnegative edge weights
    - Dijkstra’s algorithm: $O(|E| \log |V|)$
- General: Bellman-Ford: $O(|V||E|)$
- DAG: One pass of Bellman-Ford: $O(|V| + |E|)$

All-pairs shortest paths

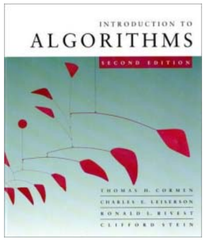
- Nonnegative edge weights
    - Dijkstra’s algorithm $|V|$ times: $O(|V||E| \log |V|)$
- General
    - Bellman-Ford $|V|$ times: $\mathrm{O}(|V|^2|E|)$
    - Floyd-Warshall: $\mathrm{O}(|V|^3)$



All-pairs shortest paths

Input: Digraph $G = (V, E)$, where $|V| = n$, with edge-weight function $w : E \to \mathbb{R}$.
Output: $n \times n$ matrix of shortest-path lengths $\delta(i, j)$ for all $i, j \in V$.

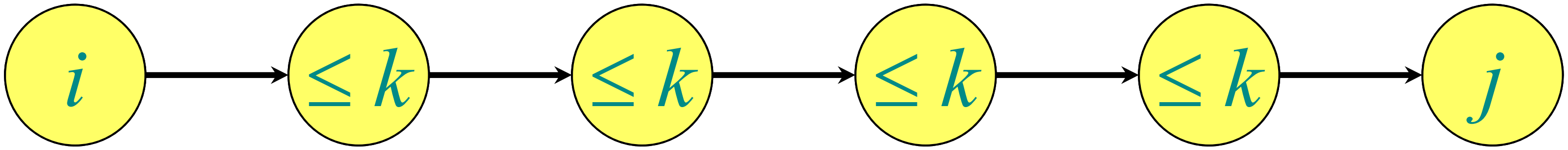
Algorithm #1:
- Run Bellman-Ford once from each vertex.
- Time $= \mathrm{O}(|V|^2|E|)$.
- But: Dense graph $\Rightarrow$ $\mathrm{O}(|V|^4)$ time.



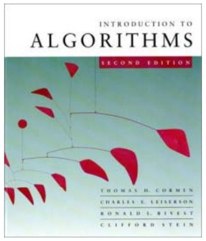
Floyd-Warshall algorithm

- Dynamic programming algorithm.
- Assume $V=\{1, 2, \ldots, n\}$, and assume G is given in an **adjacency matrix** $A=(a_{ij})_{1\le i,j\le n}$ where a_{ij} is the weight of the edge from i to j.

Define $c_{ij}^{(k)}$ = weight of a shortest path from i to j with intermediate vertices belonging to the set $\{1, 2, \ldots, k\}$.



Thus, $\delta(i, j) = c_{ij}^{(n)}$. Also, $c_{ij}^{(0)} = a_{ij}$.



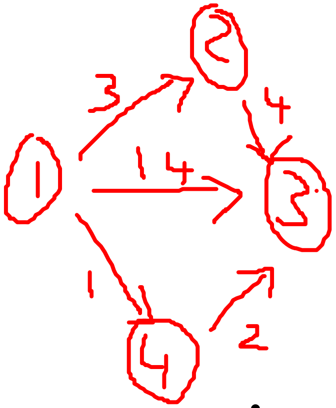
Floyd-Warshall recurrence

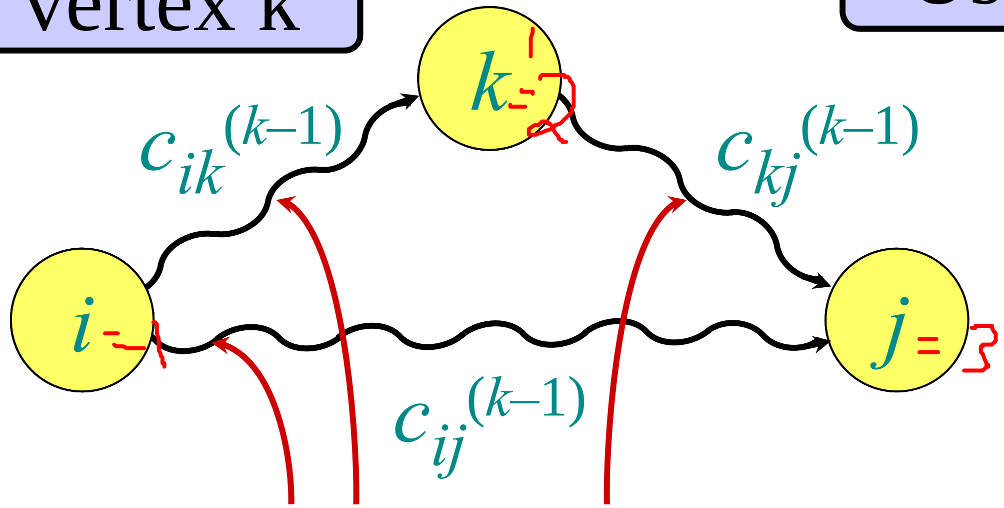
$$c_{ij}^{(k)} = \min\{c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)}\}$$

k=2

Do not use vertex k

Use vertex k





intermediate vertices in $\{1, 2, \ldots, k\text{-}1\}$

	1	2	3	4
1	0	3	14	1
2	o	0	4	o
3	o	o	o	o
4	0	2	2	o

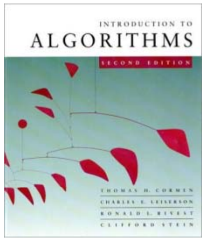
k = 0, 1

			3	
1		3	7	1
			4	
			2	

k=2, 3

			3	
1		3	3	1
			4	
			2	

k=4

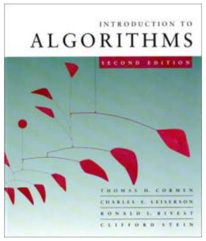


Pseudocode for Floyd-Warshall

```
for $k \leftarrow 1$ to $n$ do
  for $i \leftarrow 1$ to $n$ do
    for $j \leftarrow 1$ to $n$ do
      if $c_{ij}^{(k-1)} > c_{ik}^{(k-1)} + c_{kj}^{(k-1)}$ then
        $c_{ij}^{(k)} \leftarrow c_{ik}^{(k-1)} + c_{kj}^{(k-1)}$
      else
        $c_{ij}^{(k)} \leftarrow c_{ij}^{(k-1)}$
```

relaxation

- Runs in $\Theta(n^3)$ time and space
- Simple to code.
- Efficient in practice.



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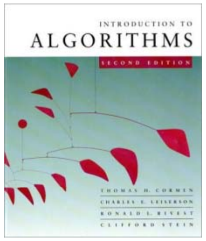
- General: Bellman-Ford: $O(|V||E|)$
- DAG: One pass of Bellman-Ford: $O(|V| + |E|)$

adj. list

All-pairs shortest paths

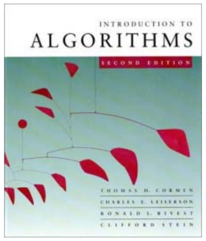
- Nonnegative edge weights — adj. list
    - Dijkstra's algorithm $|V|$ times: $O(|V||E| \log |V|)$
- General
    - Bellman-Ford $|V|$ times: $O(|V|^2|E|)$ — adj. list
    - Floyd-Warshall: $O(|V|^3)$ — adj. matrix



Johnson’s algorithm

1. Compute a weight function $\hat{w}$ from w such that $\hat{w}(u, v) \geq 0$ for all $(u, v) \in E$. (Or determine that a negative-weight cycle exists, and stop.)
   - Can be done in $O(|V||E|)$ time (details skipped)
2. Run Dijkstra’s algorithm from each vertex using $\hat{w}$.
   - Time = $O(|V||E| \log |V|)$.
3. Reweight each shortest-path length $\hat{w}(p)$ to produce the shortest-path lengths $w(p)$ of the original graph.
   - Time = $O(|V|^2)$ (details skipped)

Total time = $O(|V||E| \log |V|)$.



Shortest paths

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- DAG: One pass of Bellman-Ford: $O(|V| + |E|)$

} adj. list

All-pairs shortest paths

- Nonnegative edge weights — adj. list
    - Dijkstra’s algorithm $|V|$ times: $O(|V||E| \log |V|)$
- General
    - Bellman-Ford $|V|$ times: $\mathrm{O}(|V|^2|E|)$ — adj. list
    - Floyd-Warshall: $\mathrm{O}(|V|^3)$ — adj. matrix
    - Johnson’s algorithm: $O(|V| |E| \log |V|)$ — adj. list