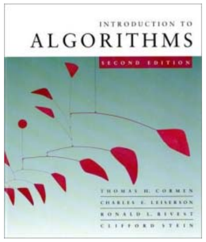

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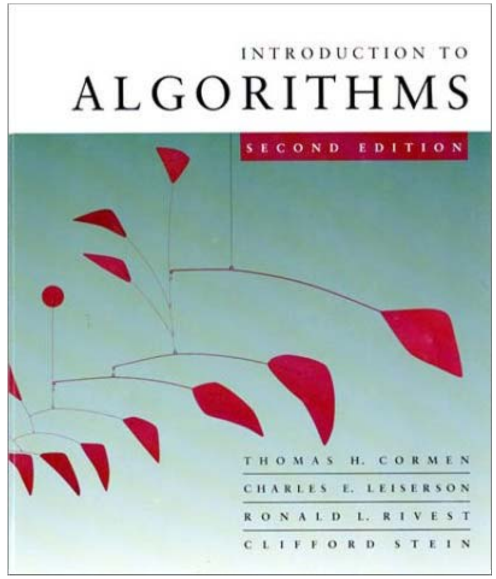
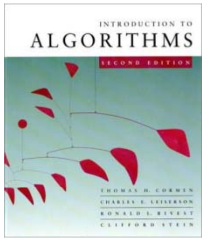


Dynamic Programming

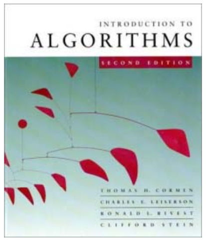
Carola Wenk

Slides courtesy of Charles Leiserson with small changes by Carola Wenk



Dynamic programming

- Algorithm design technique
- A technique for solving problems that have
    1. an optimal substructure property (recursion)
    2. overlapping subproblems
- **Idea:** Do not repeatedly solve the same subproblems, but solve them only once and store the solutions in a **dynamic programming table**



Example: Fibonacci numbers

- F(0)=0; F(1)=1; F(*n*)=F(*n*-1)+F(*n*-2) for $n \geq 2$

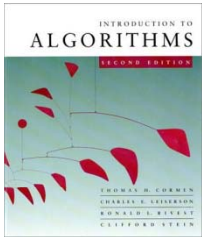
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

Dynamic-programming hallmark #1

Optimal substructure

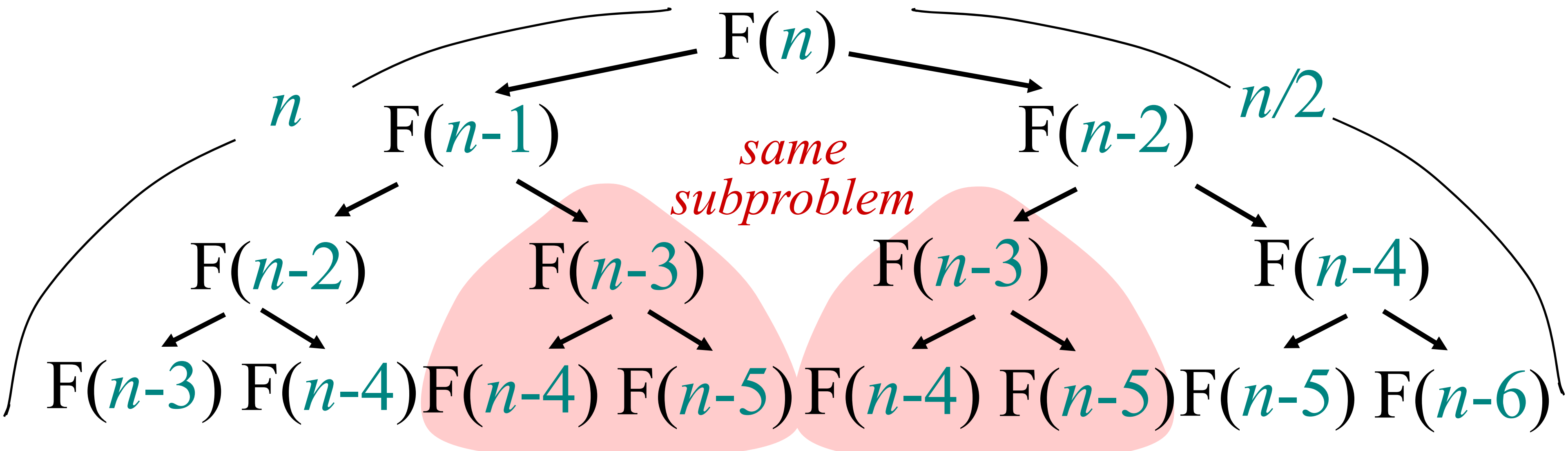
An optimal solution to a problem (instance) contains optimal solutions to subproblems.

→ ***Recursion***

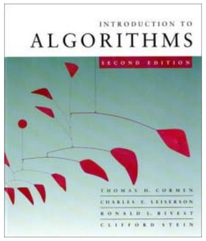


Example: Fibonacci numbers

- F(0)=0; F(1)=1; F(n)=F(n-1)+F(n-2) for $n \geq 2$
- Implement this recursion directly:



- Runtime is exponential: $2^{n/2} \leq T(n) \leq 2^n$
- But we are repeatedly solving the same subproblems

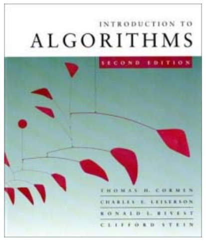


Dynamic-programming hallmark #2

Overlapping subproblems

A recursive solution contains a "small" number of distinct subproblems repeated many times.

The number of distinct Fibonacci subproblems is only n.

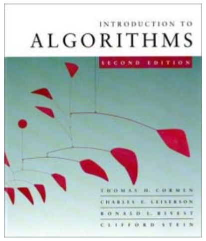


Dynamic-programming

There are two variants of dynamic programming:

1. Bottom-up dynamic programming (often referred to as “dynamic programming”)

2. Memoization



Bottom-up dynamic-programming algorithm

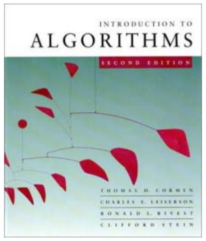
- Store 1D DP-table and fill bottom-up:

F:

0	1	1	2	3	5	8				

fibBottomUpDP(n)
    F[0] ← 0
    F[1] ← 1
    for ($i \leftarrow 2$, $i \leq n$, i++)
        F[i] ← F[i-1]+F[i-2]
    return F[n]

- Time = $\Theta(n)$, space = $\Theta(n)$

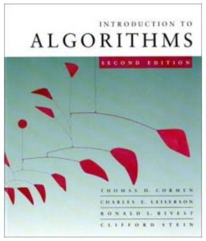


Memoization algorithm

Memoization: Use recursive algorithm. After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

```
fibMemoizationRec(n,F)
    if (F[n]= null)
            if (n=0) F[n] ← 0
            if (n=1) F[n] ← 1
            F[n] ← fibMemoizationRec(n-1,F)
                      + fibMemoizationRec(n-2,F)
    return F[n];
```

- Time = $\Theta(n)$
- Space = $\Theta(n)$



Longest Common Subsequence

Example: ***Longest Common Subsequence (LCS)***

- Given two sequences $x[1 \,..\, m]$ and $y[1 \,..\, n]$, find a longest subsequence common to them both.

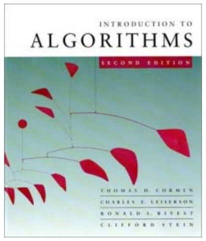
"a" *not* "the"

x: A B C B D A B

y: B D C A B A

BCBA = LCS(x, y)

functional notation, but not a function

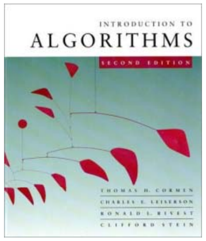


Brute-force LCS algorithm

Check every subsequence of $x[1 \,.\,.\, m]$ to see if it is also a subsequence of $y[1 \,.\,.\, n]$.

Analysis

- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).
- Hence, the runtime would be exponential !



Towards a better algorithm

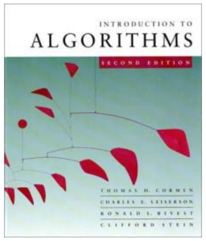
Two-Step Approach:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|s|$.

Strategy: Consider ***prefixes*** of x and y.

- Define $c[i, j] = |\text{LCS}(x[1 \,.\,.\, i], y[1 \,.\,.\, j])|$.
- Then, $c[m, n] = |\text{LCS}(x, y)|$.

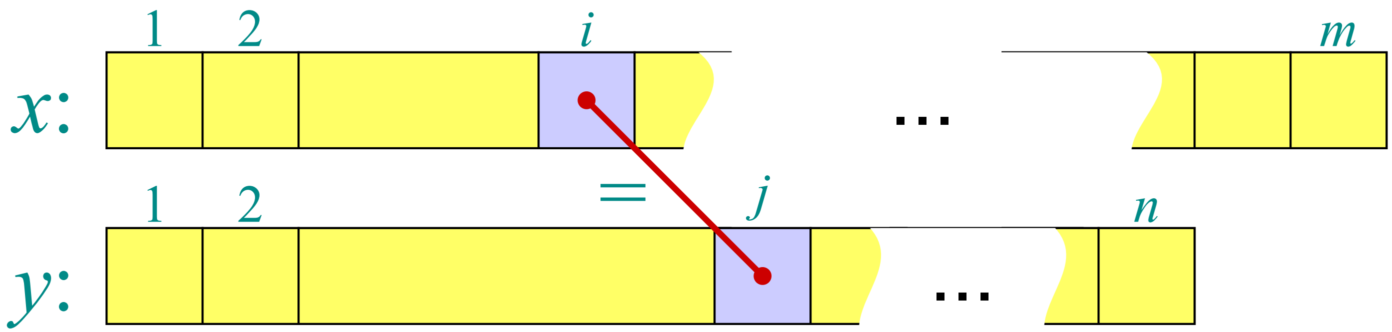


Recursive formulation

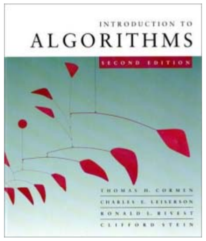
Theorem.

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

Proof. Case $x[i] = y[j]$:



Let $z[1 \mathinner{.\,.} k] = \text{LCS}(x[1 \mathinner{.\,.} i], y[1 \mathinner{.\,.} j])$, where $c[i, j] = k$. Then, $z[k] = x[i]$, or else z could be extended. Thus, $z[1 \mathinner{.\,.} k-1]$ is CS of $x[1 \mathinner{.\,.} i-1]$ and $y[1 \mathinner{.\,.} j-1]$.



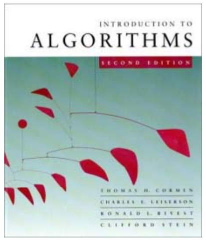
Proof (continued)

Claim: $z[1 \,.\,.\, k-1] = \mathrm{LCS}(x[1 \,.\,.\, i-1], y[1 \,.\,.\, j-1])$.

Suppose w is a longer CS of $x[1 \,.\,.\, i-1]$ and $y[1 \,.\,.\, j-1]$, that is, $|w| > k-1$. Then, ***cut and paste***: $w \,||\, z[k]$ (w concatenated with $z[k]$) is a common subsequence of $x[1 \,.\,.\, i]$ and $y[1 \,.\,.\, j]$ with $\left| w \,||\, z[k] \right| > k$. Contradiction, proving the claim.

Thus, $c[i-1, j-1] = k-1$, which implies that $c[i, j] = c[i-1, j-1] + 1$.

Other cases are similar. ☐



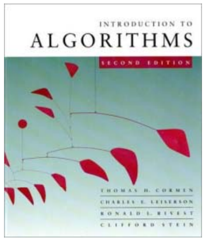
Dynamic-programming hallmark #1

Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

→ ***Recursion***

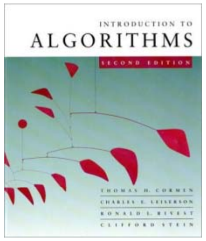
If $z = \mathrm{LCS}(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y.



Recursive algorithm for LCS

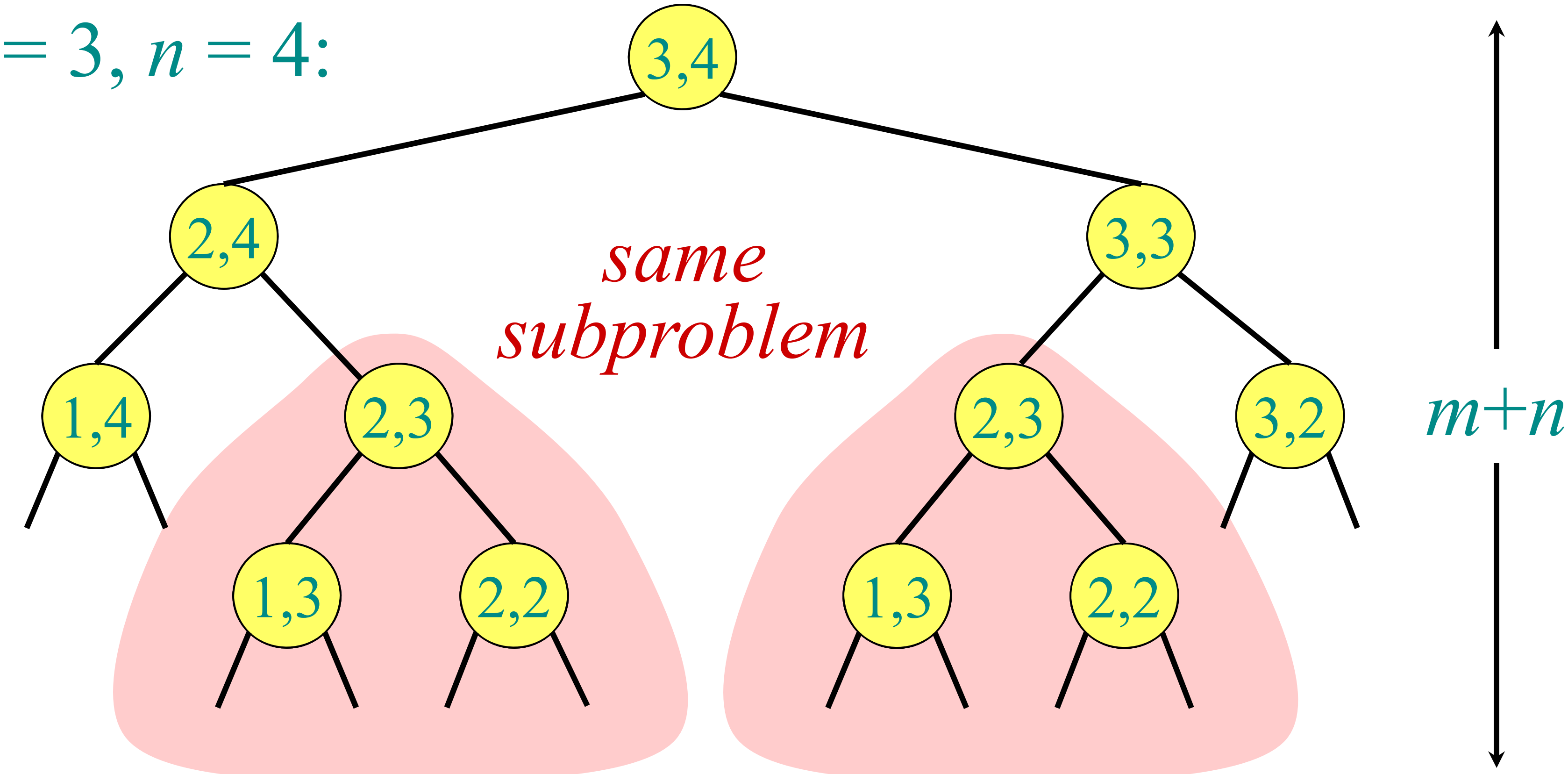
LCS(x, y, i, j)
    if $x[i] = y[j]$
        then $c[i, j] \leftarrow$ LCS$(x, y, i-1, j-1) + 1$
        else $c[i, j] \leftarrow \max\{LCS(x, y, i-1, j)$,
                                LCS$(x, y, i, j-1)\}$

Worst-case: $x[i] \neq y[j]$, in which case the algorithm evaluates two subproblems, each with only one parameter decremented.

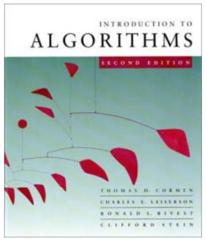


Recursion tree

$m = 3,\ n = 4$:



Height $= m + n \Rightarrow$ work potentially exponential, but we're solving subproblems already solved!

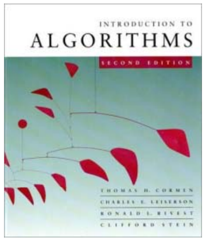


Dynamic-programming hallmark #2

Overlapping subproblems

A recursive solution contains a "small" number of distinct subproblems repeated many times.

The number of distinct LCS subproblems for two strings of lengths m and n is only mn.



Memoization algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

for all i, j**:** $c[i,0]=0$ **and** $c[0, j]=0$

LCS(x, y, i, j)

  if $c[i, j] = \text{NIL}$

    then if $x[i] = y[j]$

      then $c[i, j] \leftarrow \text{LCS}(x, y, i-1, j-1) + 1$

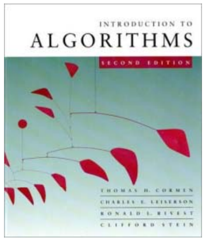
      else $c[i, j] \leftarrow \max\{\text{LCS}(x, y, i-1, j), \text{LCS}(x, y, i, j-1)\}$

  return $c[i,j]$

same as before (applies to the **then if** … **else** lines)

Time $= \Theta(m\ n)$ = constant work per table entry.

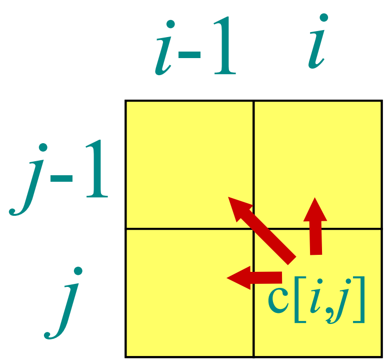
Space $= \Theta(m\ n)$.

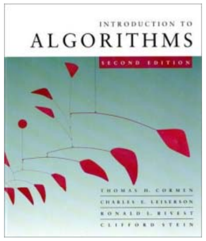


Recursive formulation

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

c:





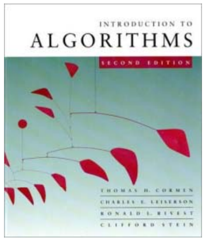
Bottom-up dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4



Bottom-up dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct LCS by back-tracing.

Space = $\Theta(mn)$.

Exercise: $O(\min\{m, n\})$.

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4