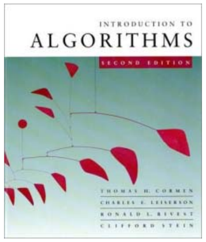
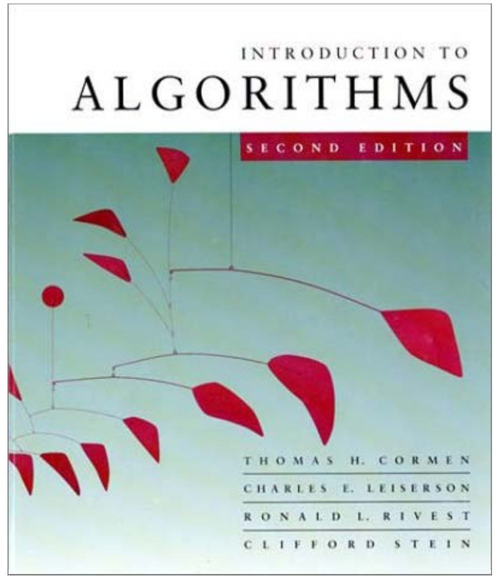


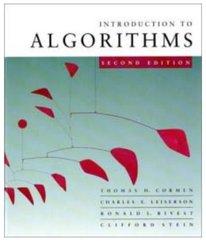

# CS 5633 – Spring 12




# *Minimum Spanning Trees*

**Carola Wenk**

Slides courtesy of Charles Leiserson with changes and additions by Carola Wenk



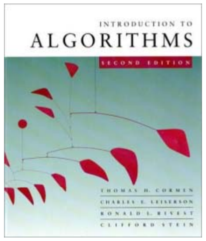

# Minimum spanning trees

**Input:** A connected, undirected graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{R}$.

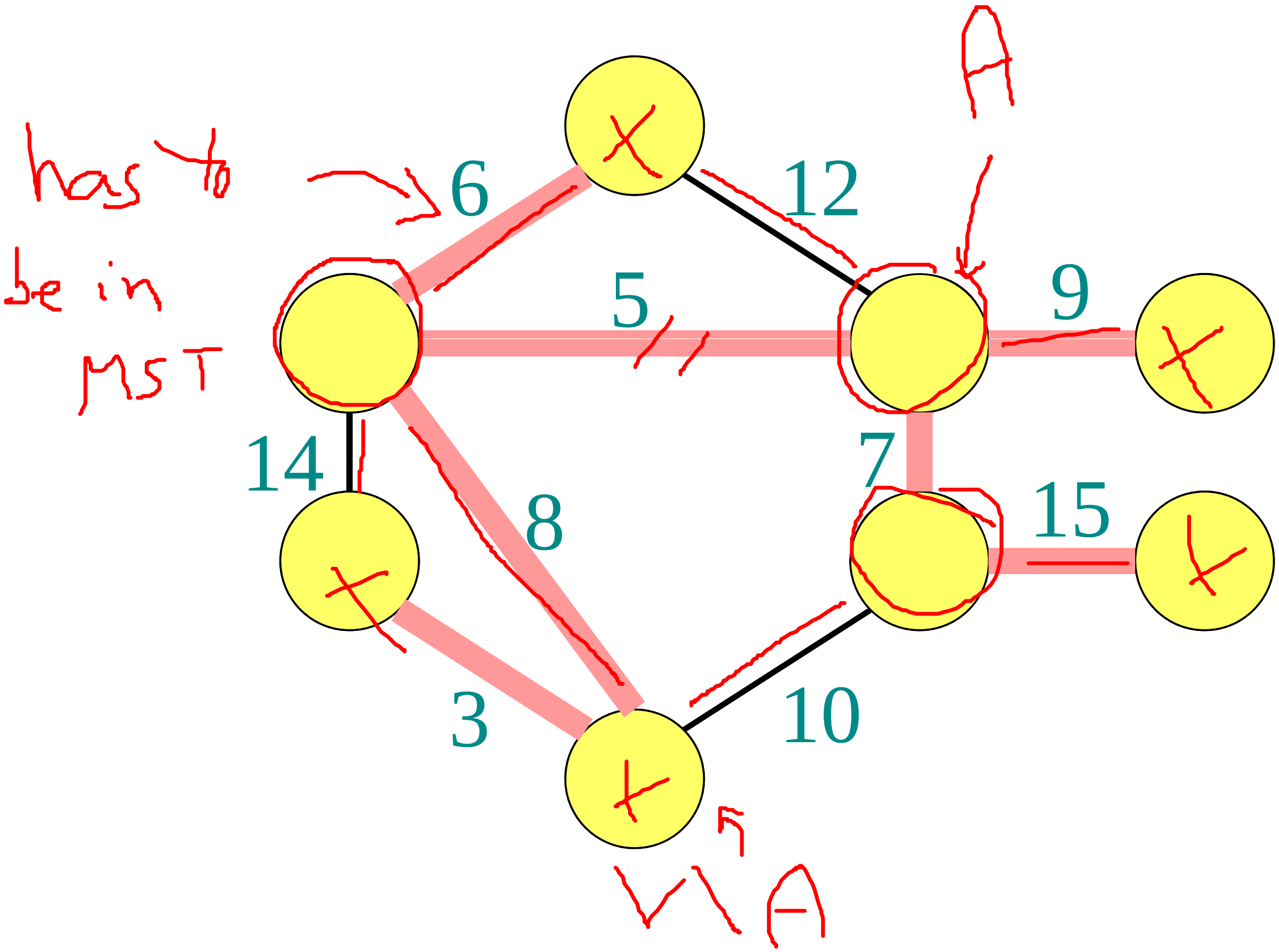
- For simplicity, assume that all edge weights are distinct. (CLRS covers the general case.)

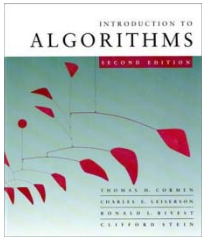
**Output:** A ***spanning tree*** $T$ — a tree that connects all vertices — of minimum weight:

$$w(T) = \sum_{(u,v) \in T} w(u,v).$$




# Example of MST



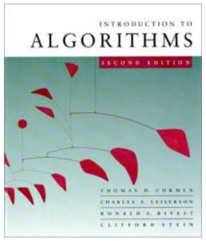



# Hallmark for "greedy" algorithms

***Greedy-choice property***

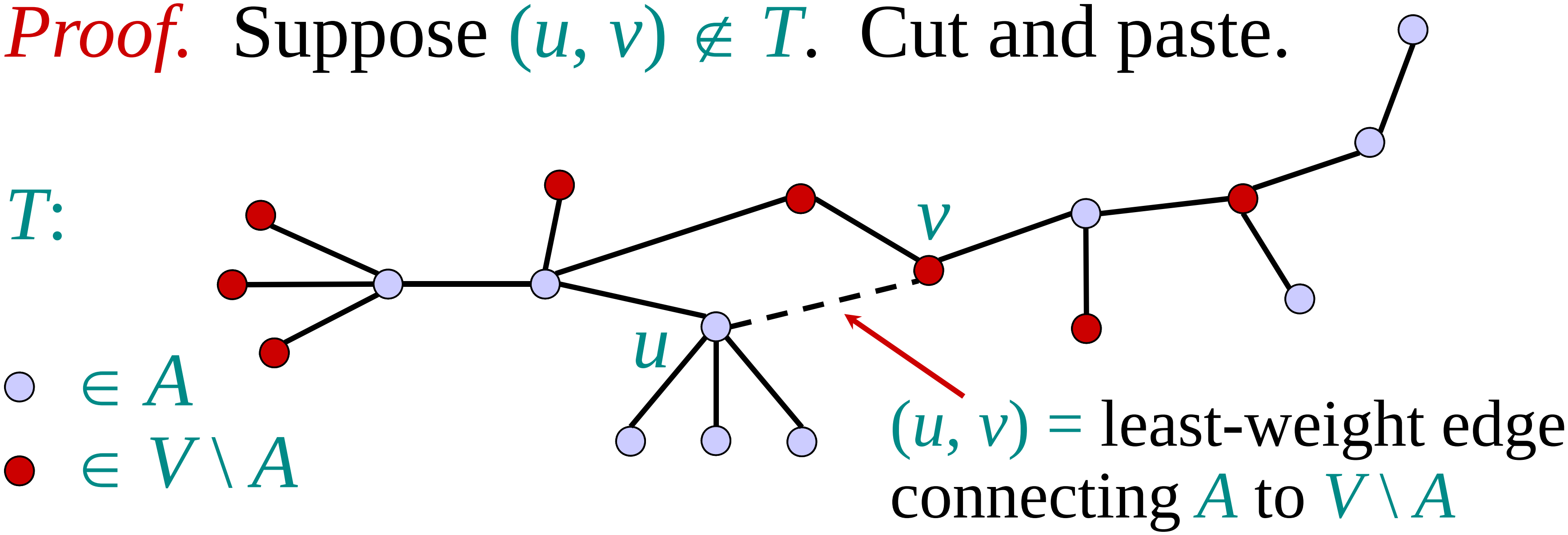
*A locally optimal choice is globally optimal.*

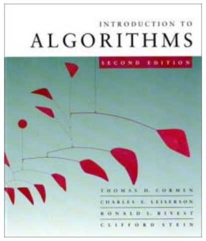
**Theorem.** Let $T$ be the MST of $G = (V, E)$, and let $A \subseteq V$. Suppose that $(u, v) \in E$ is the least-weight edge connecting $A$ to $V \setminus A$. Then, $(u, v) \in T$.




# Proof of theorem

*Proof.* Suppose $(u, v) \notin T$. Cut and paste.



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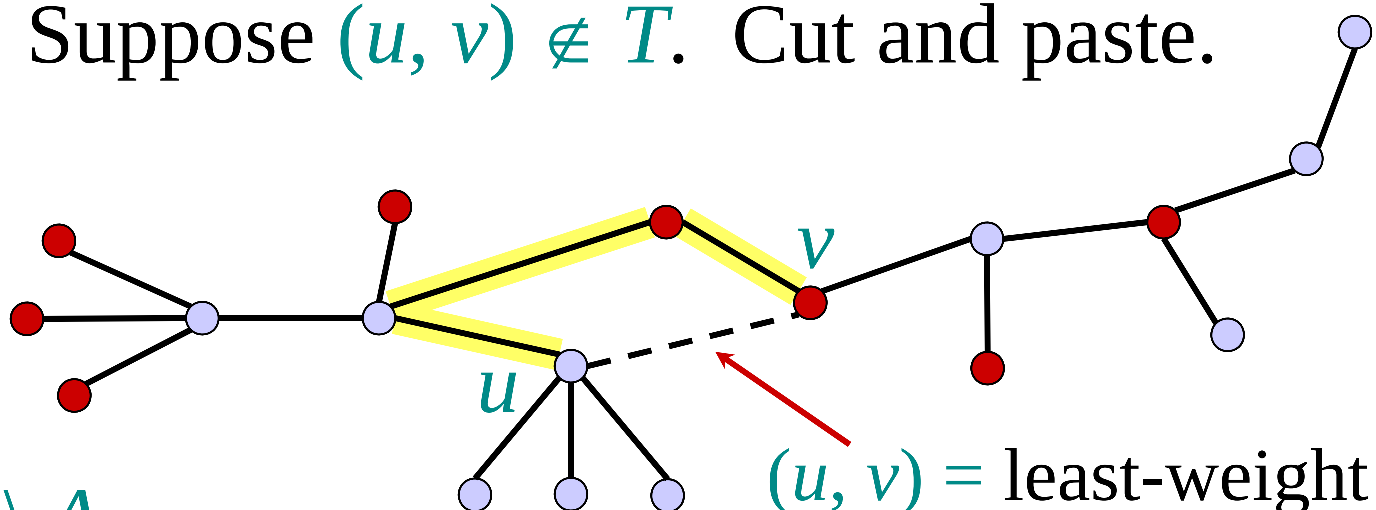
$T$:

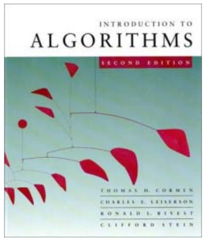



○ $\in A$

● $\in V \setminus A$

$(u, v)$ = least-weight edge connecting $A$ to $V \setminus A$

Consider the unique simple path from $u$ to $v$ in $T$.

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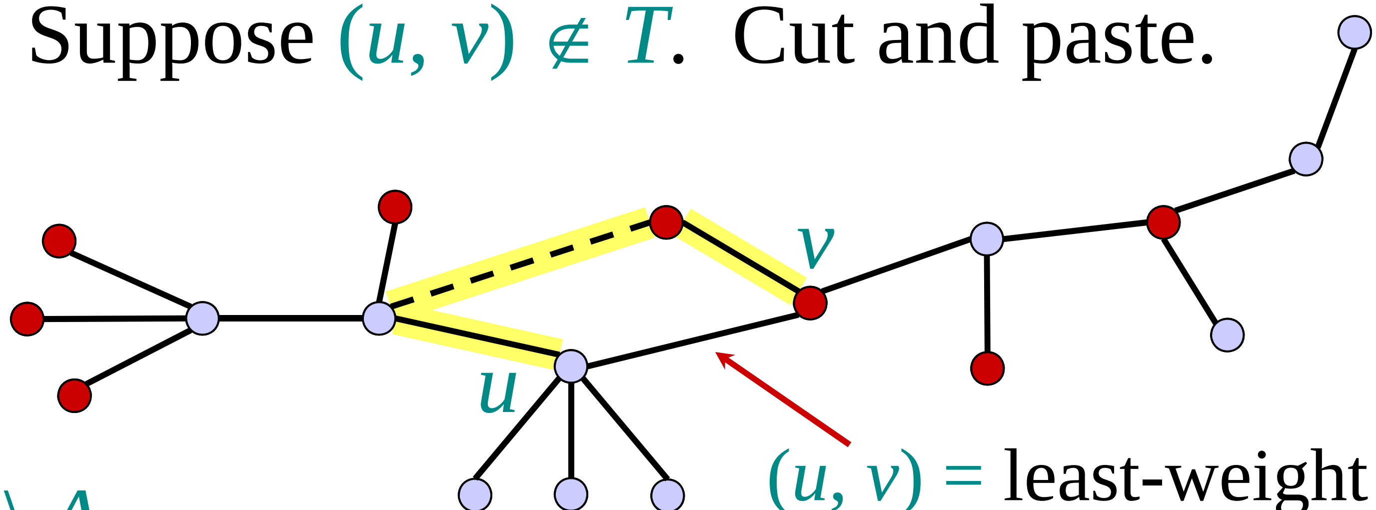


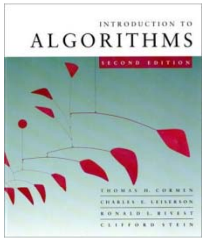

○ $\in A$

● $\in V \setminus A$

$(u, v)$ = least-weight edge connecting $A$ to $V \setminus A$

Consider the unique simple path from $u$ to $v$ in $T$.

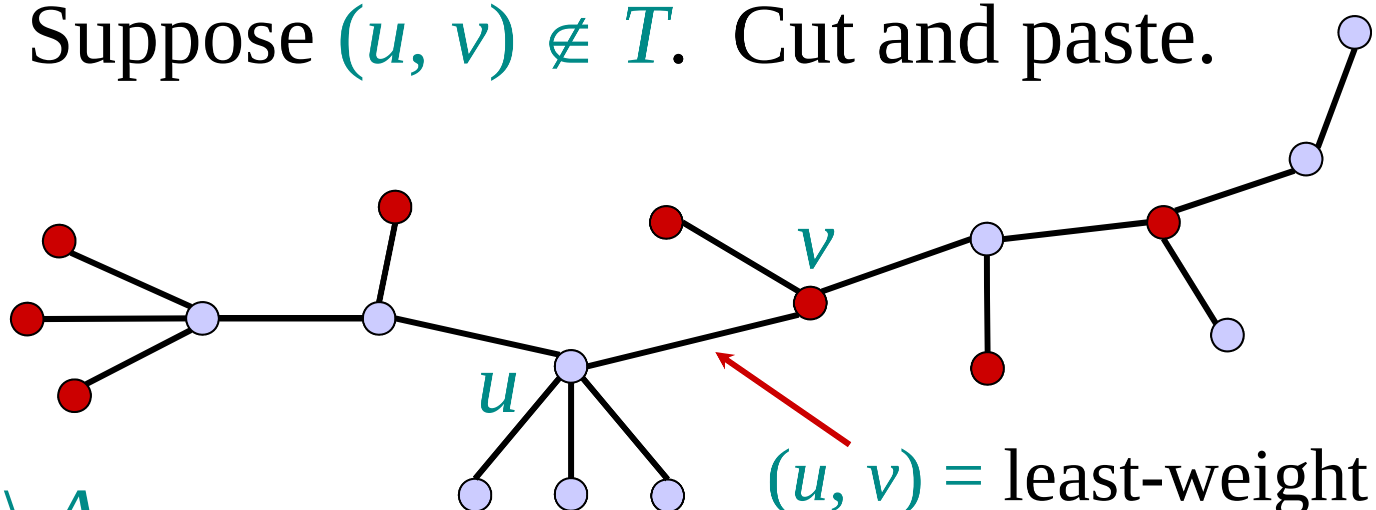
Swap $(u, v)$ with the first edge on this path that connects a vertex in $A$ to a vertex in $V \setminus A$.




# Proof of theorem

*Proof.* Suppose $(u, v) \notin T$. Cut and paste.

$T'$:




○ $\in A$

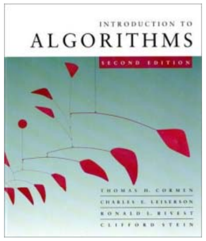
● $\in V \setminus A$

$(u, v)$ = least-weight edge connecting $A$ to $V \setminus A$

Consider the unique simple path from $u$ to $v$ in $T$.

Swap $(u, v)$ with the first edge on this path that connects a vertex in $A$ to a vertex in $V \setminus A$.

A lighter-weight spanning tree than $T$ results. □

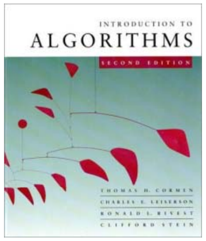



# Prim's algorithm

**IDEA:** Maintain $V \setminus A$ as a priority queue $Q$. Key each vertex in $Q$ with the weight of the least-weight edge connecting it to a vertex in $A$.

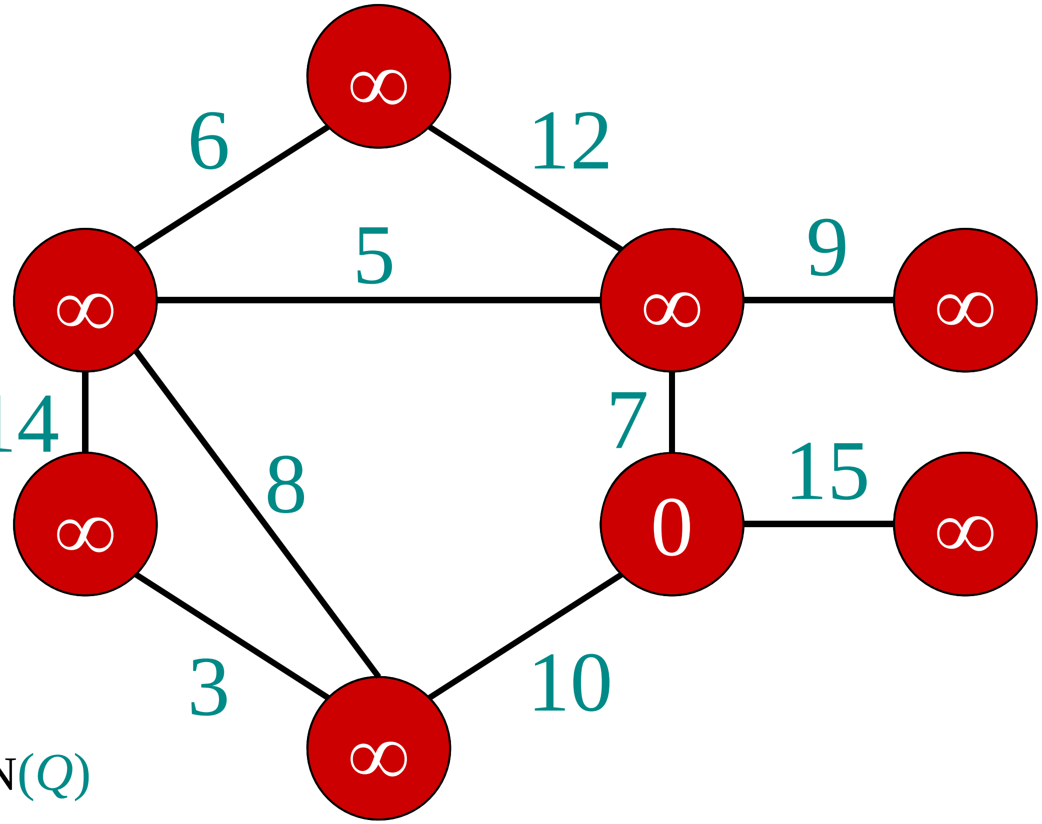
$Q \leftarrow V$
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
**while** $Q \neq \emptyset$
  **do** $u \leftarrow$ EXTRACT-MIN($Q$)
    **for** each $v \in Adj[u]$
      **do if** $v \in Q$ and $w(u, v) < key[v]$
        **then** $key[v] \leftarrow w(u, v)$  ⊳ DECREASE-KEY
          $\pi[v] \leftarrow u$

At the end, $\{(v, \pi[v])\}$ forms the MST edges.




# Example of Prim's algorithm

- ○ $\in A$
- ● $\in V \setminus A$




$u \leftarrow$ EXTRACT-MIN$(Q)$

**for** each $v \in Adj[u]$

  **do if** $v \in Q$ and $w(u, v) < key[v]$

    **then** $key[v] \leftarrow w(u, v)$ ⊳ DECREASE-KEY

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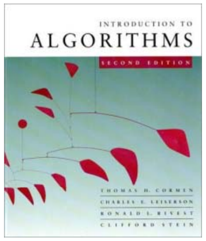
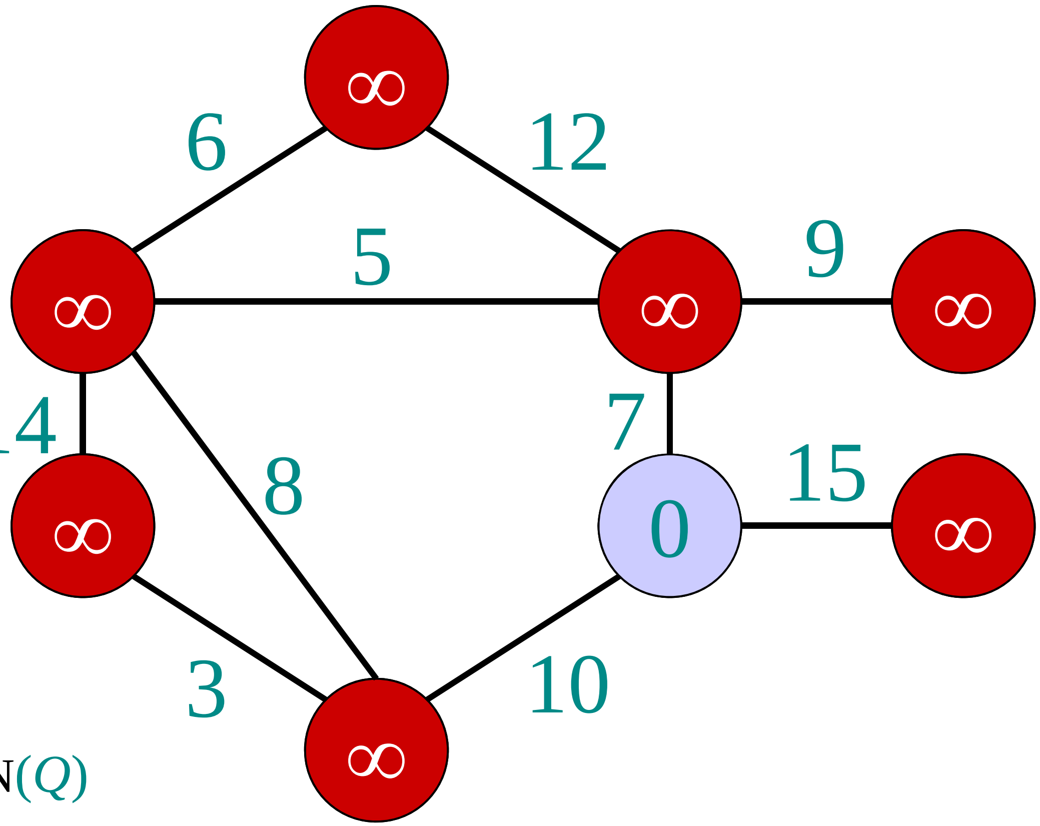



# Example of Prim's algorithm

- $\in A$
- $\in V \setminus A$

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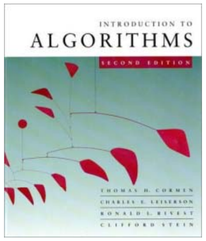


# Example of Prim's algorithm

- $\in A$
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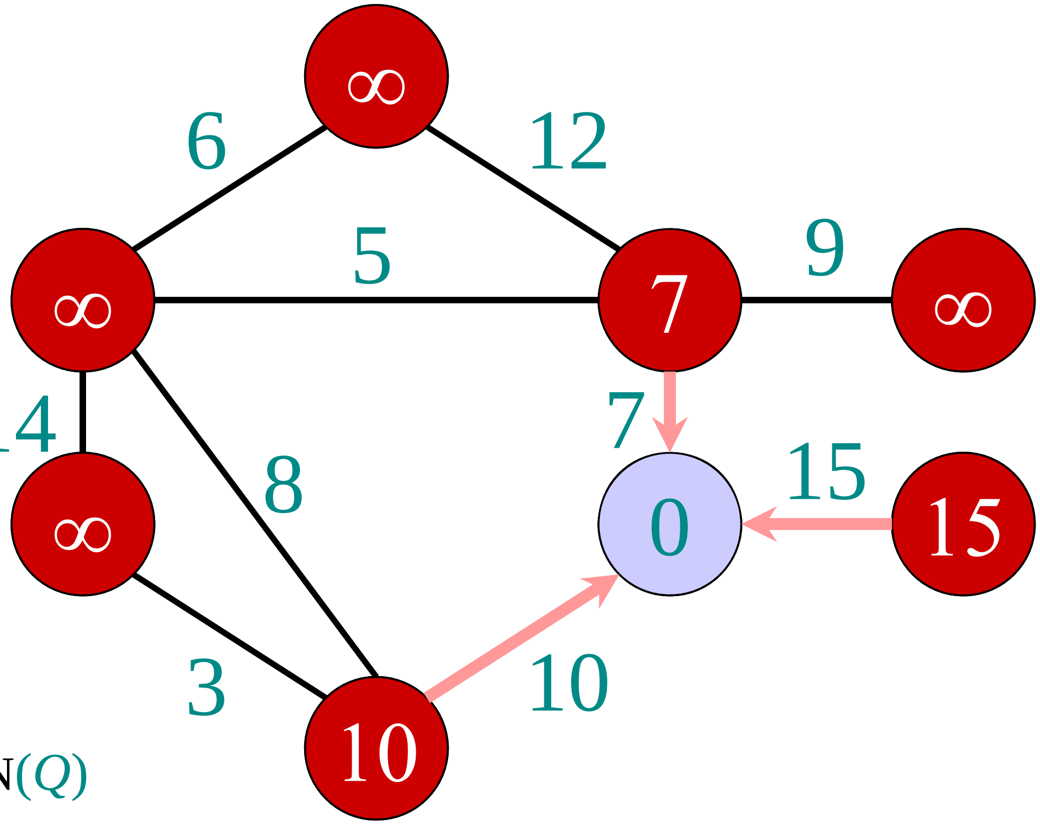


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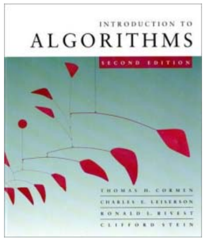
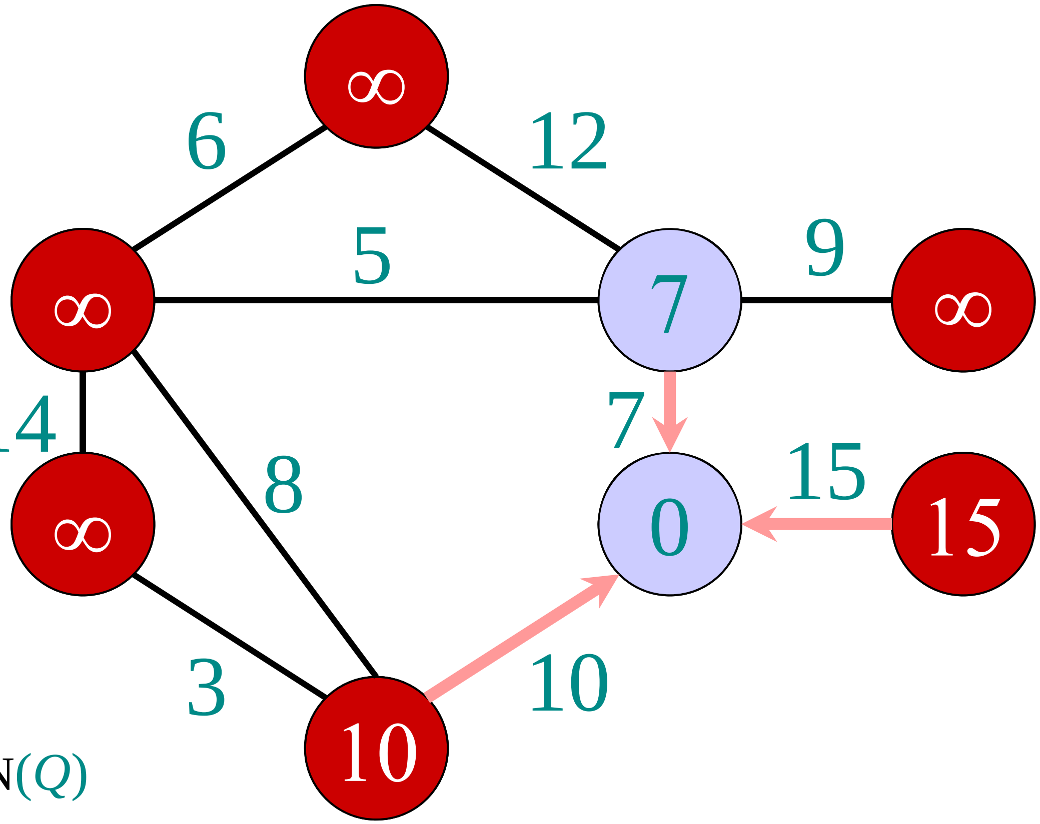



# Example of Prim's algorithm

- $\in A$
- $\in V \setminus A$

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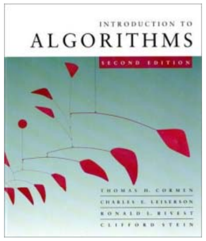
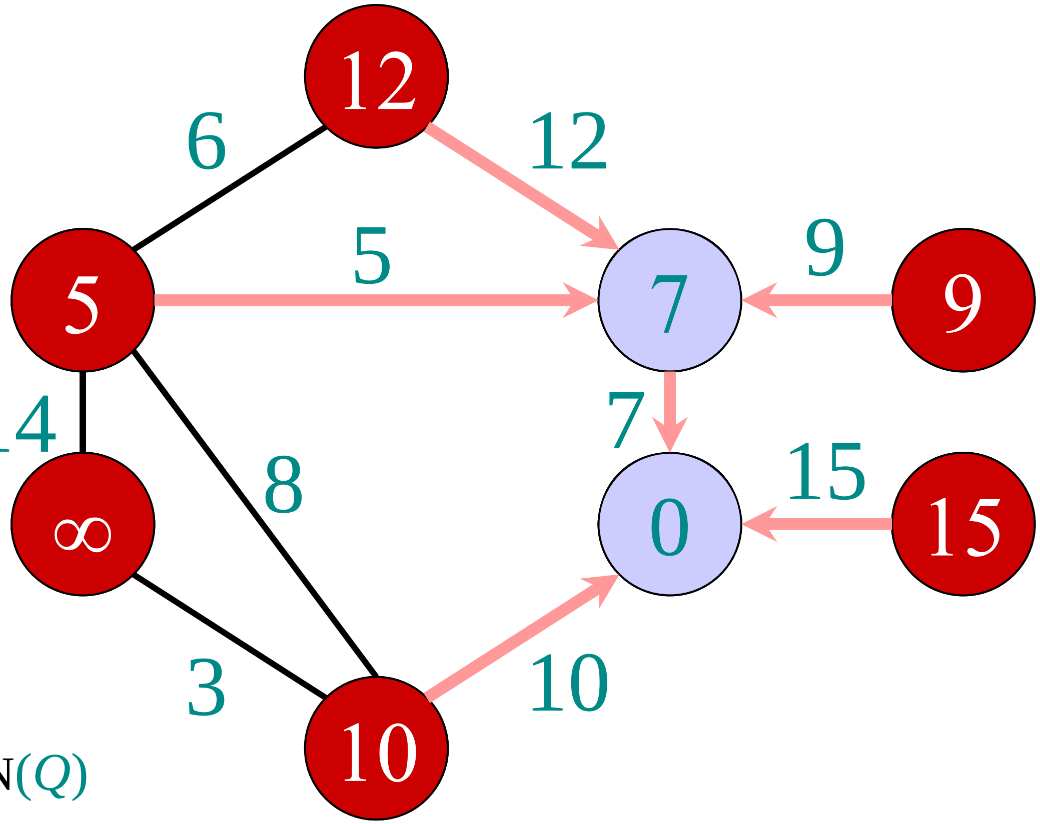



# Example of Prim's algorithm

○ $\in A$

● $\in V \setminus A$

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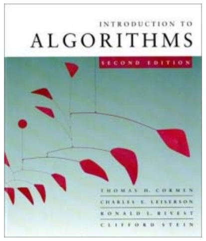


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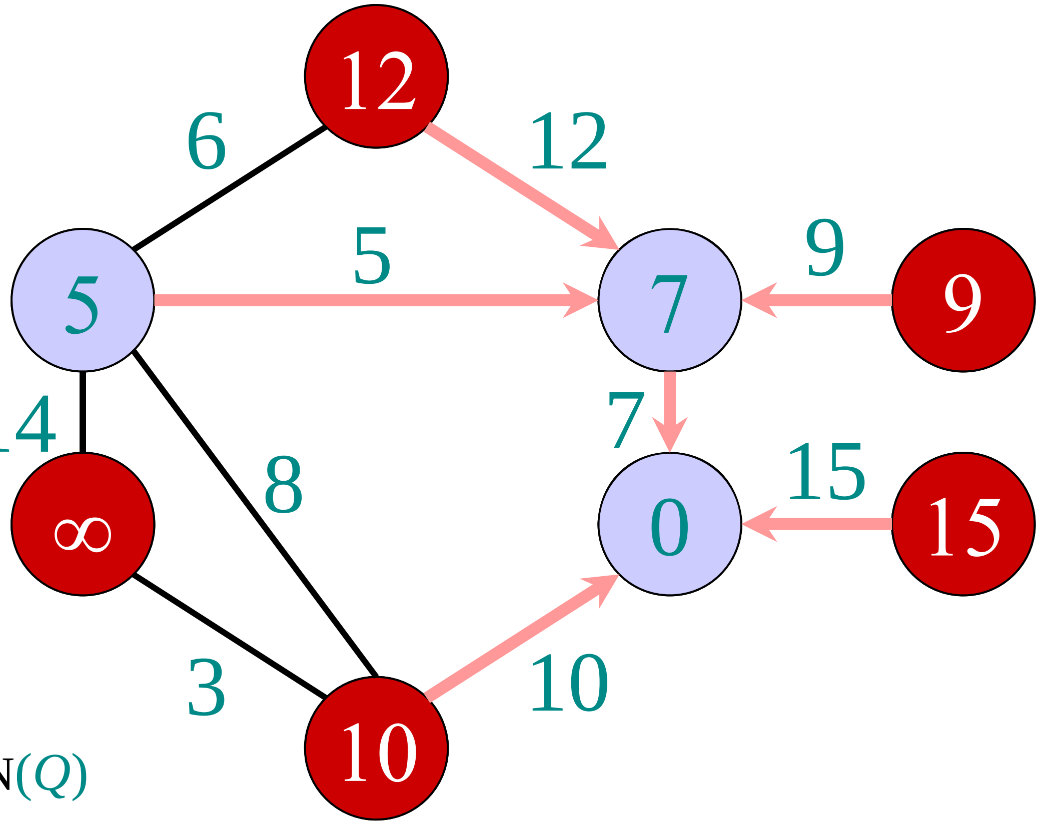


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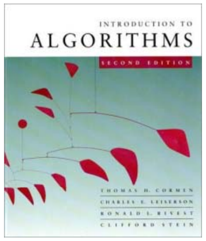

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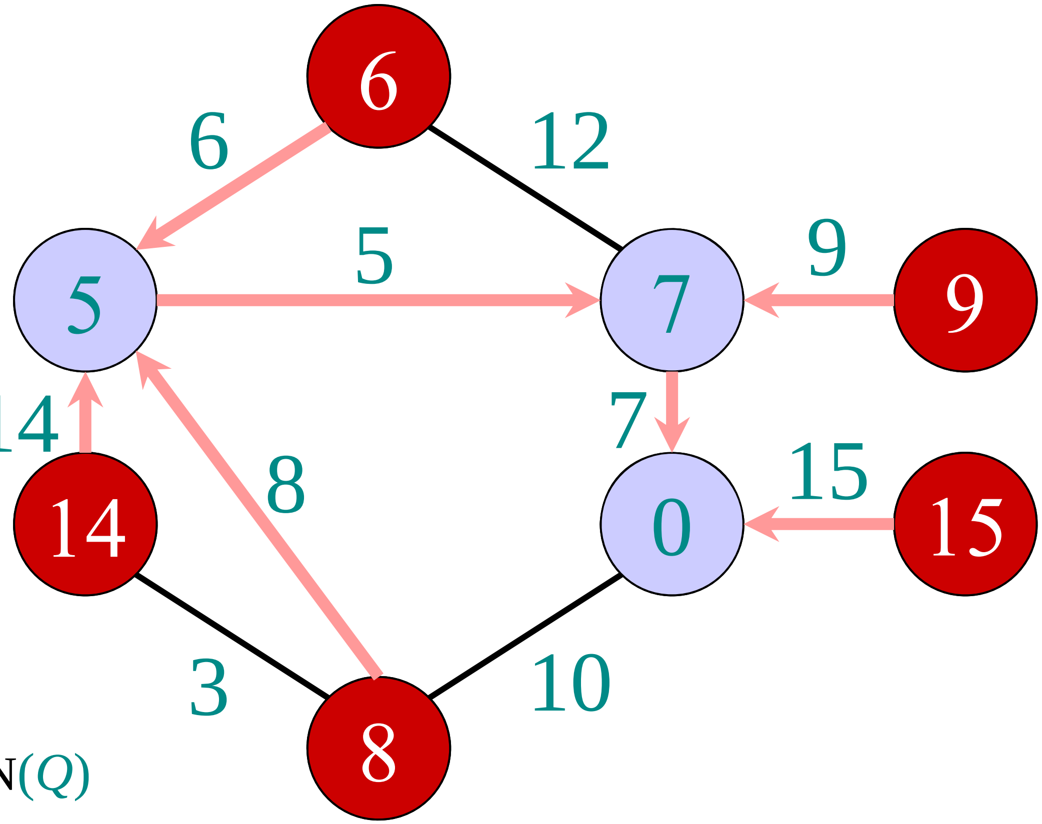


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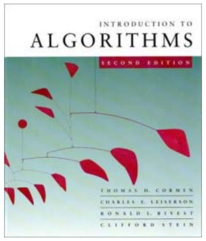


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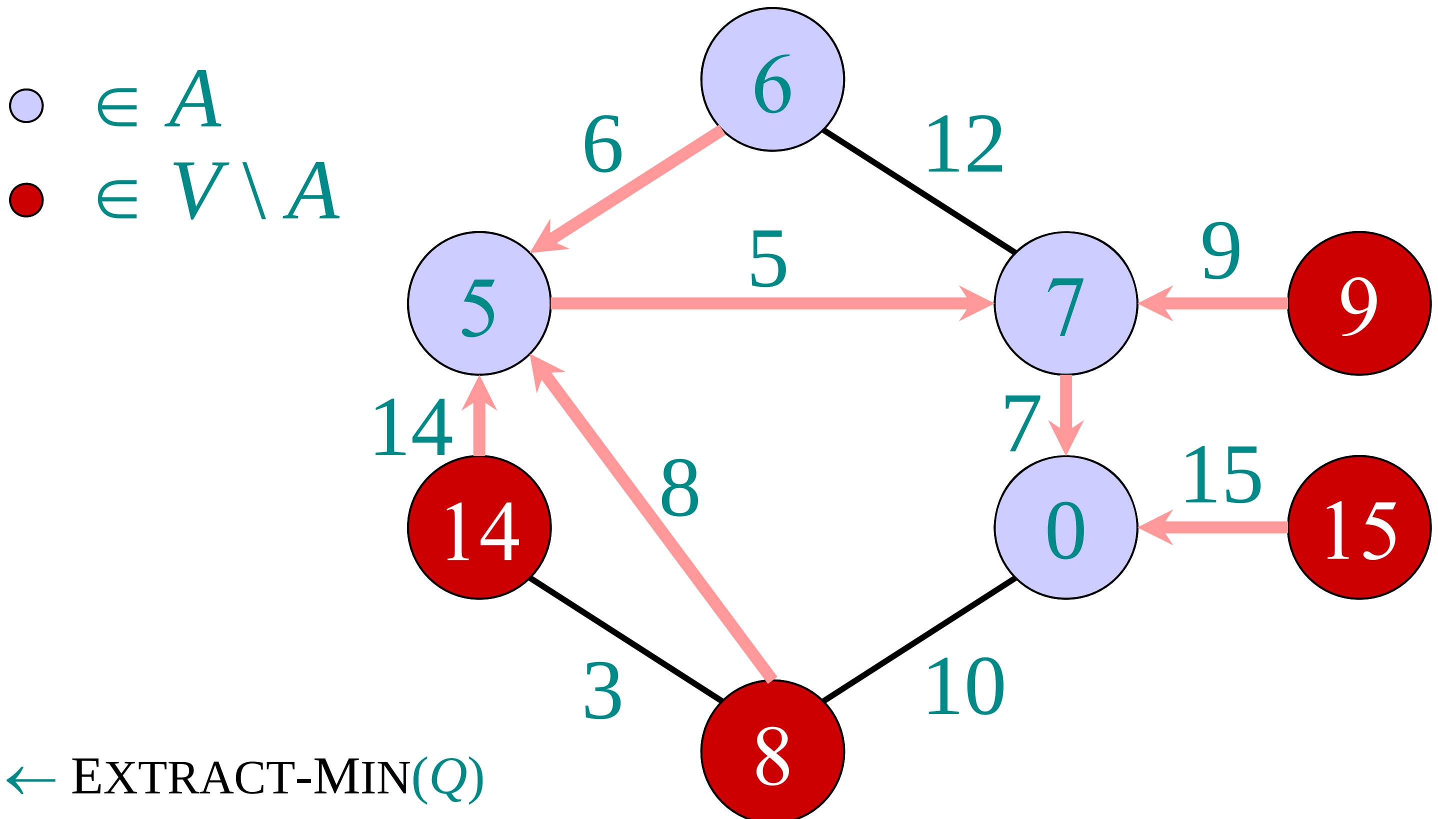


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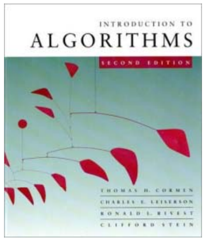
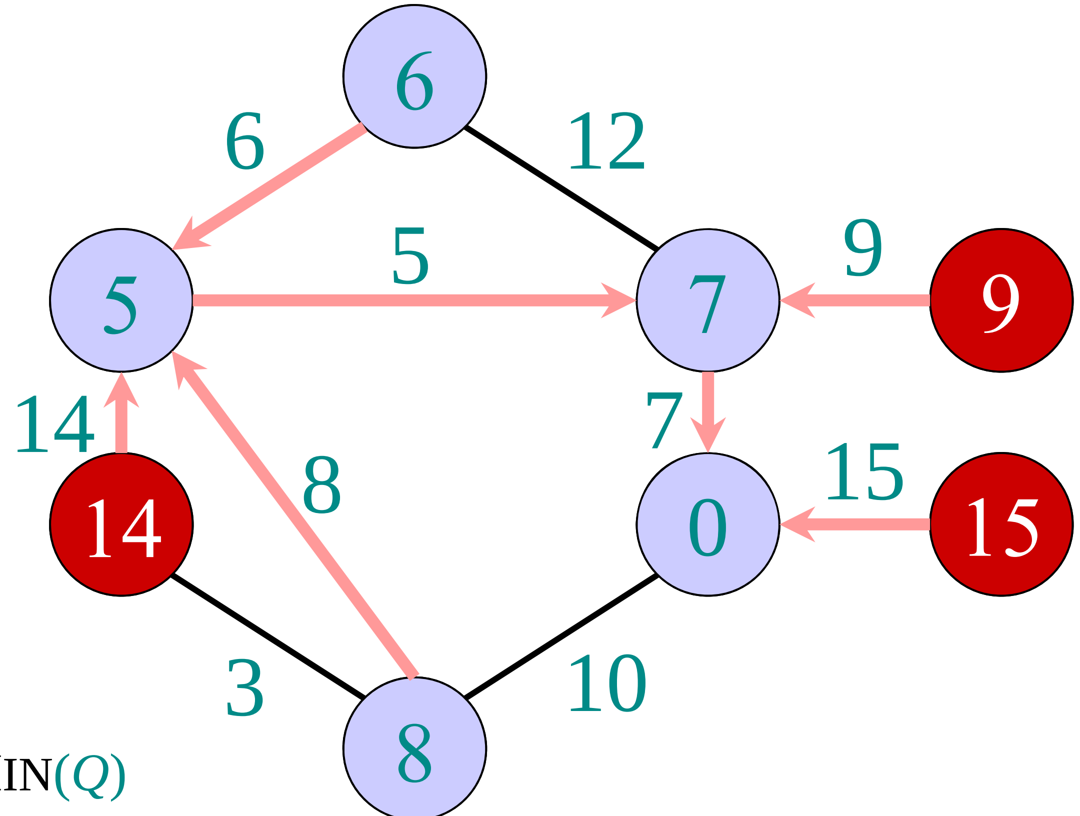



# Example of Prim's algorithm

○ $\in A$
● $\in V \setminus A$

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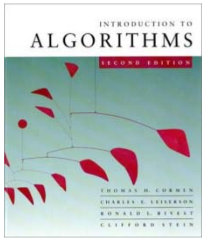


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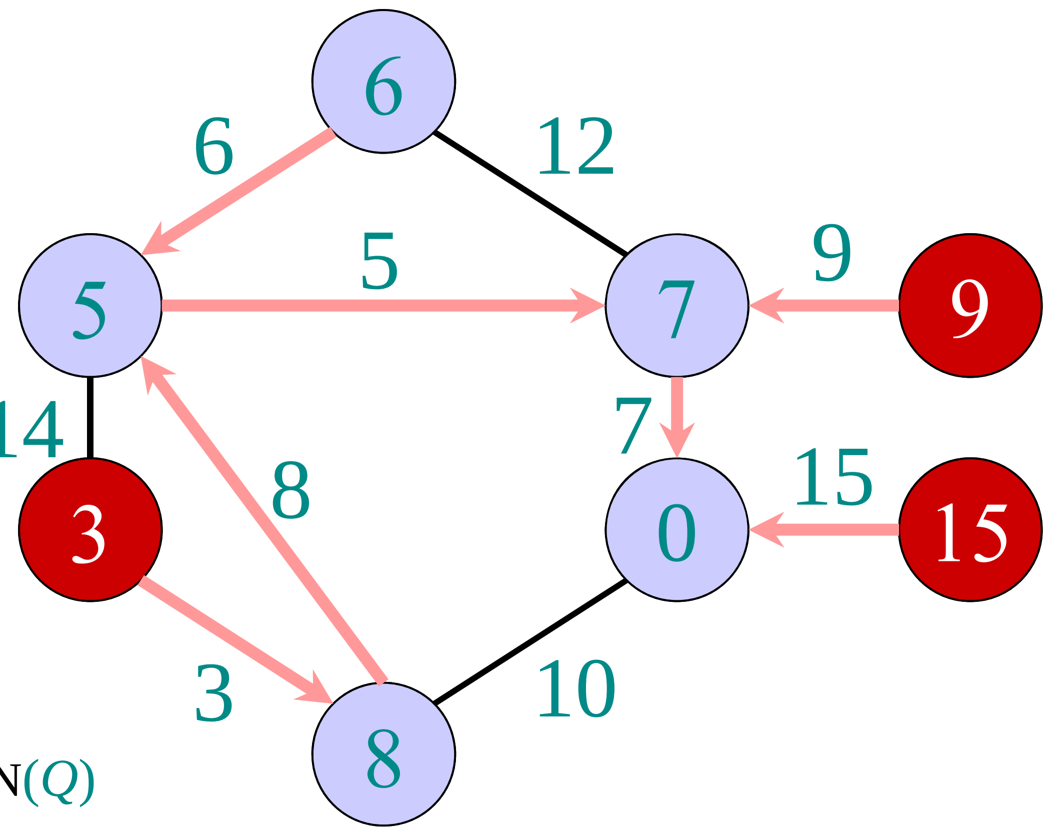


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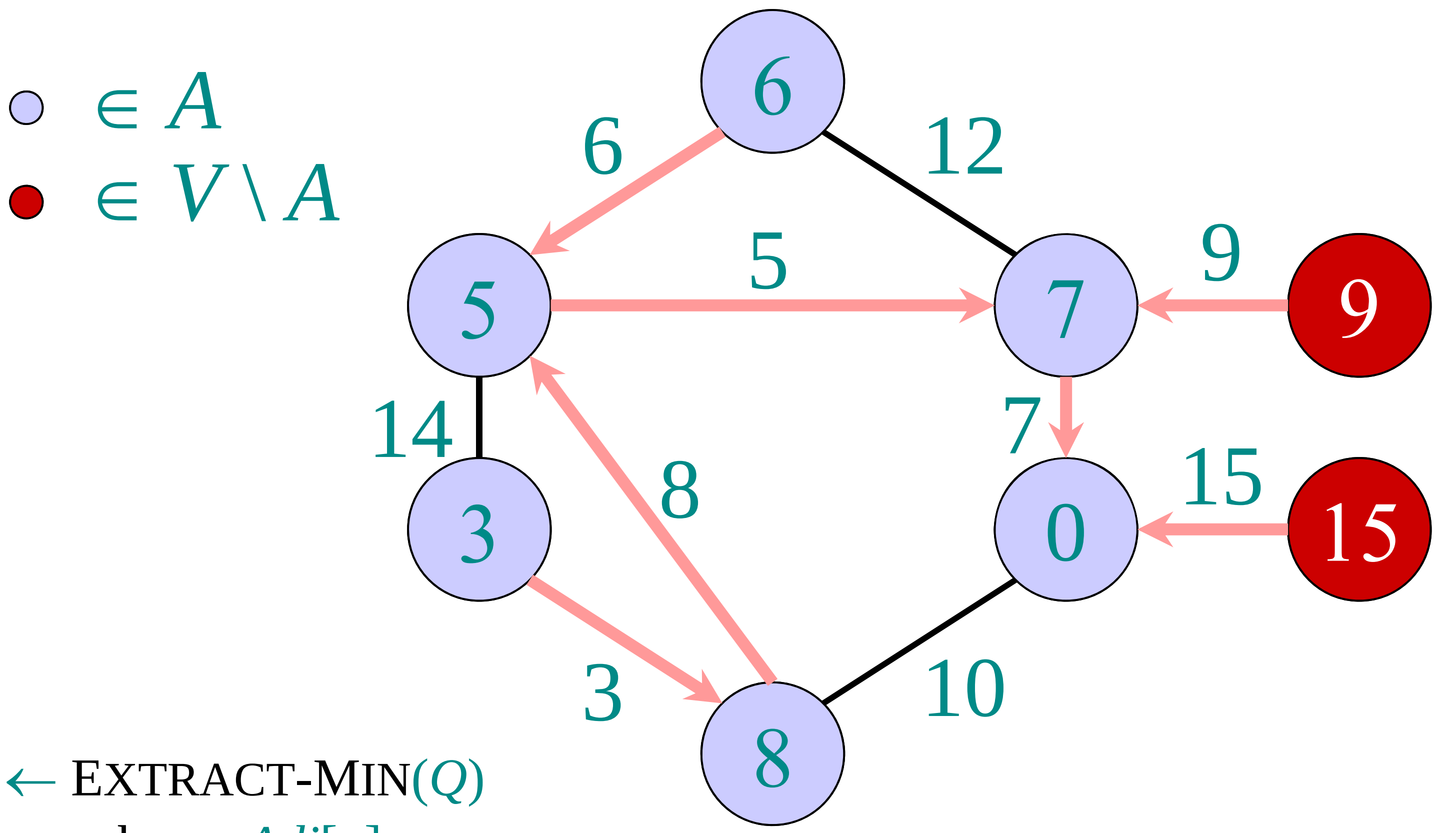


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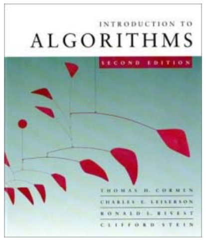
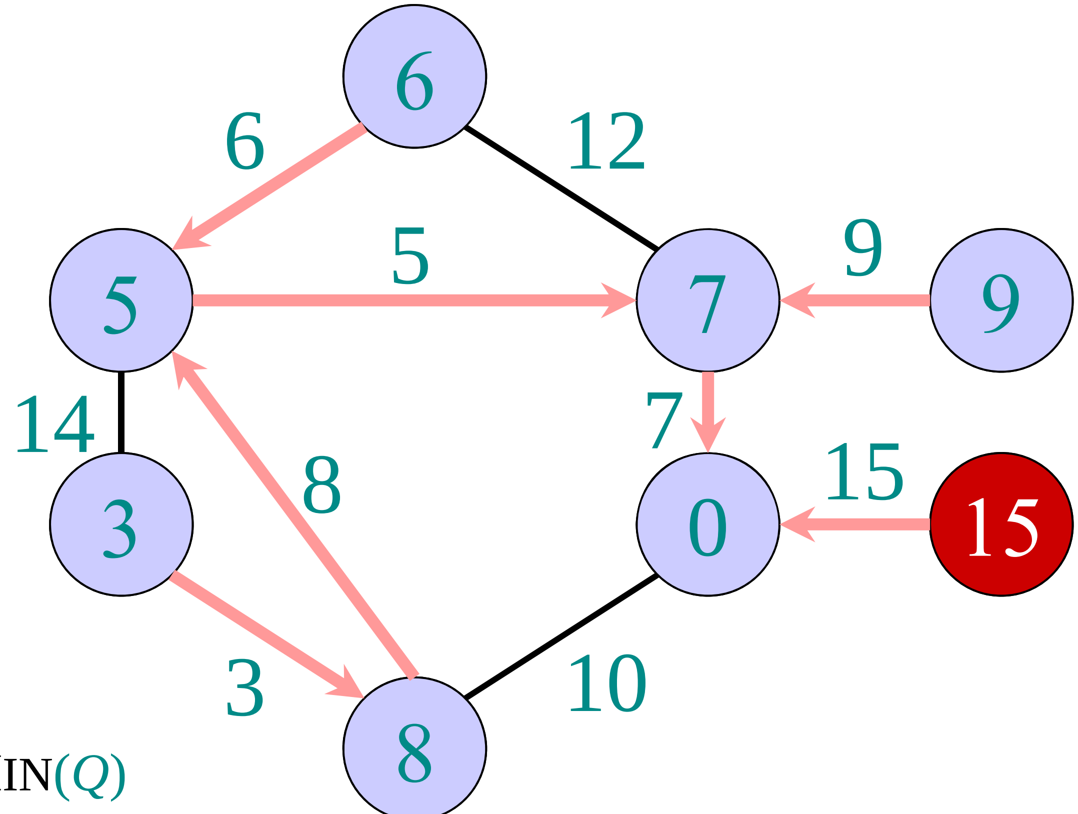



# Example of Prim's algorithm

● $\in A$
● $\in V \setminus A$

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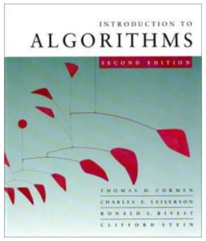


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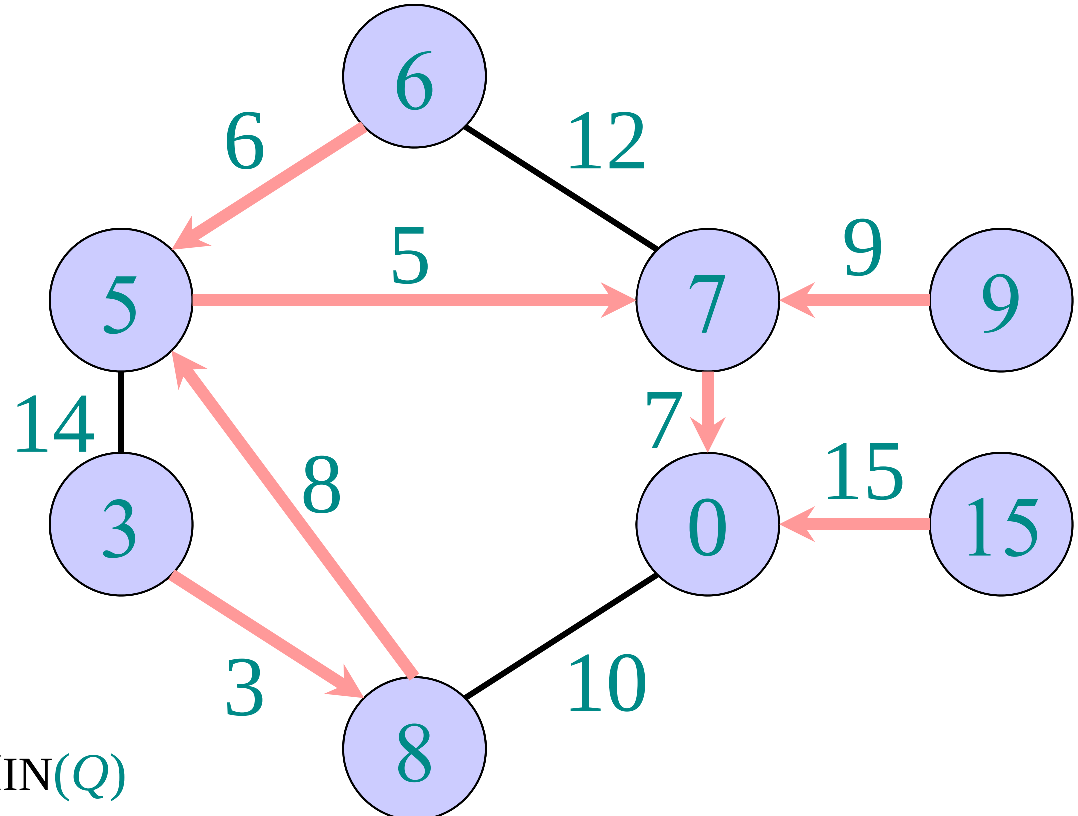


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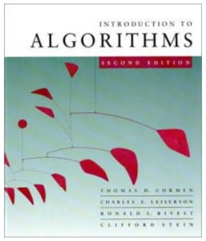



# Analysis of Prim

$\Theta(|V|)$ total:

- $Q \leftarrow V$
- $key[v] \leftarrow \infty$ for all $v \in V$
- $key[s] \leftarrow 0$ for some arbitrary $s \in V$

$|V|$ times:

**while** $Q \neq \emptyset$

  **do** $u \leftarrow$ EXTRACT-MIN$(Q)$

  $degree(u)$ times:

   **for** each $v \in Adj[u]$

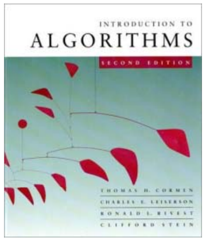
    **do if** $v \in Q$ and $w(u, v) < key[v]$

     **then** $key[v] \leftarrow w(u, v)$

      $\pi[v] \leftarrow u$

Handshaking Lemma $\Rightarrow \Theta(|E|)$ implicit DECREASE-KEY's.

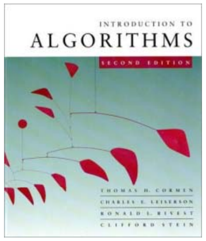
Time $= \Theta(|V|) \cdot T_{\text{EXTRACT-MIN}} + \Theta(|E|) \cdot T_{\text{DECREASE-KEY}}$




# Analysis of Prim (continued)

Time = $\Theta(|V|)\cdot T_{\text{EXTRACT-MIN}} + \Theta(|E|)\cdot T_{\text{DECREASE-KEY}}$

| $Q$ | $T_{\text{EXTRACT-MIN}}$ | $T_{\text{DECREASE-KEY}}$ | Total |
|---|---|---|---|
| array | $O(\|V\|)$ | $O(1)$ | $O(\|V\|^2)$ |
| binary heap | $O(\log \|V\|)$ | $O(\log \|V\|)$ | $O(\|E\| \log \|V\|)$ |
| Fibonacci heap | $O(\log \|V\|)$ amortized | $O(1)$ amortized | $O(\|E\| + \|V\| \log \|V\|)$ worst case |

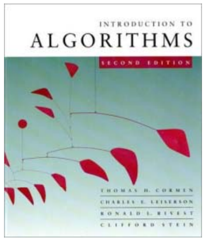



# Kruskal's algorithm

**IDEA (again greedy):**
Repeatedly pick edge with smallest weight as long as it does not form a cycle.

- The algorithm creates a set of trees (a **forest**)
- During the algorithm the added edges merge the trees together, such that in the end only one tree remains

- The correctness of this greedy strategy is not obvious and needs to be proven. (Proof skipped here.)

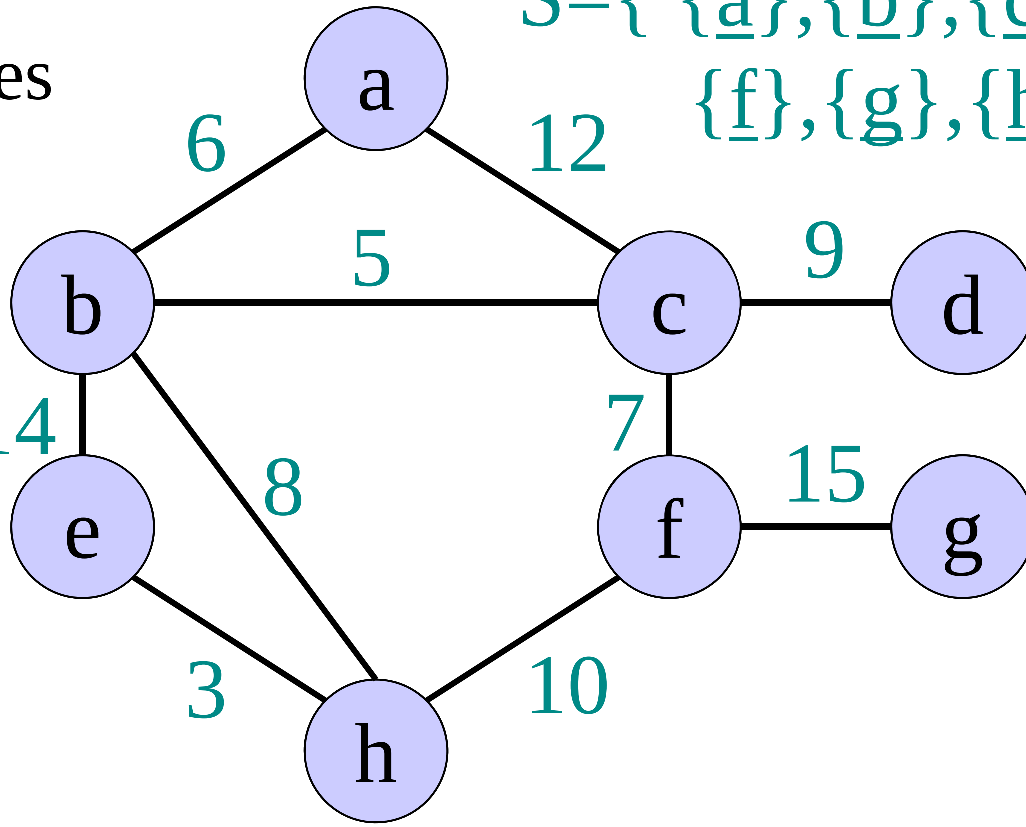



# Example of Kruskal's algorithm

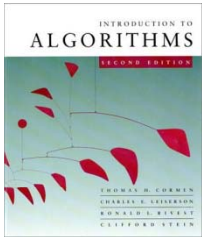
MST edges

a set repr.

S={ {a},{b},{c},{d},{e} {f},{g},{h} }




Every node is a single tree.

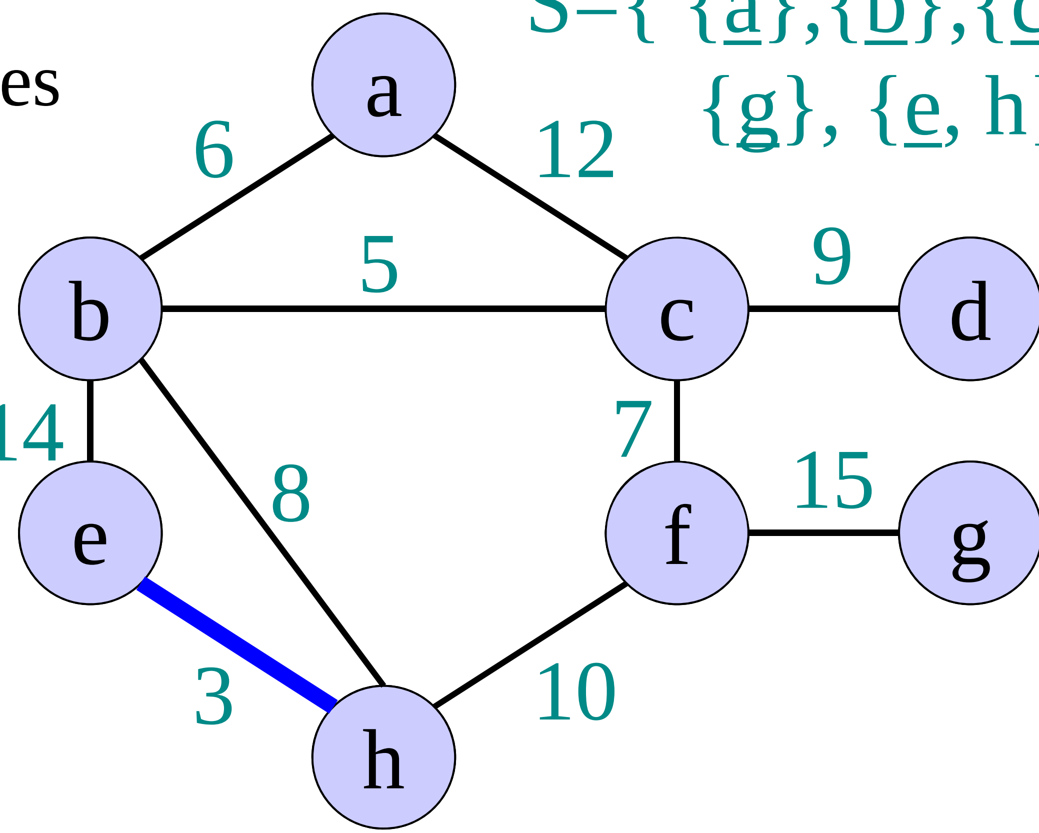



# Example of Kruskal's algorithm

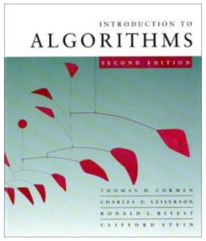
MST edges

a set repr.

S={ {a},{b},{c},{d},{f}
{g}, {e, h} }




Edge 3 merged two singleton trees.

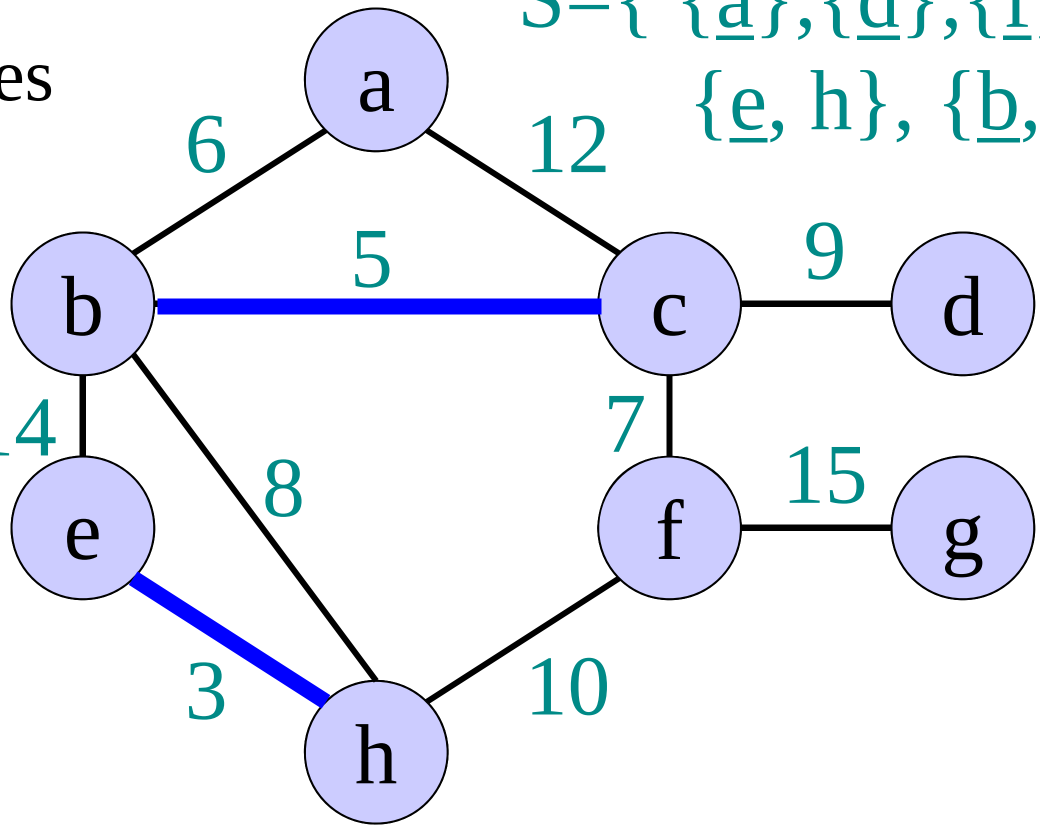


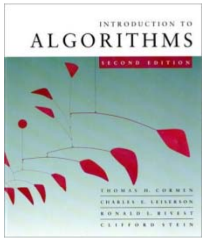
# Example of Kruskal's algorithm

MST edges

a set repr.

S={ {a},{d},{f}, {g}
{e, h}, {b, c} }



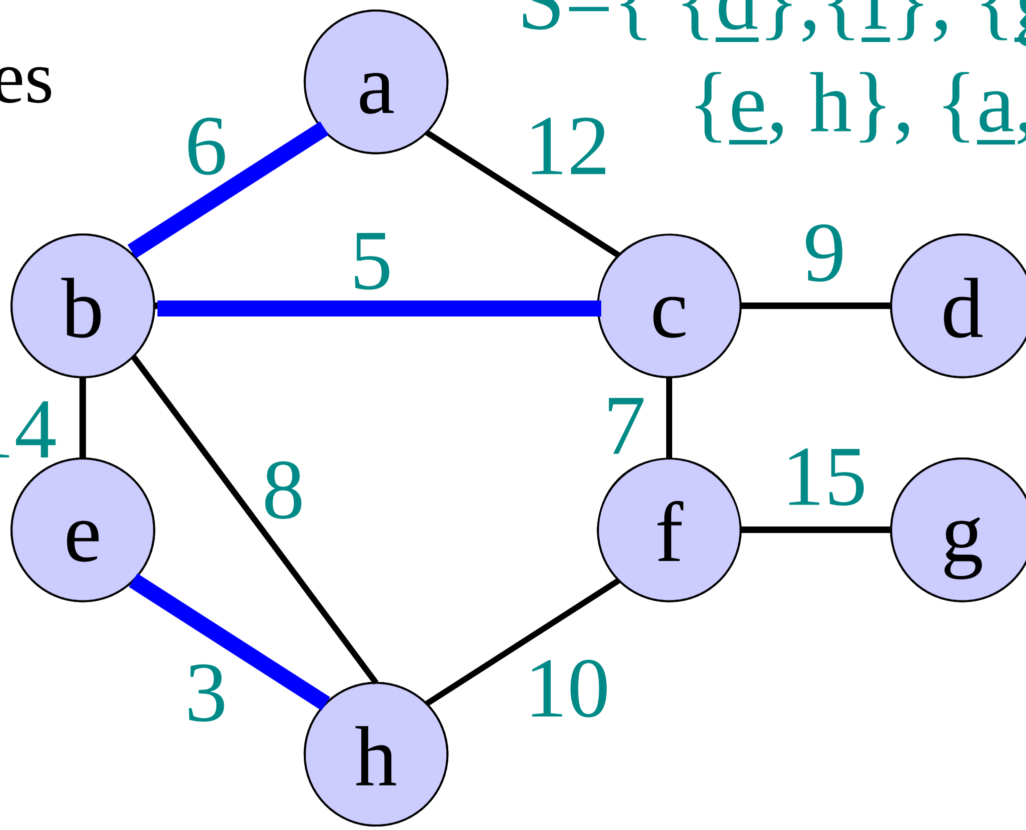


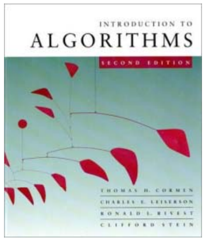
# Example of Kruskal's algorithm

MST edges

a set repr.

S={ {d},{f}, {g}
{e, h}, {a, b, c} }



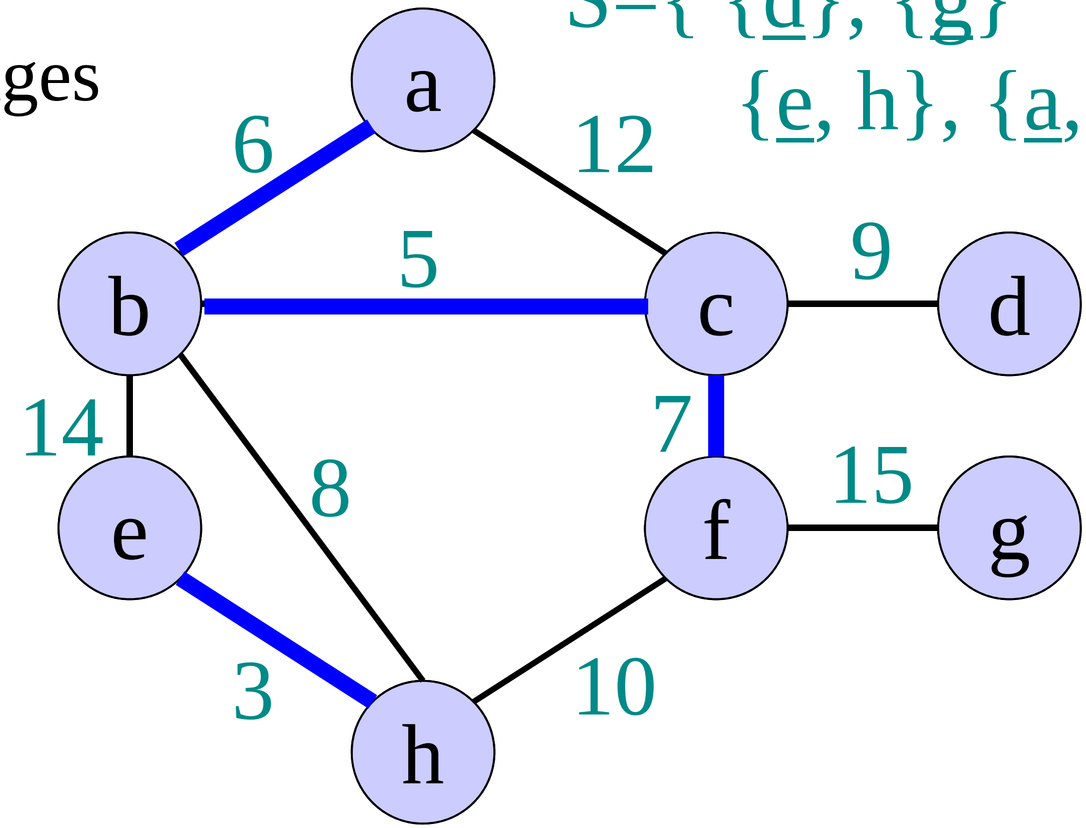


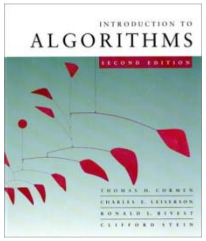

# Example of Kruskal's algorithm

MST edges

a set repr.

S={ {d}, {g}
{e, h}, {a, b, c, f} }



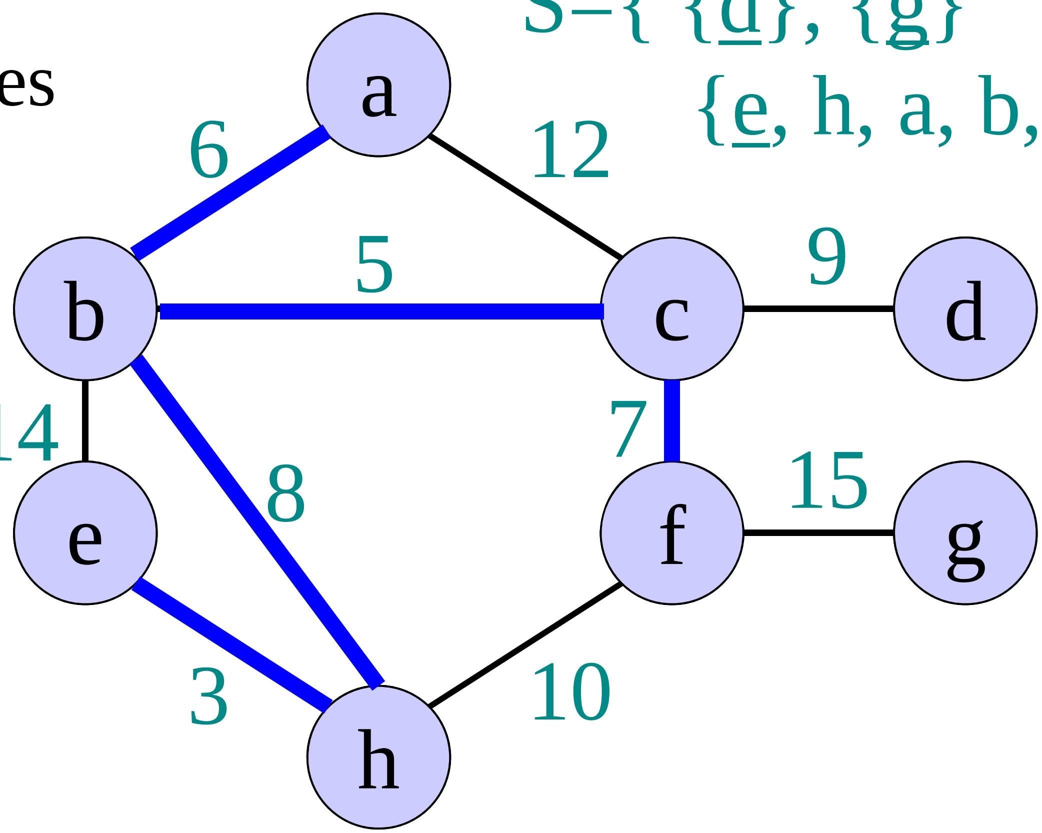



# Example of Kruskal's algorithm

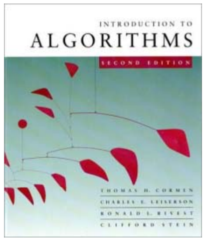
MST edges

a set repr.

S={ {d}, {g}
{e, h, a, b, c, f} }




Edge 8 merged the two bigger trees.

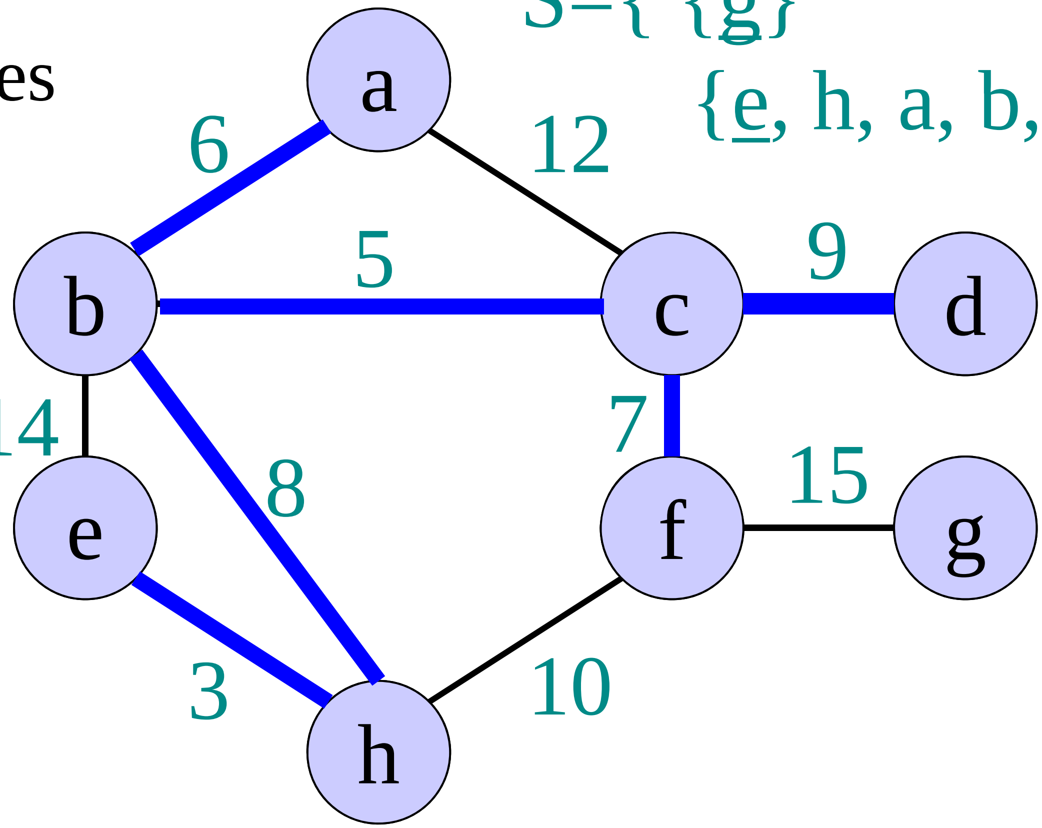


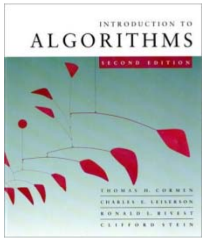

# Example of Kruskal's algorithm

MST edges

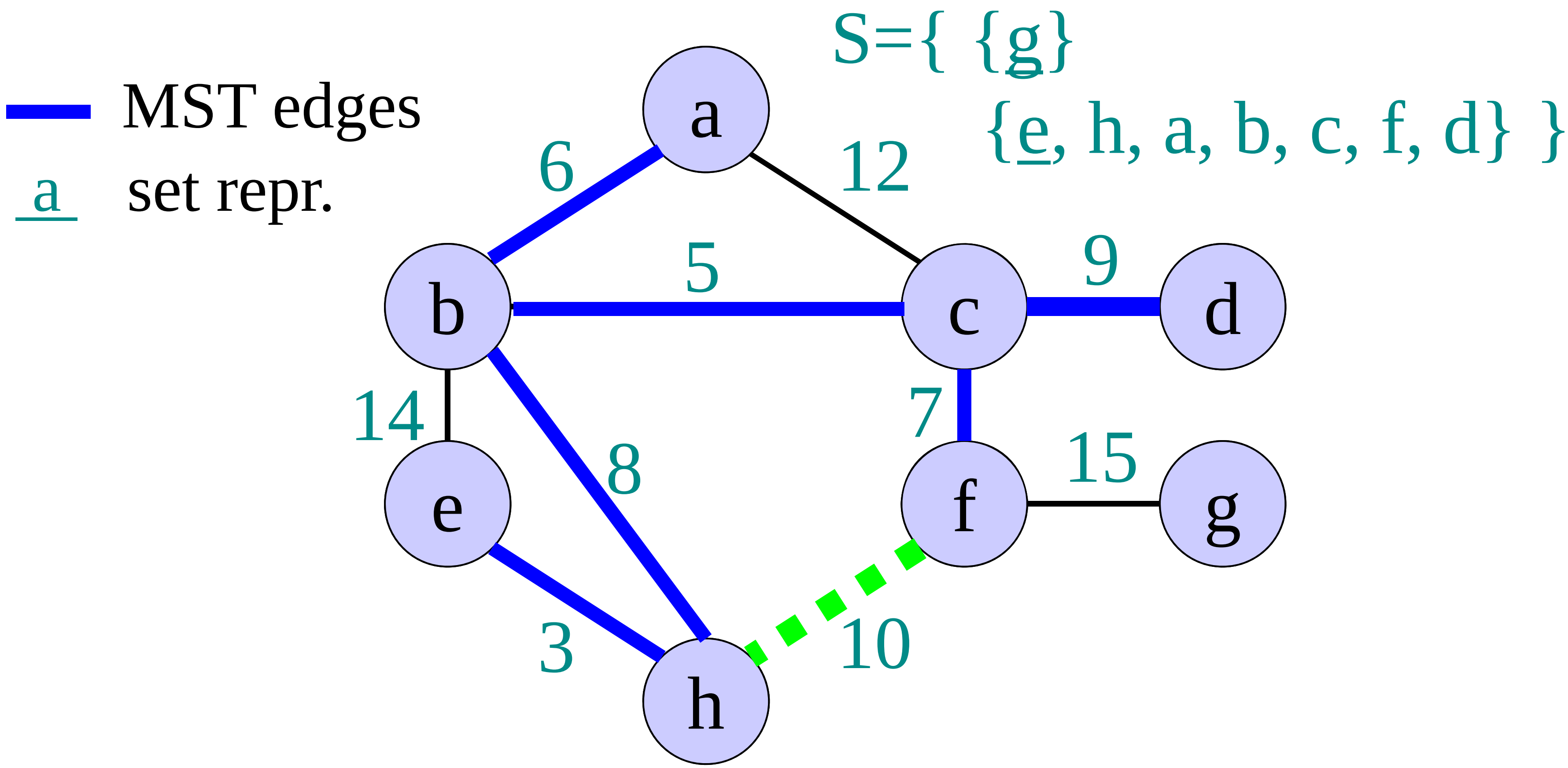
a set repr.

S={ {g}
{e, h, a, b, c, f, d} }

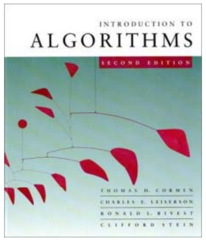





# Example of Kruskal's algorithm




Skip edge 10 as it would cause a cycle.

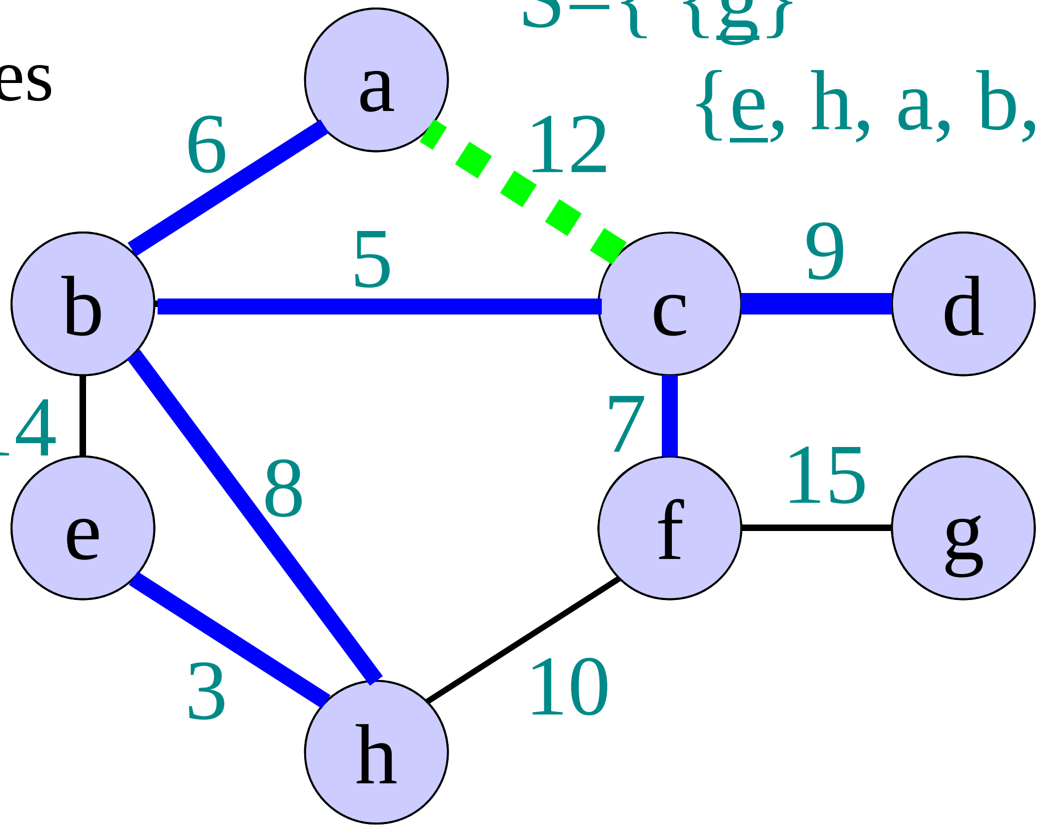


# Example of Kruskal's algorithm

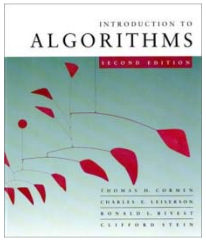
MST edges

a set repr.

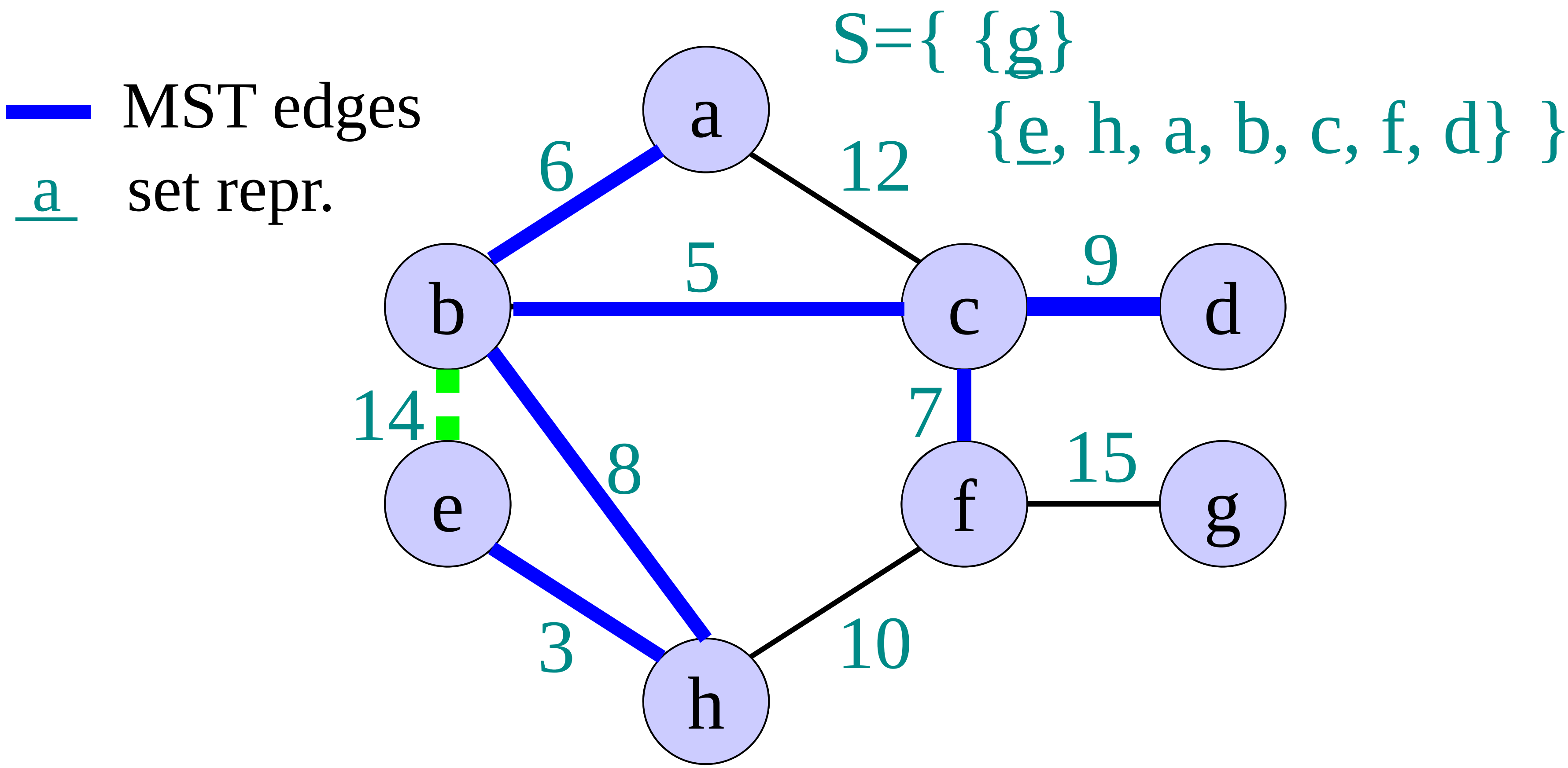
S={ {g}
{e, h, a, b, c, f, d} }



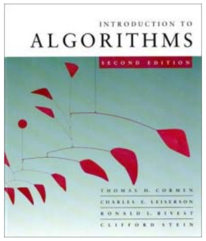

Skip edge 12 as it would cause a cycle.




# Example of Kruskal's algorithm




Skip edge 14 as it would cause a cycle.

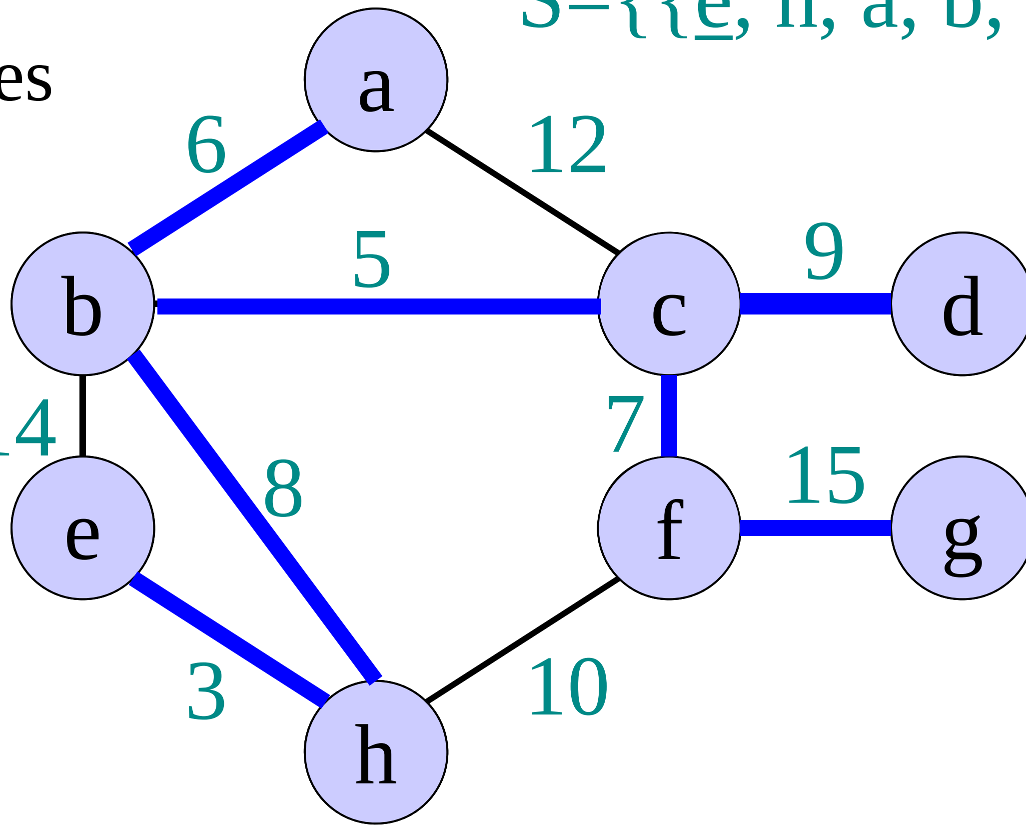


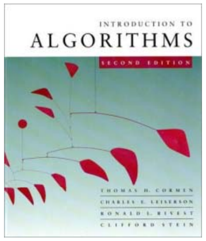

# Example of Kruskal’s algorithm

S={{<u>e</u>, h, a, b, c, f, d, g} }

MST edges

<u>a</u> set repr.






# Kruskal's algorithm

**IDEA:** Repeatedly pick edge with smallest weight as long as it does not form a cycle.

$S \leftarrow \emptyset$ ⊳ $S$ will contain all MST edges

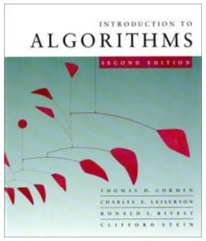
$O(|V|)$ **for** each $v \in V$ **do** MAKE-SET($v$)

$O(|E|\log|E|)$ Sort edges of $E$ in non-decreasing order according to $w$

$O(|E|)$ **For** each $(u,v) \in E$ taken in this order **do**

$O(\alpha(|V|))$ {
- **if** FIND-SET($u$) $\neq$ FIND-SET($v$) ⊳ $u,v$ in different trees
  - $S \leftarrow S \cup \{(u,v)\}$
  - UNION($u,v$) ⊳ Edge $(u,v)$ connects the two trees

**Runtime:** $O(|V|+|E|\log|E|+|E|\alpha(|V|)) = O(|E| \log |E|)$




# MST algorithms

- Prim's algorithm:
  - Maintains one tree
  - Runs in time $O(|E| \log |V|)$, with binary heaps.
- Kruskal's algorithm:
  - Maintains a forest and uses the disjoint-set data structure
  - Runs in time $O(|E| \log |E|)$
- Best to date: Randomized algorithm by Karger, Klein, Tarjan [1993]. Runs in expected time $O(|V| + |E|)$