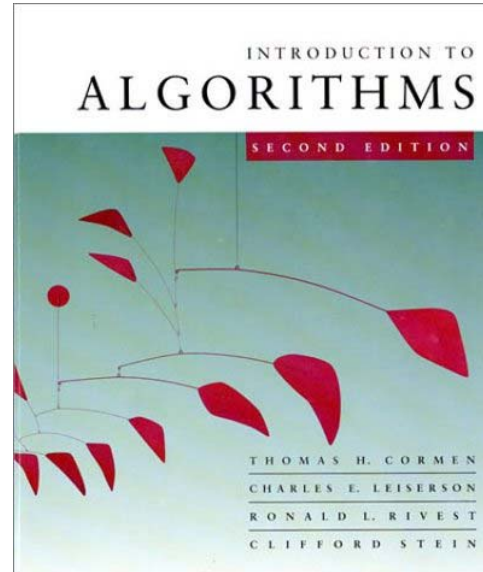




CS 5633 -- Spring 2010



Minimum Spanning Trees

Carola Wenk

Slides courtesy of Charles Leiserson with changes and additions by Carola Wenk



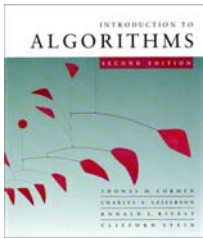
Minimum spanning trees

Input: A connected, undirected graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{R}$.

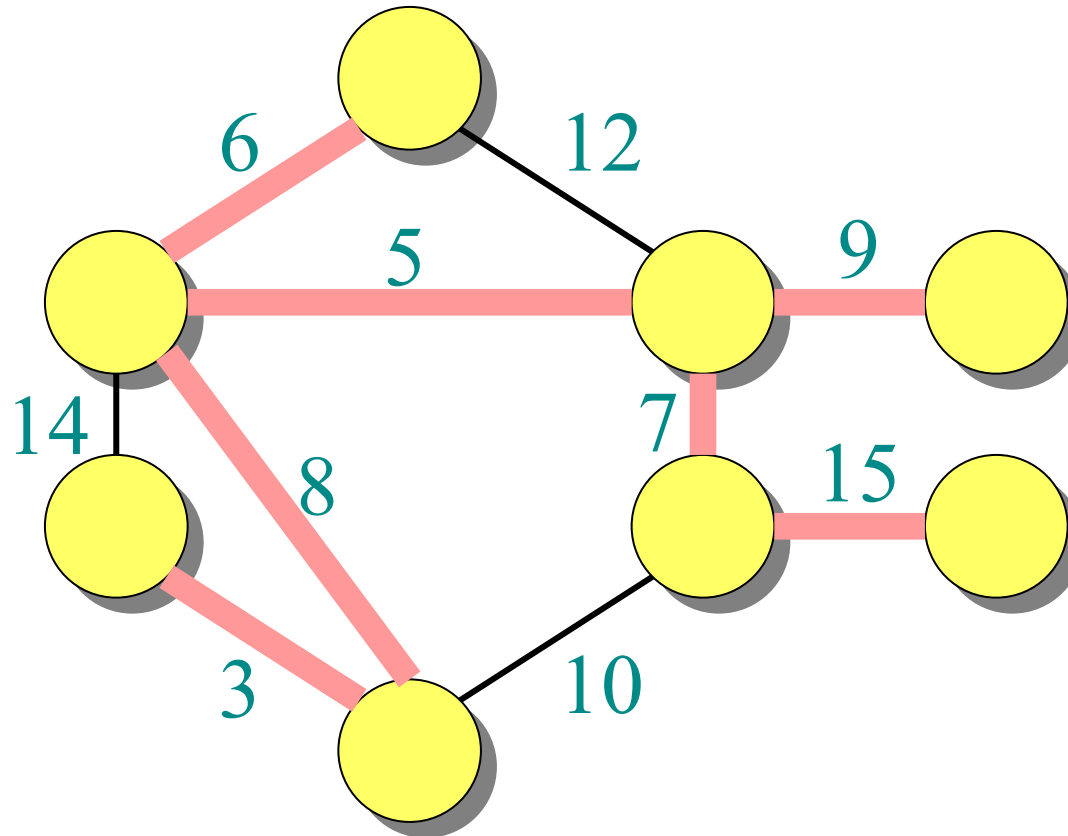
- For simplicity, assume that all edge weights are distinct. (CLRS covers the general case.)

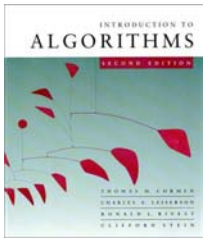
Output: A *spanning tree* T — a tree that connects all vertices — of minimum weight:

$$w(T) = \sum_{(u,v) \in T} w(u,v).$$



Example of MST





Hallmark for “greedy” algorithms

***Greedy-choice property**
A locally optimal choice
is globally optimal.*

Theorem. Let T be the MST of $G = (V, E)$, and let $A \subseteq V$. Suppose that $(u, v) \in E$ is the least-weight edge connecting A to $V \setminus A$. Then, $(u, v) \in T$.

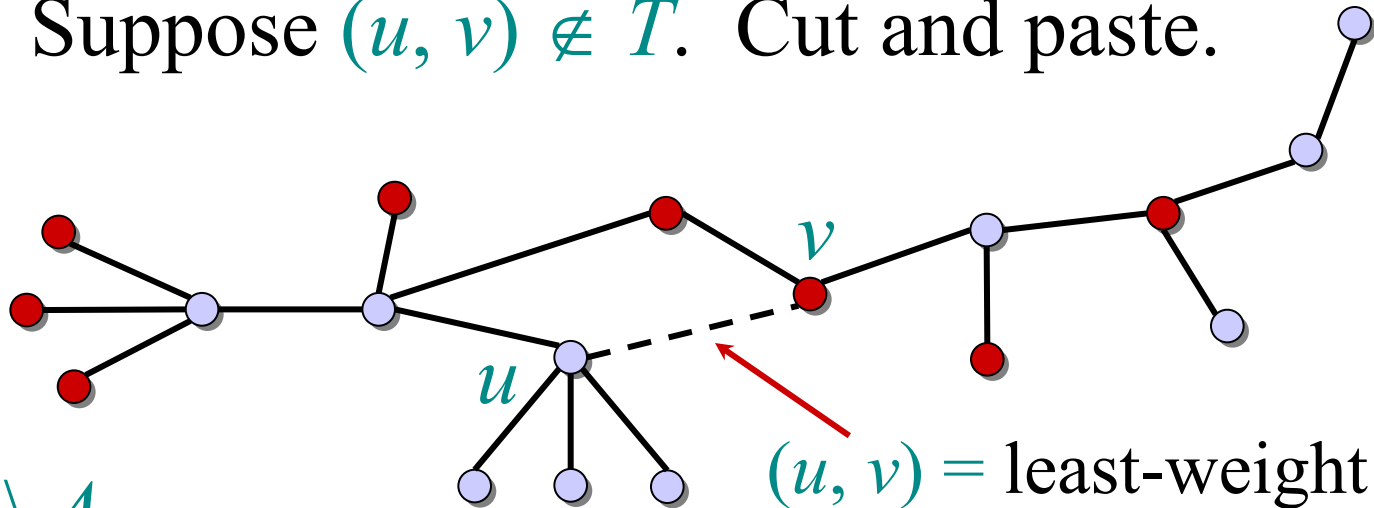


Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.

T :

$\circ \in A$
 $\bullet \in V \setminus A$



$(u, v) =$ least-weight edge connecting A to $V \setminus A$

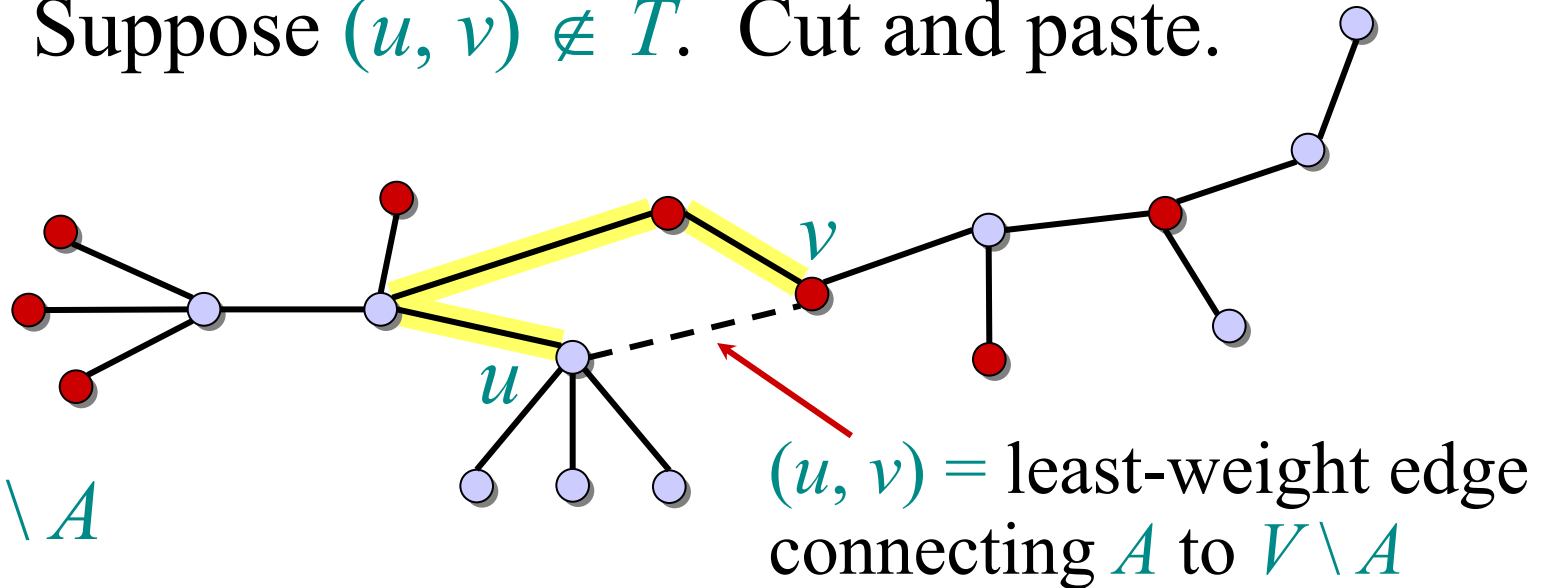


Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.

T :

$\bullet \in A$
 $\bullet \in V \setminus A$



Consider the unique simple path from u to v in T .

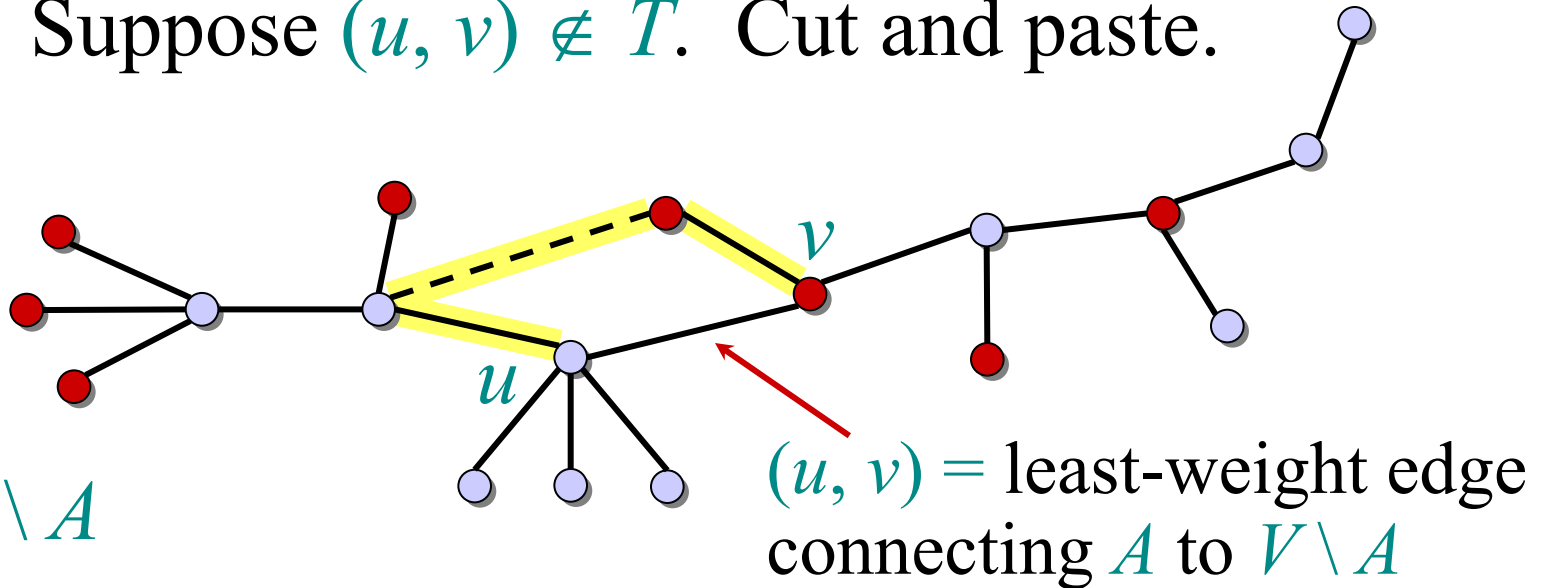


Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.

T :

- $\bullet \in A$
- $\bullet \in V \setminus A$



Consider the unique simple path from u to v in T .

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V \setminus A$.

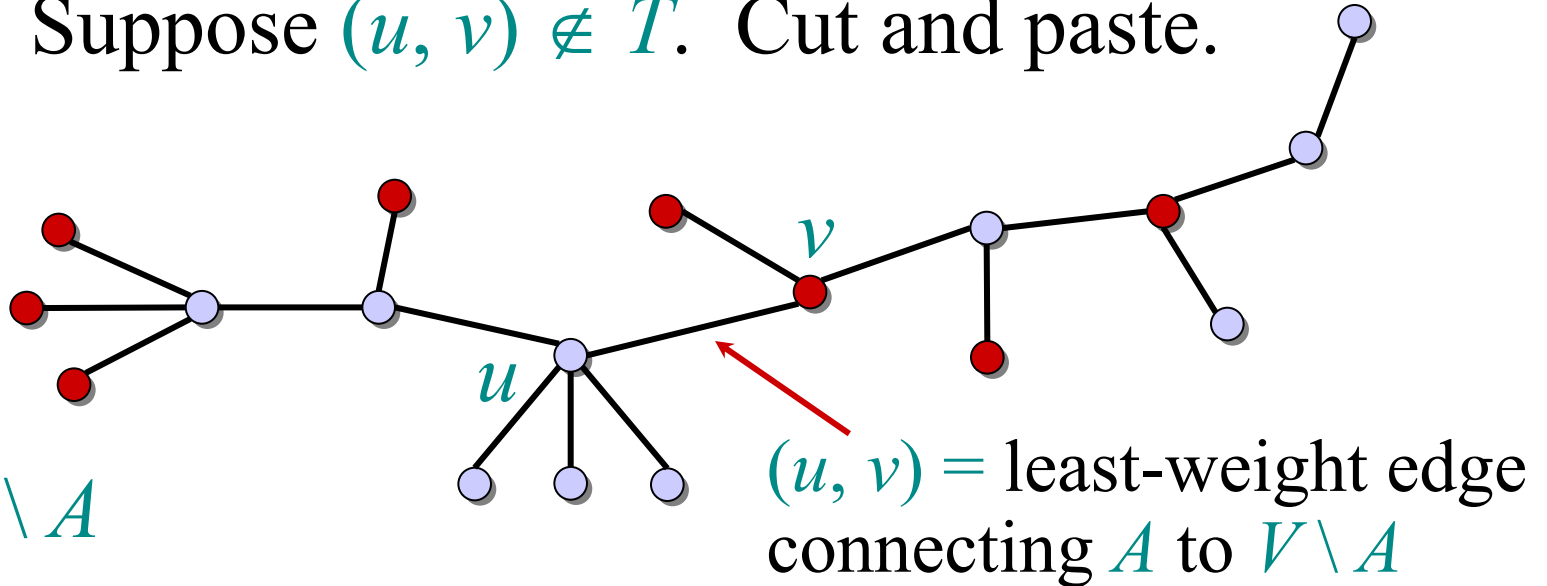


Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.

T' :

- $\bullet \in A$
- $\bullet \in V \setminus A$



Consider the unique simple path from u to v in T .

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V \setminus A$.

A lighter-weight spanning tree than T results. □



Prim's algorithm

IDEA: Maintain $V \setminus A$ as a priority queue Q . Key each vertex in Q with the weight of the least-weight edge connecting it to a vertex in A .

$Q \leftarrow V$

$key[v] \leftarrow \infty$ for all $v \in V$

$key[s] \leftarrow 0$ for some arbitrary $s \in V$

while $Q \neq \emptyset$

do $u \leftarrow \text{EXTRACT-MIN}(Q)$

for each $v \in \text{Adj}[u]$

do if $v \in Q$ and $w(u, v) < key[v]$

then $key[v] \leftarrow w(u, v)$ \triangleright DECREASE-KEY

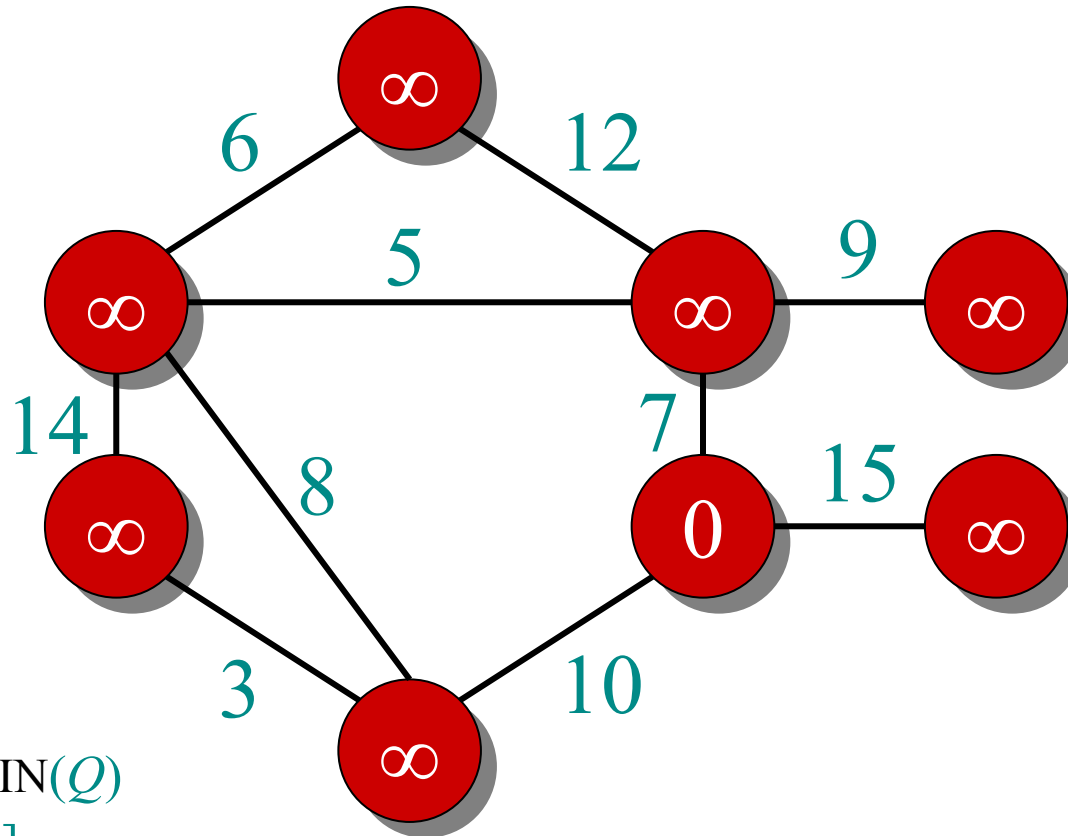
$\pi[v] \leftarrow u$

At the end, $\{(v, \pi[v])\}$ forms the MST.



Example of Prim's algorithm

- $\in A$
- $\in V \setminus A$



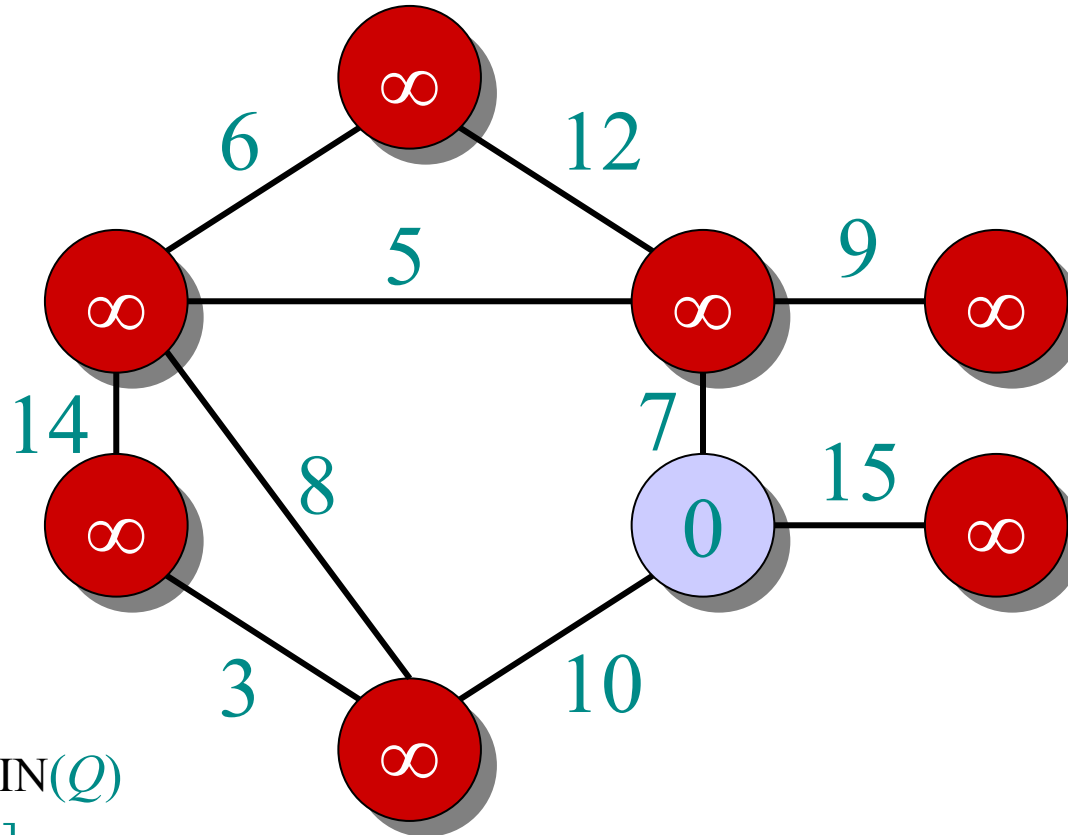
```

u ← EXTRACT-MIN(Q)
for each v ∈ Adj[u]
  do if v ∈ Q and w(u, v) < key[v]
     then key[v] ← w(u, v) ▷ DECREASE-KEY
        π[v] ← u
  
```



Example of Prim's algorithm

- $\circ \in A$
- $\bullet \in V \setminus A$

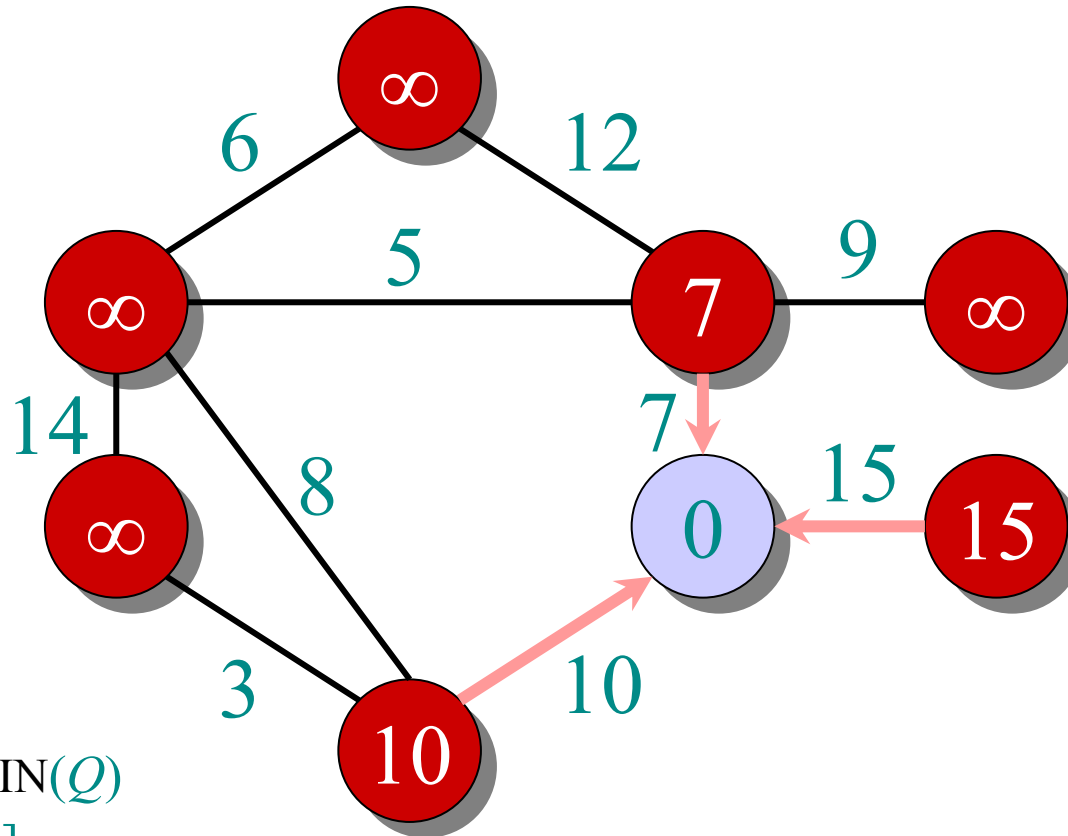


```
 $u \leftarrow \text{EXTRACT-MIN}(Q)$   
for each  $v \in \text{Adj}[u]$   
  do if  $v \in Q$  and  $w(u, v) < \text{key}[v]$   
    then  $\text{key}[v] \leftarrow w(u, v) \triangleright \text{DECREASE-KEY}$   
         $\pi[v] \leftarrow u$ 
```



Example of Prim's algorithm

- $\in A$
- $\in V \setminus A$



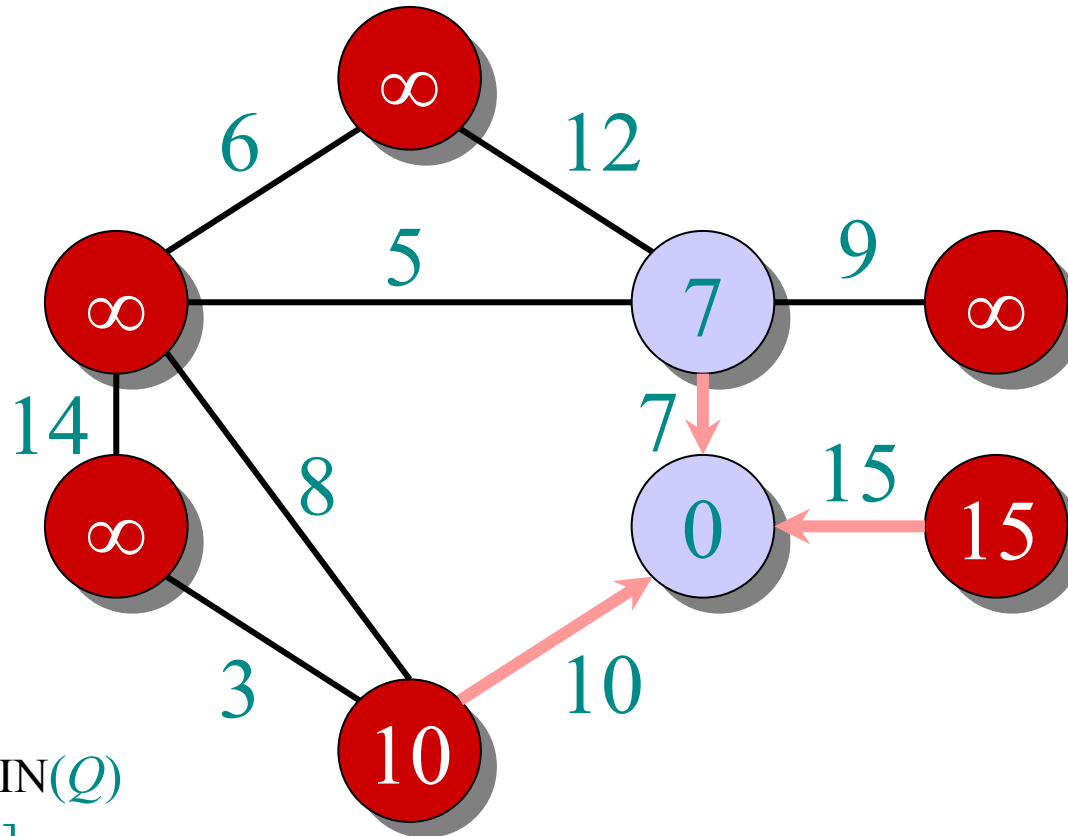
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Example of Prim's algorithm

- $\in A$
- $\in V \setminus A$



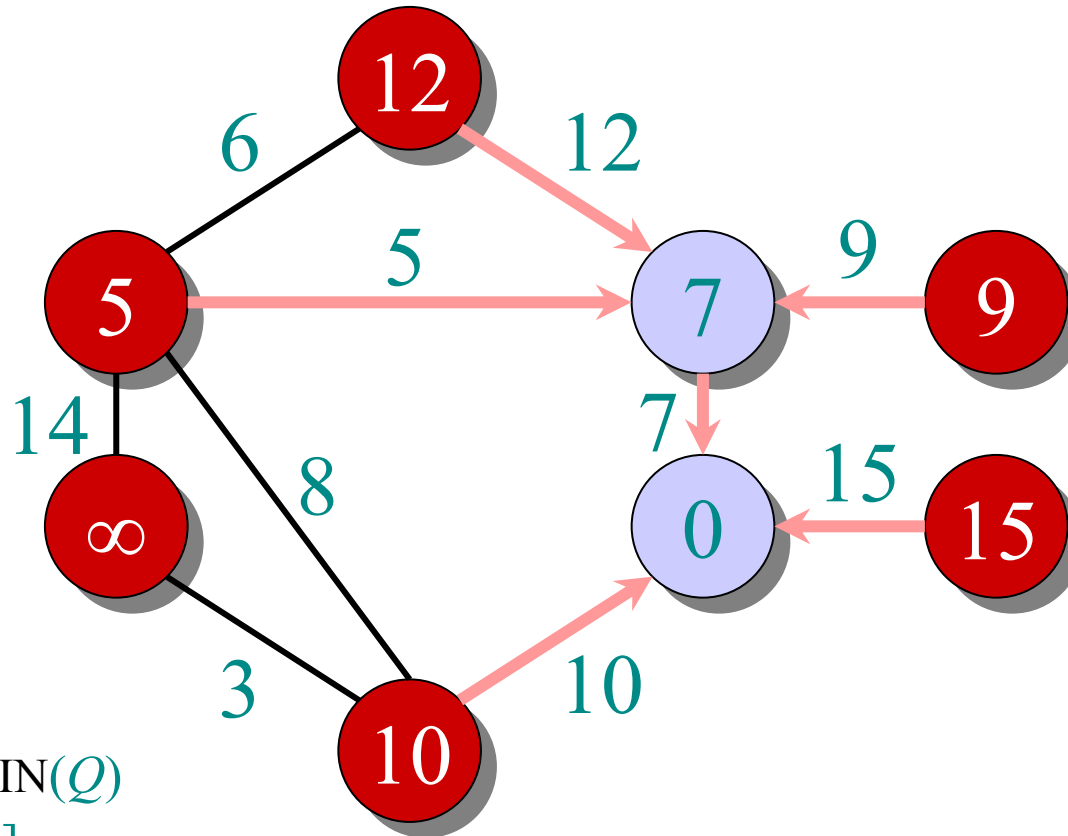
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Example of Prim's algorithm

- $\in A$
- $\in V \setminus A$



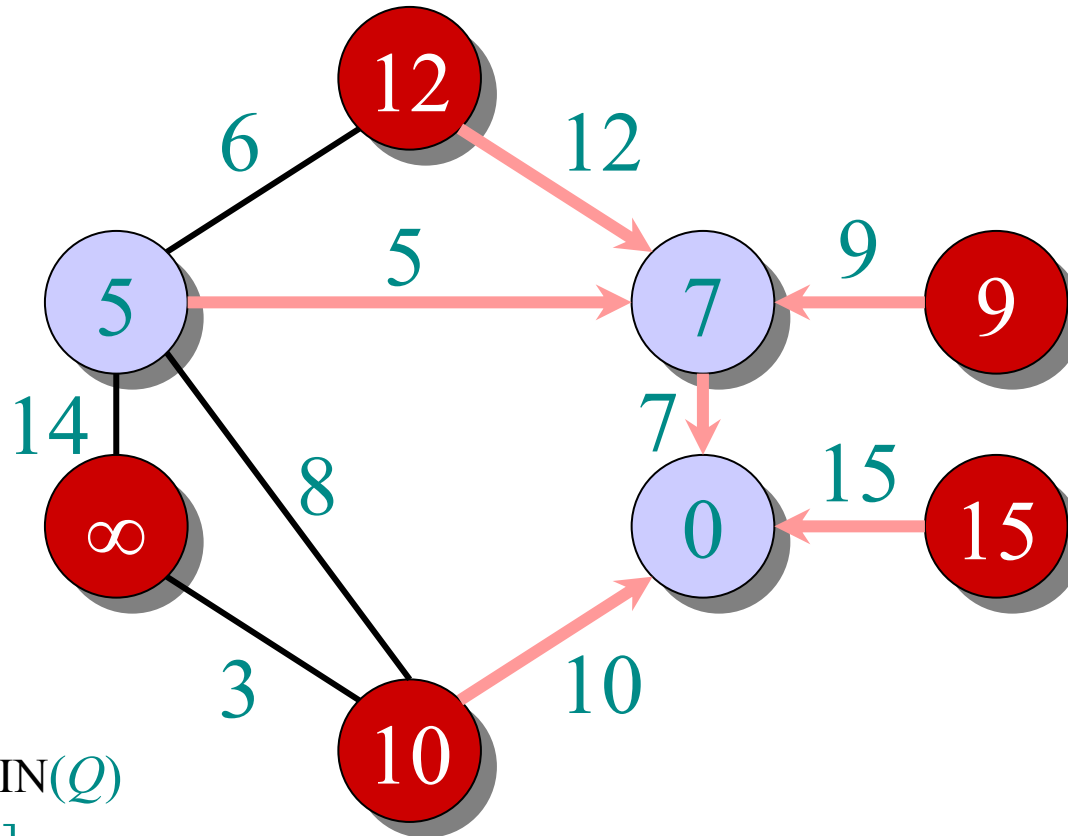
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Example of Prim's algorithm

- $\in A$
- $\in V \setminus A$



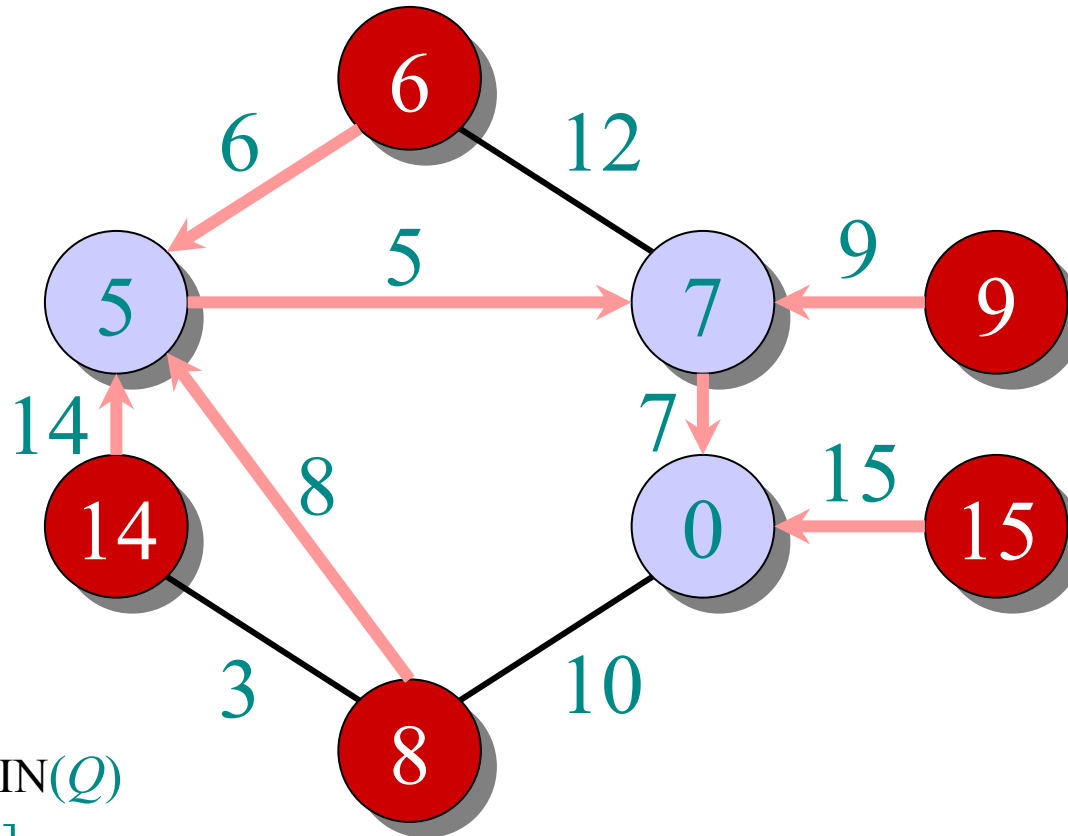
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Example of Prim's algorithm

● $\in A$
● $\in V \setminus A$



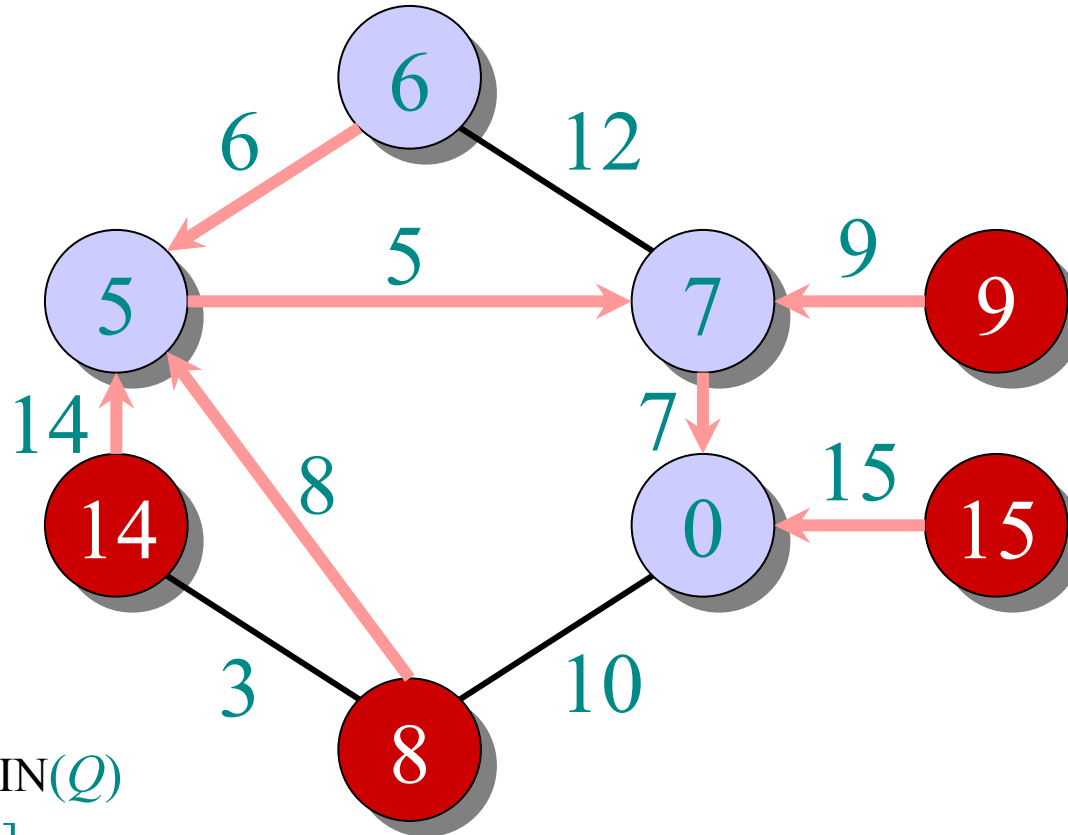
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Example of Prim's algorithm

- $\in A$
- $\in V \setminus A$



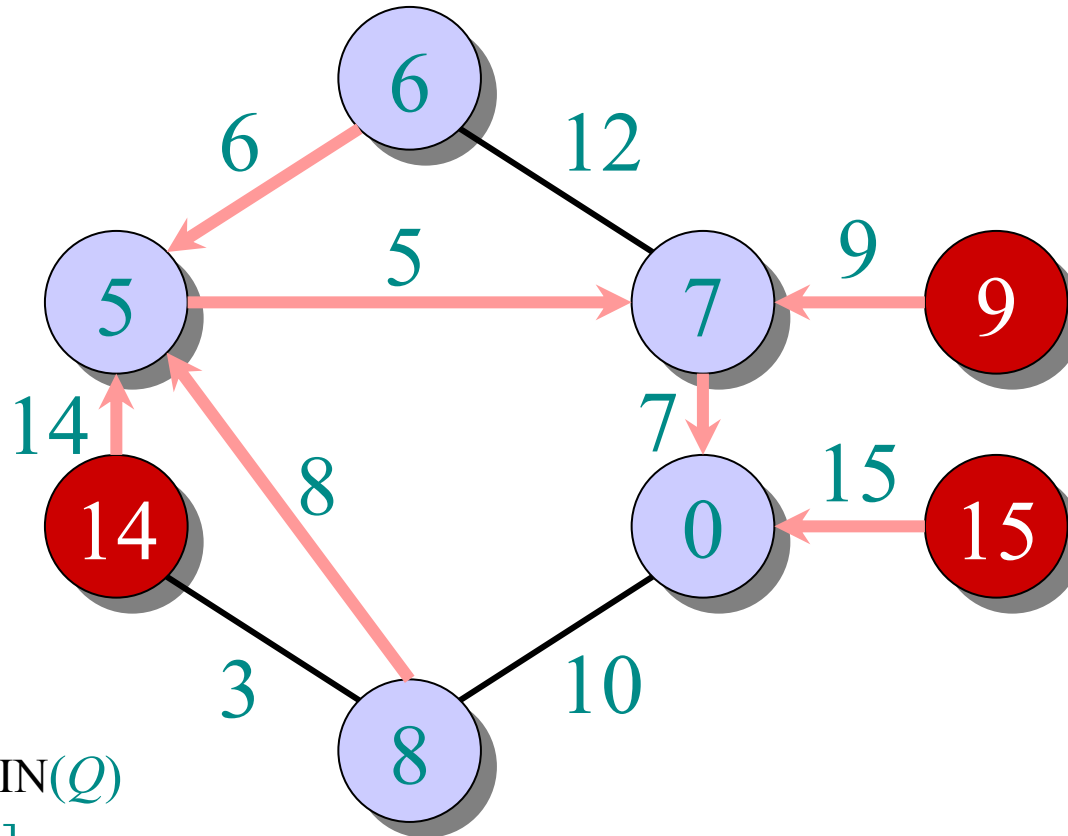
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Example of Prim's algorithm

- $\in A$
- $\in V \setminus A$



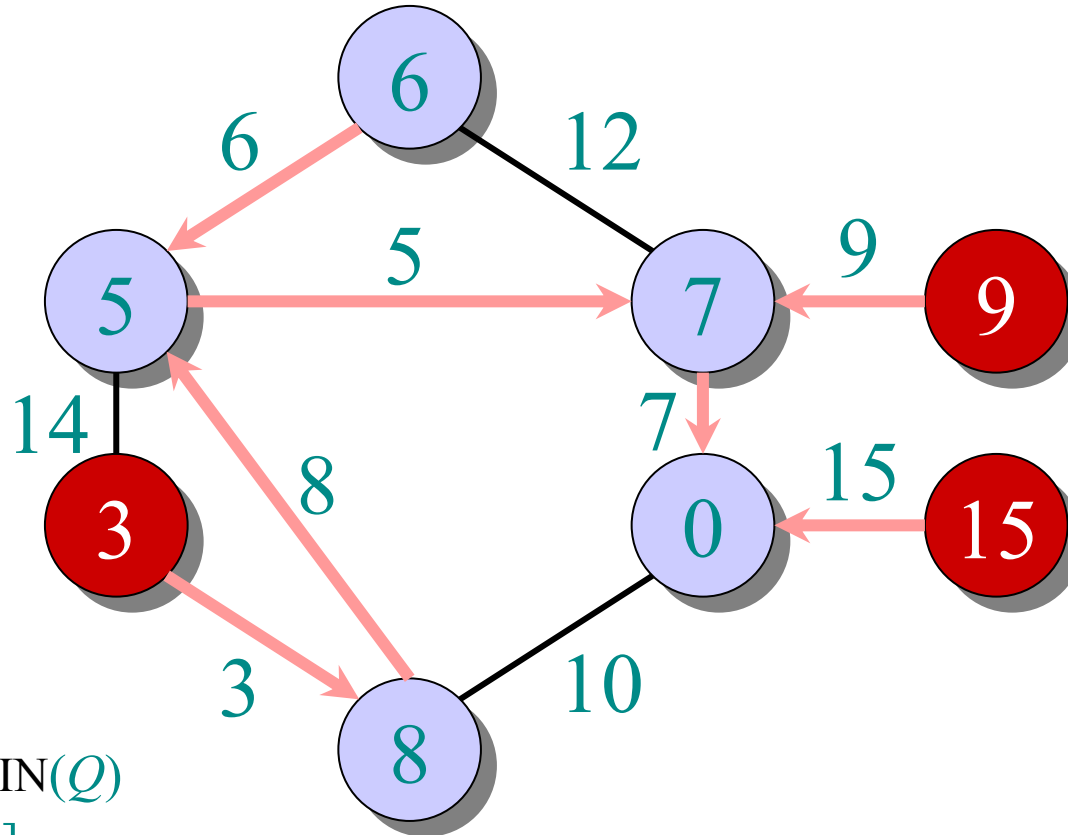
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Example of Prim's algorithm

- $\in A$
- $\in V \setminus A$



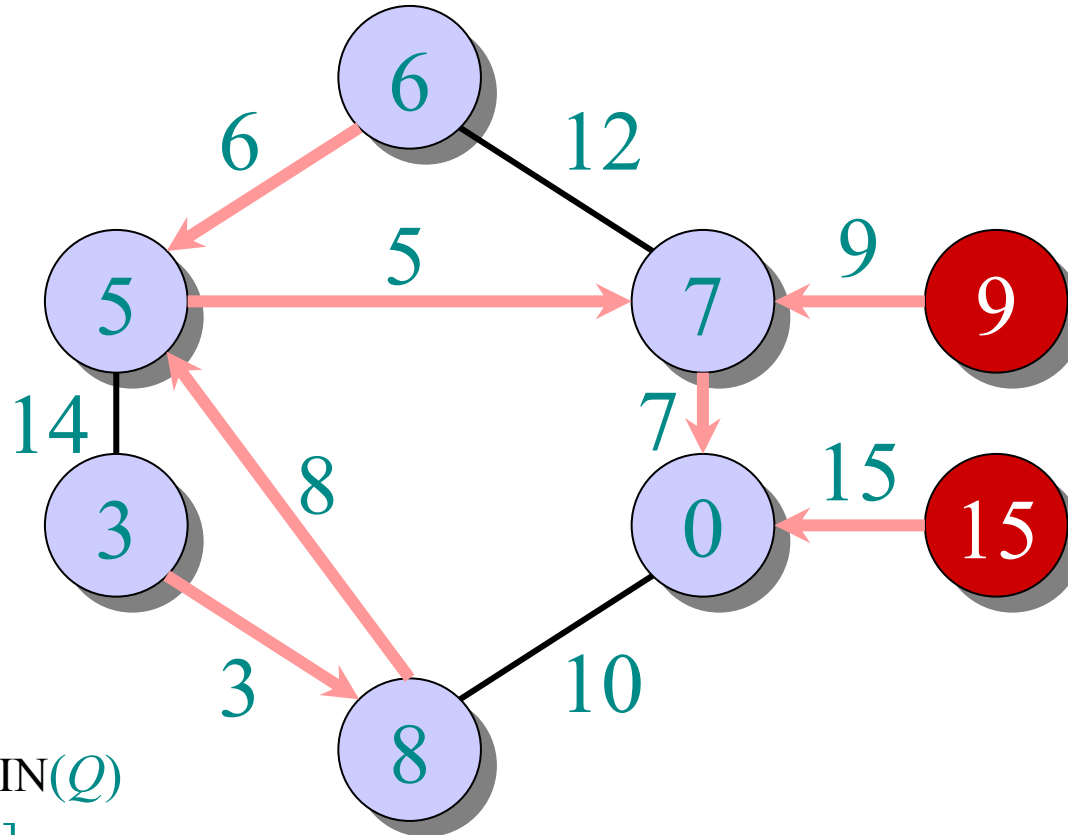
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```



Example of Prim's algorithm

● $\in A$
● $\in V \setminus A$



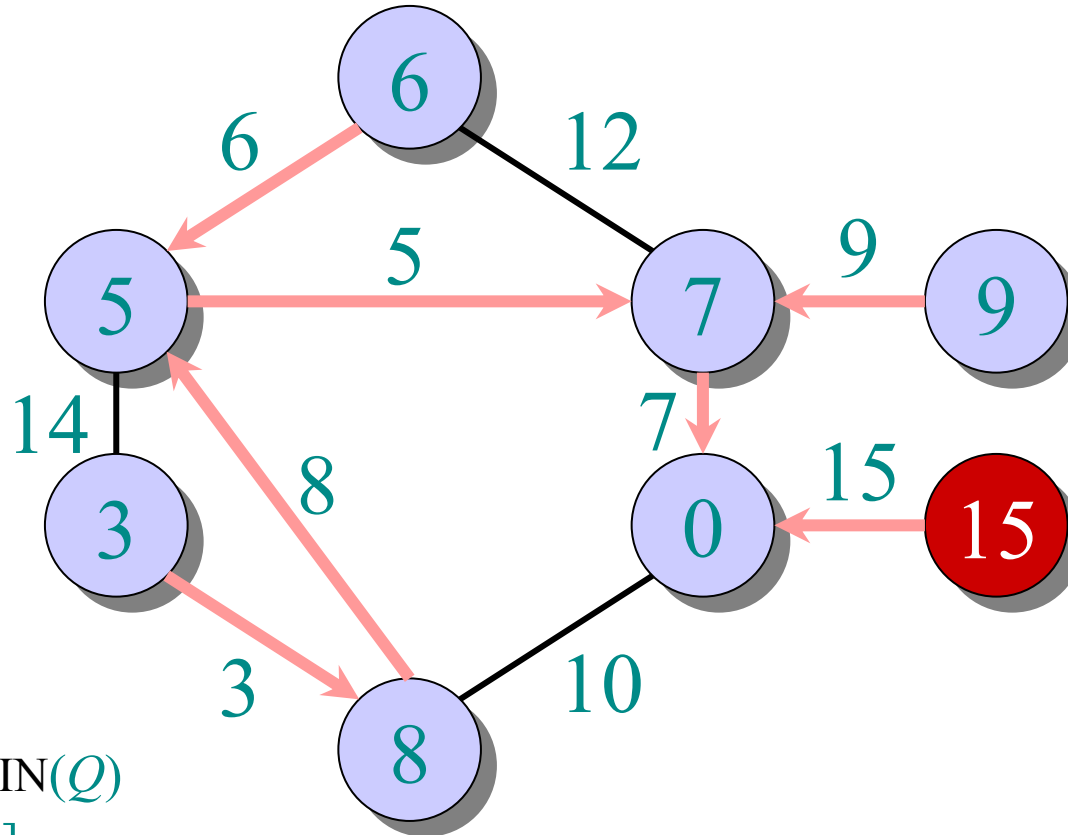
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Example of Prim's algorithm

- $\in A$
- $\in V \setminus A$



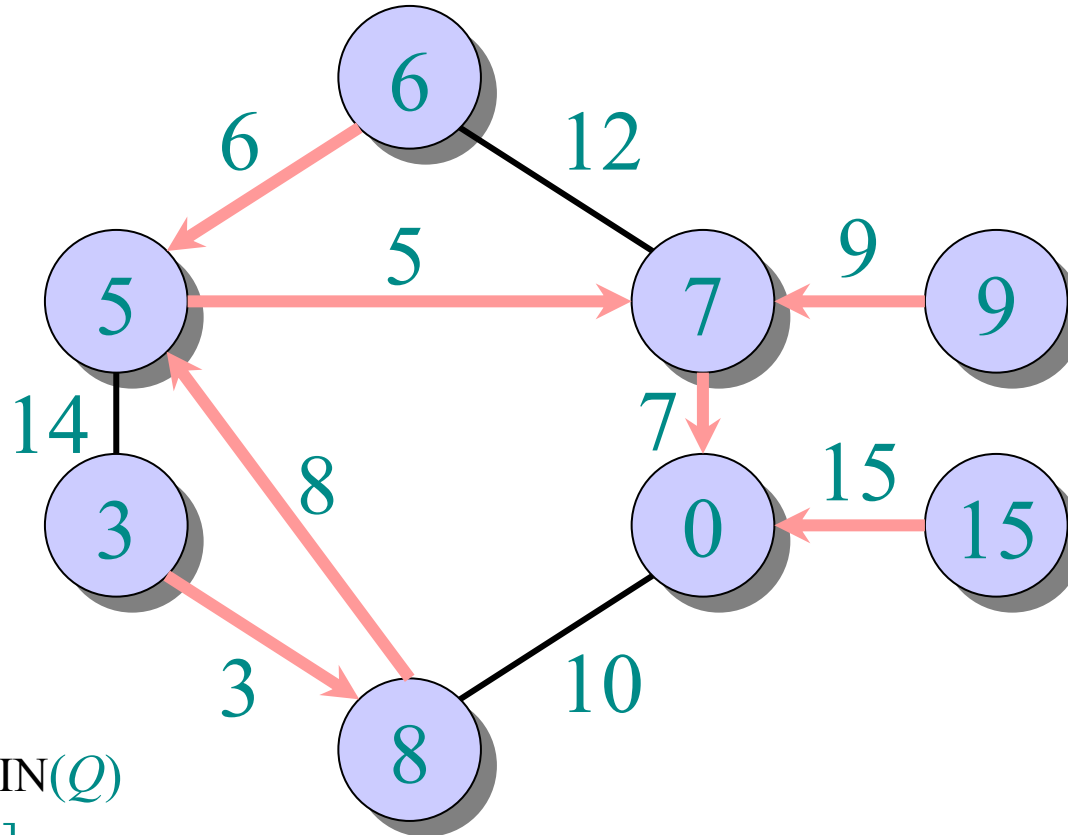
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Example of Prim's algorithm

- $\in A$
- $\in V \setminus A$



```

u ← EXTRACT-MIN(Q)
for each v ∈ Adj[u]
  do if v ∈ Q and w(u, v) < key[v]
     then key[v] ← w(u, v) ▷ DECREASE-KEY
        π[v] ← u
  
```



Analysis of Prim

$\Theta(|V|)$ total

$Q \leftarrow V$
 $key[v] \leftarrow \infty$ for all $v \in V$
 $key[s] \leftarrow 0$ for some arbitrary $s \in V$

while $Q \neq \emptyset$

do $u \leftarrow \text{EXTRACT-MIN}(Q)$

for each $v \in \text{Adj}[u]$

do if $v \in Q$ and $w(u, v) < key[v]$

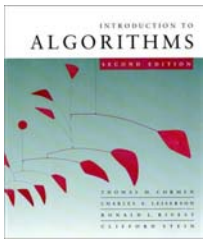
then $key[v] \leftarrow w(u, v)$
 $\pi[v] \leftarrow u$

$|V|$ times

$degree(u)$ times

Handshaking Lemma $\Rightarrow \Theta(|E|)$ implicit DECREASE-KEY's.

$$\text{Time} = \Theta(|V|) \cdot T_{\text{EXTRACT-MIN}} + \Theta(|E|) \cdot T_{\text{DECREASE-KEY}}$$



Analysis of Prim (continued)

$$\text{Time} = \Theta(|V|) \cdot T_{\text{EXTRACT-MIN}} + \Theta(|E|) \cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
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array	$O(V)$	$O(1)$	$O(V ^2)$
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binary heap	$O(\log V)$	$O(\log V)$	$O(E \log V)$
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Fibonacci heap	$O(\log V)$ amortized	$O(1)$ amortized	$O(E + V \log V)$ worst case
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Kruskal's algorithm

IDEA (again greedy):

Repeatedly pick edge with smallest weight as long as it does not form a cycle.

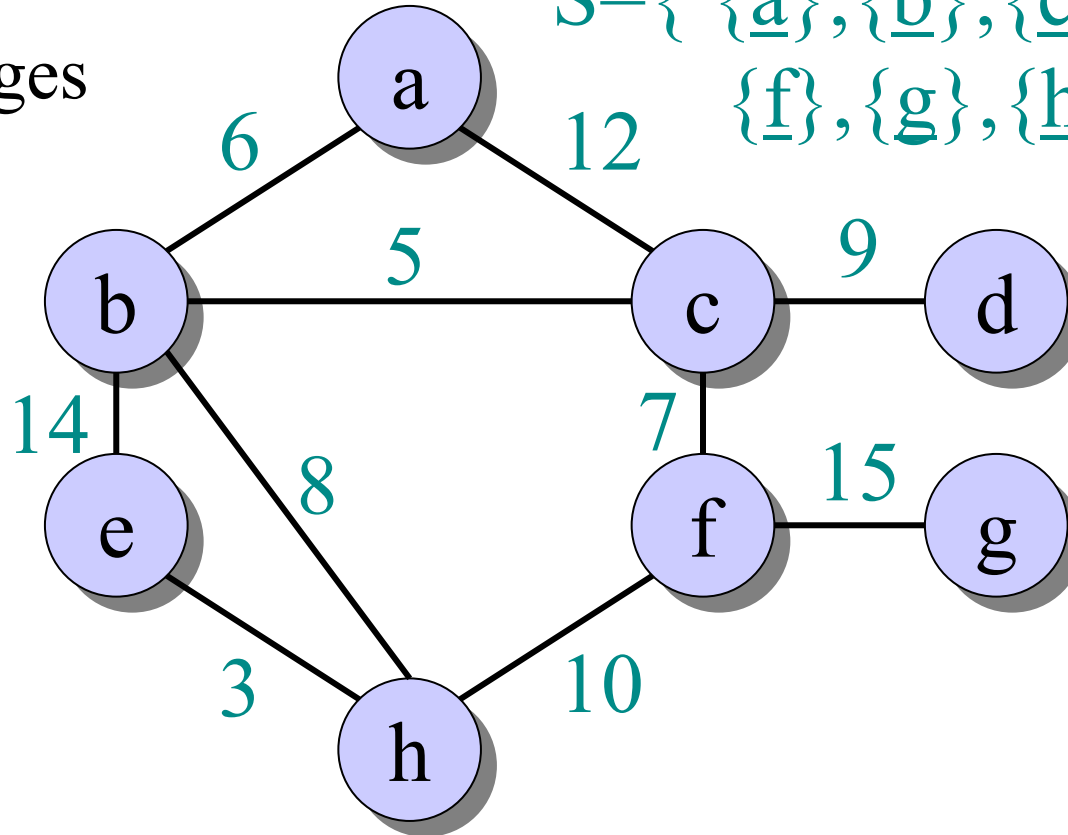
- The algorithm creates a set of trees (a **forest**)
- During the algorithm the added edges merge the trees together, such that in the end only one tree remains
- The correctness of this greedy strategy is not obvious and needs to be proven. (Proof skipped here.)



Example of Kruskal's algorithm

— MST edges
a set repr.

$S = \{ \{\underline{a}\}, \{\underline{b}\}, \{\underline{c}\}, \{\underline{d}\}, \{\underline{e}\} \}$
 $\{ \{\underline{f}\}, \{\underline{g}\}, \{\underline{h}\} \}$



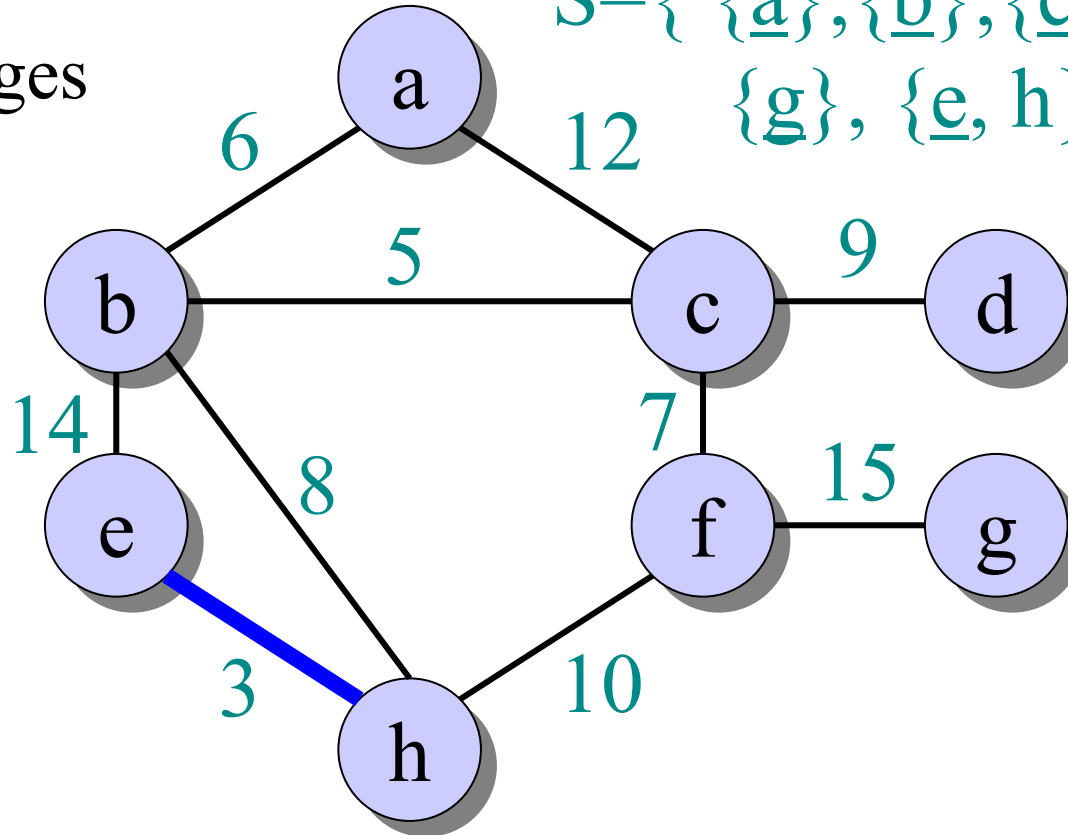
Every node is a single tree.



Example of Kruskal's algorithm

— MST edges
a set repr.

$S = \{ \{\underline{a}\}, \{\underline{b}\}, \{\underline{c}\}, \{\underline{d}\}, \{\underline{f}\} \}$
 $\{ \underline{g}\}, \{\underline{e}, h\} \}$

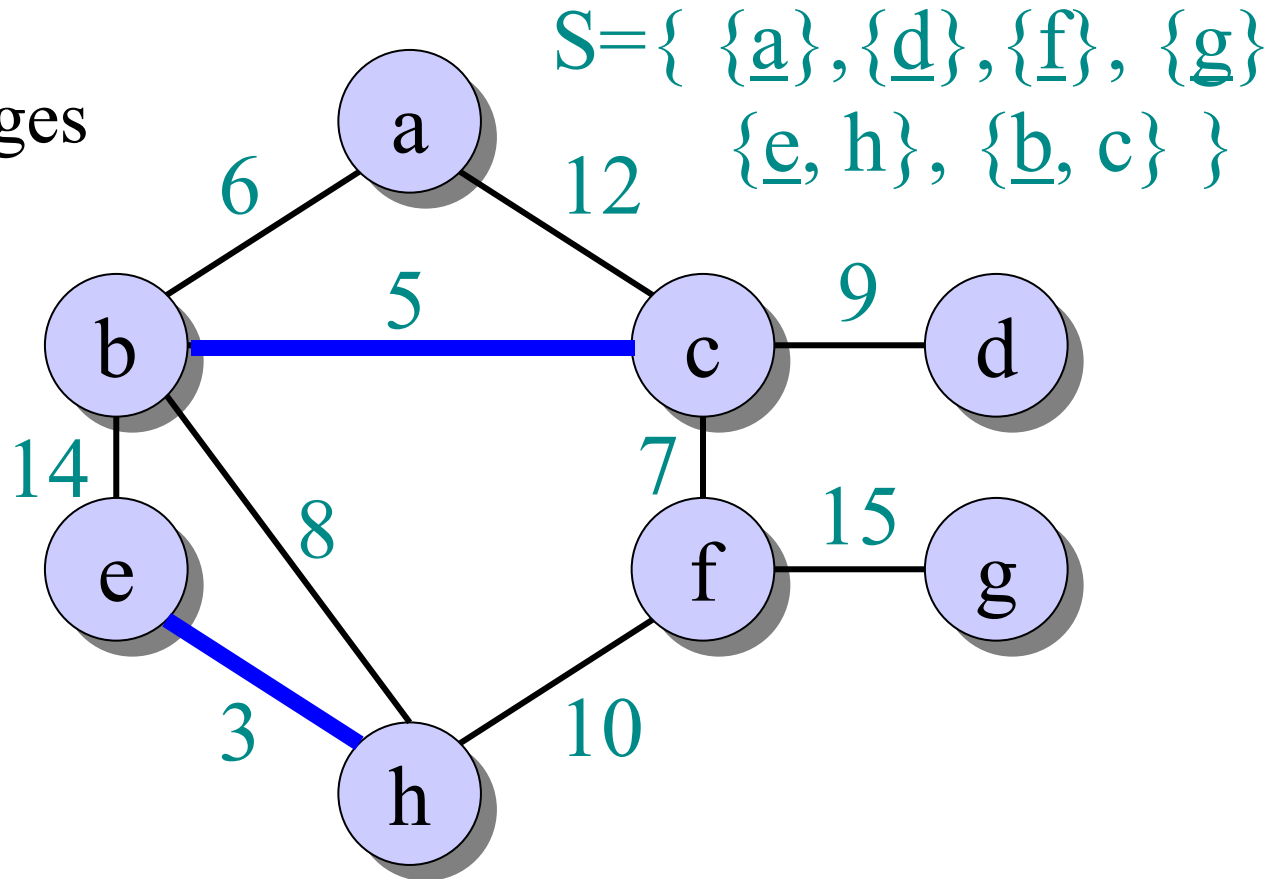


Edge 3 merged two singleton trees.



Example of Kruskal's algorithm

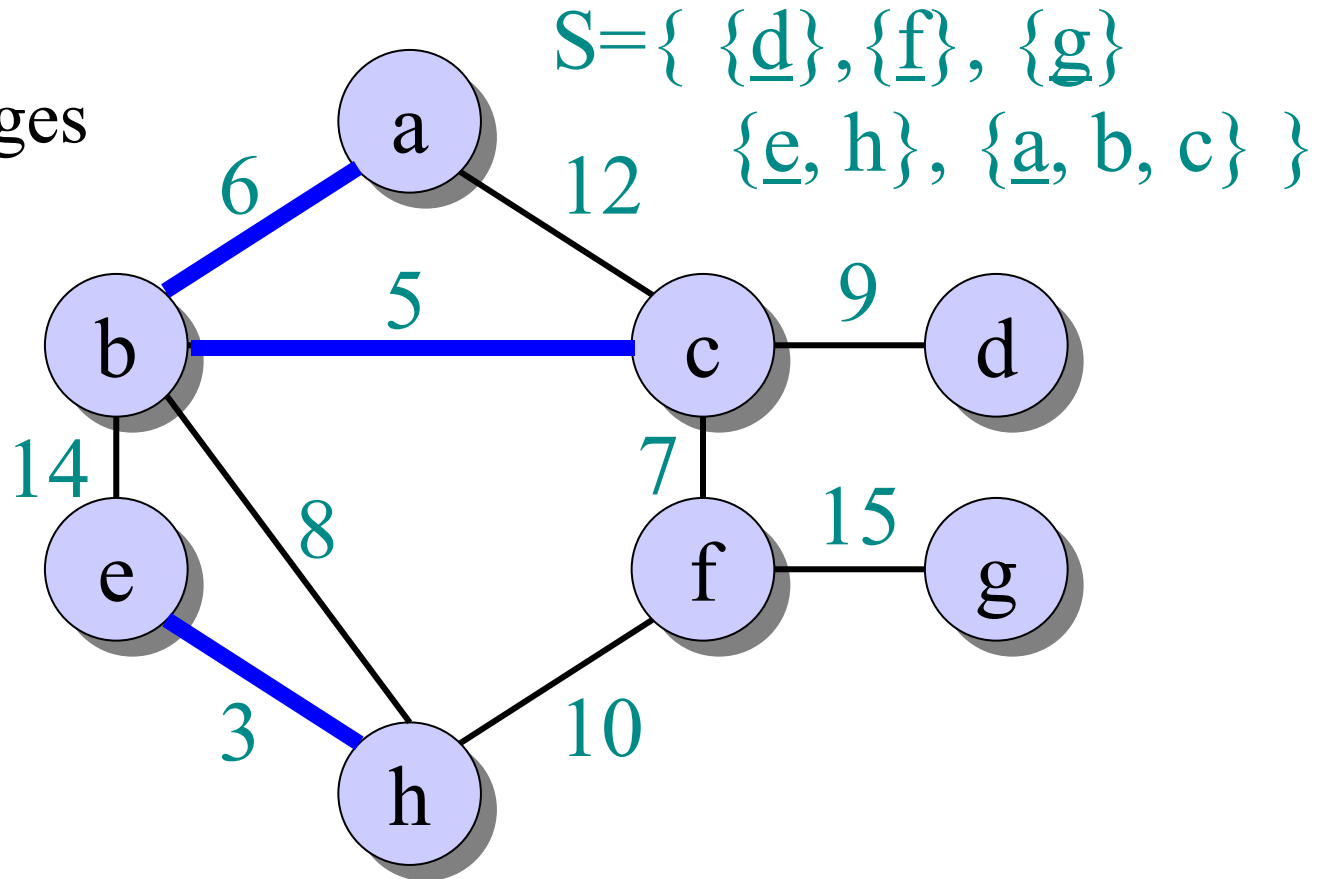
— MST edges
a set repr.





Example of Kruskal's algorithm

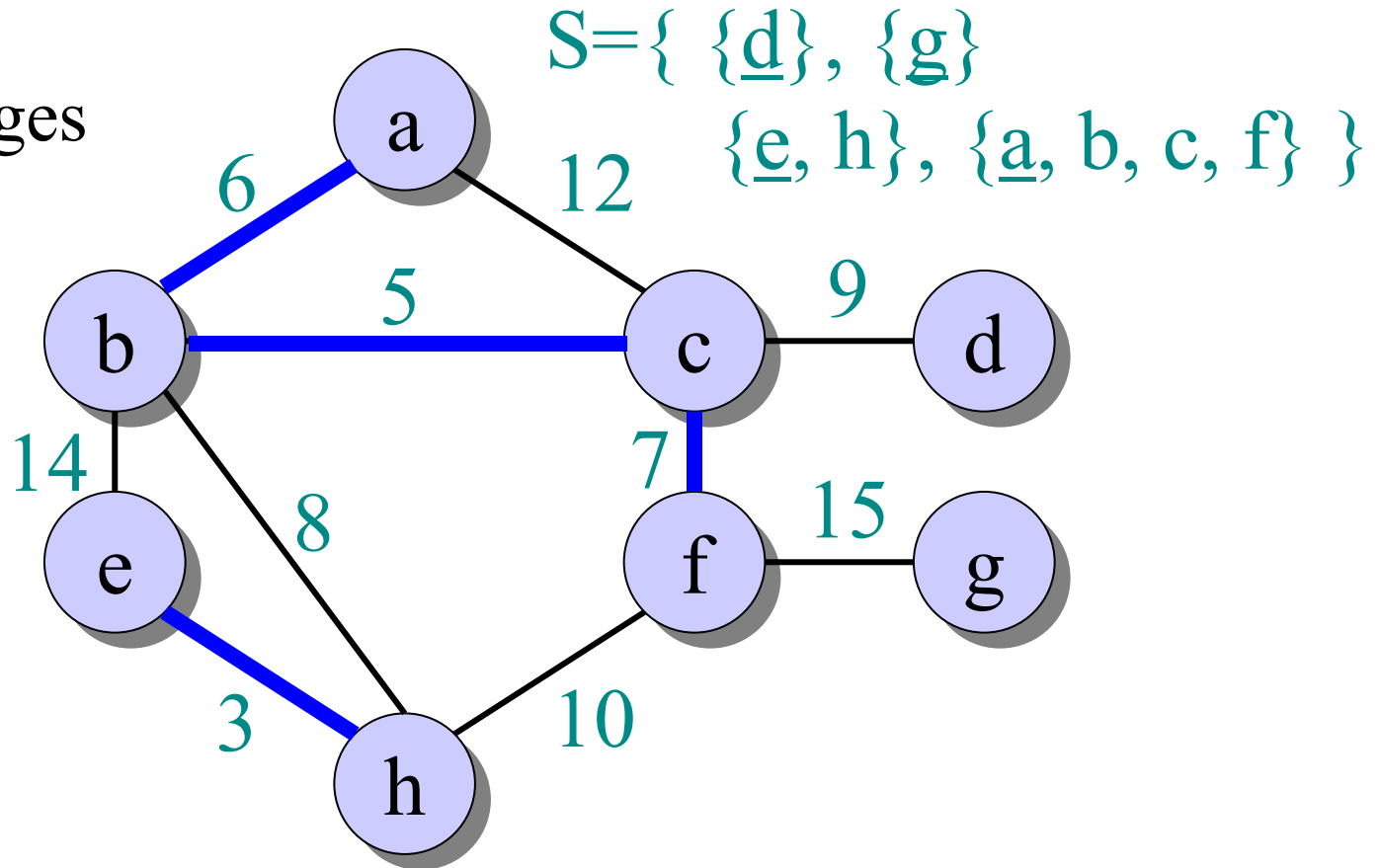
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Example of Kruskal's algorithm

— MST edges
a set repr.

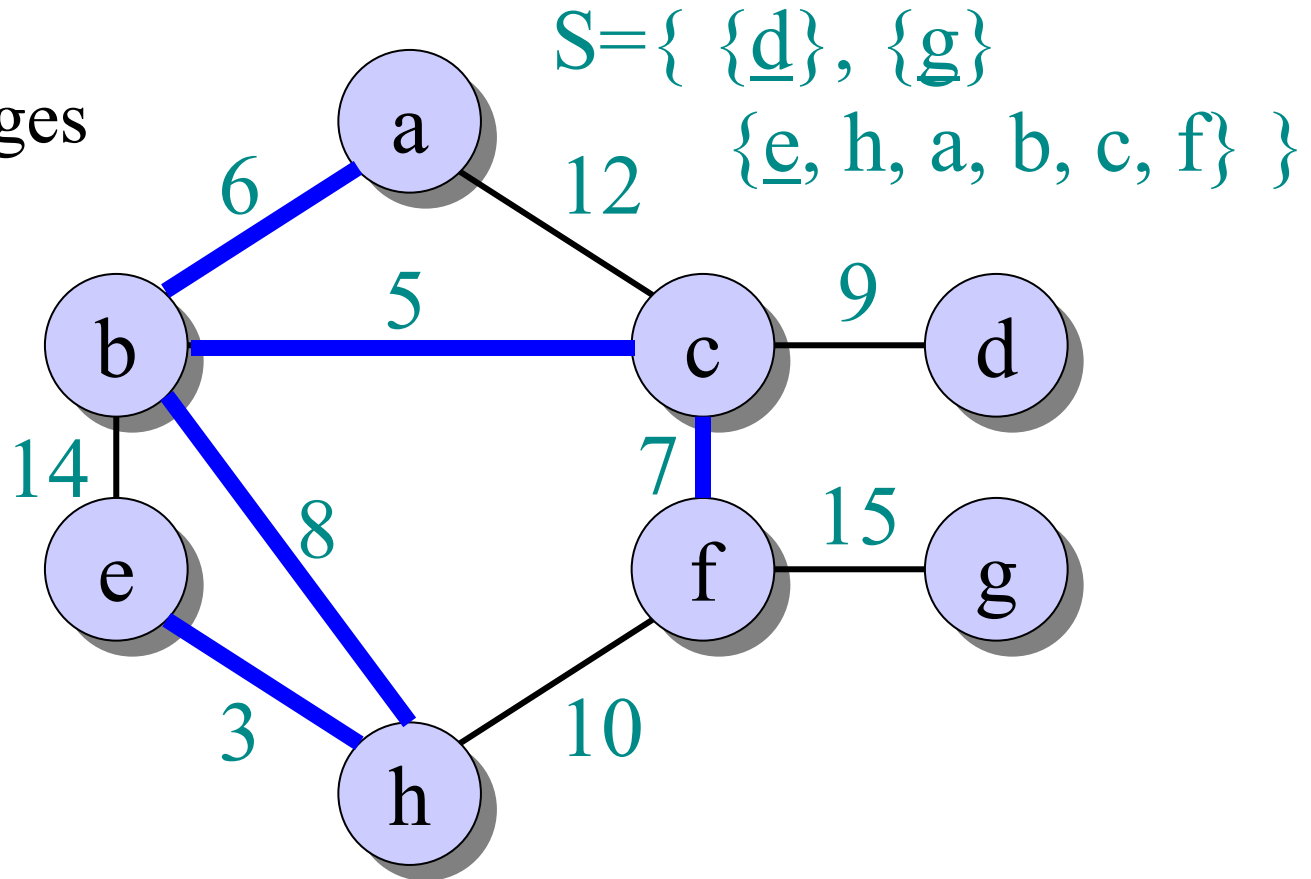




Example of Kruskal's algorithm

— MST edges

a set repr.

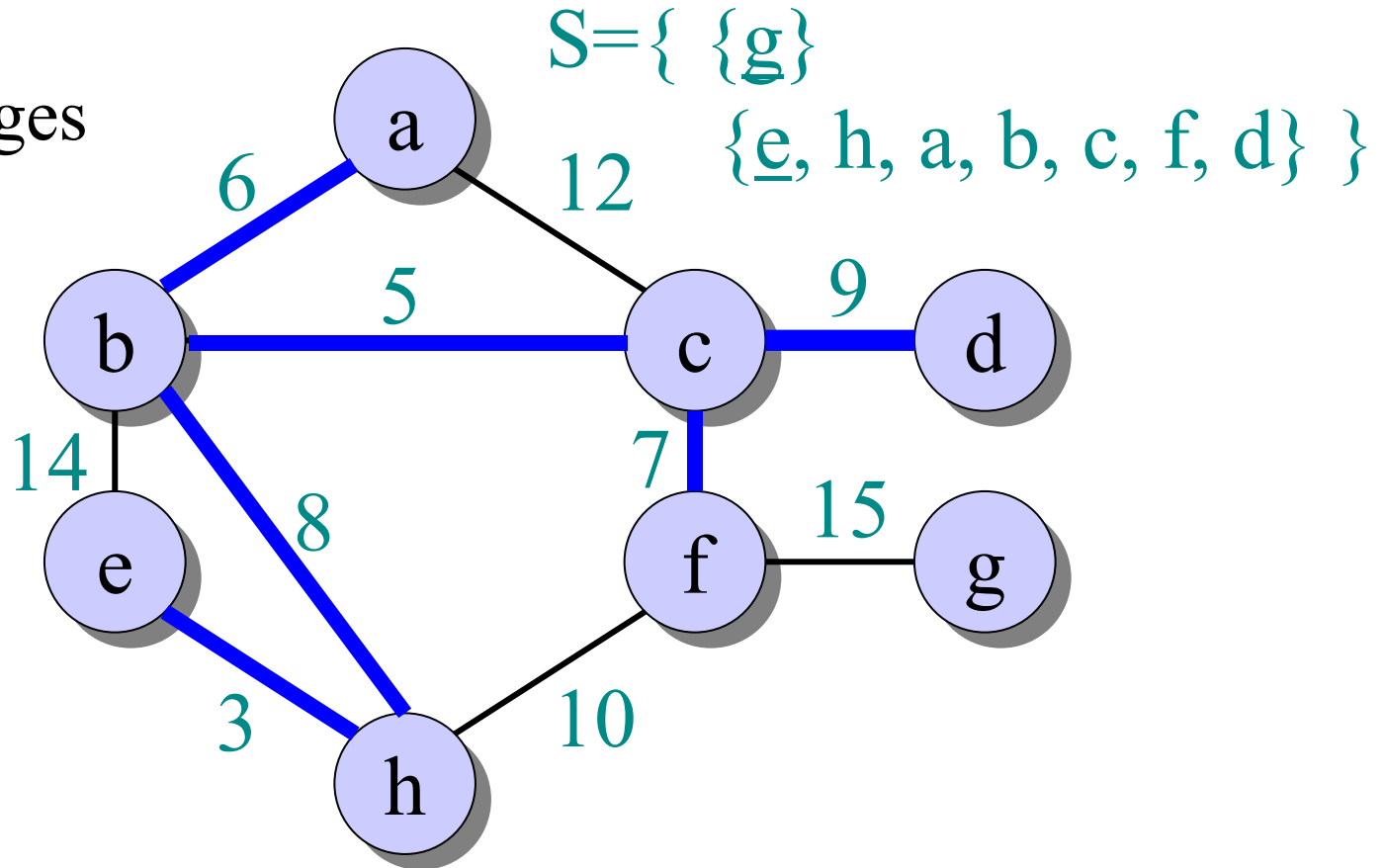


Edge 8 merged the two bigger trees.



Example of Kruskal's algorithm

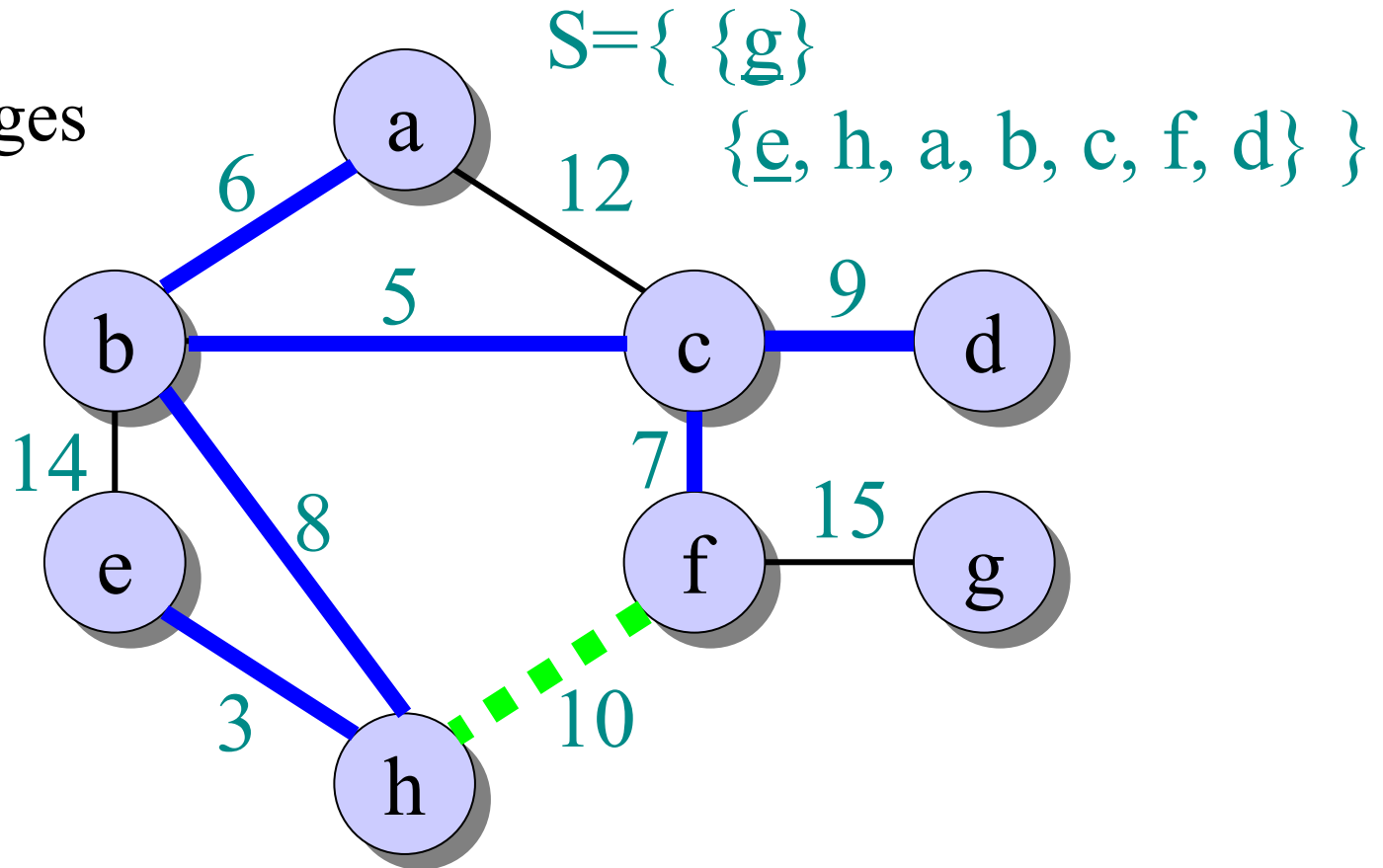
— MST edges
a set repr.





Example of Kruskal's algorithm

— MST edges
a set repr.

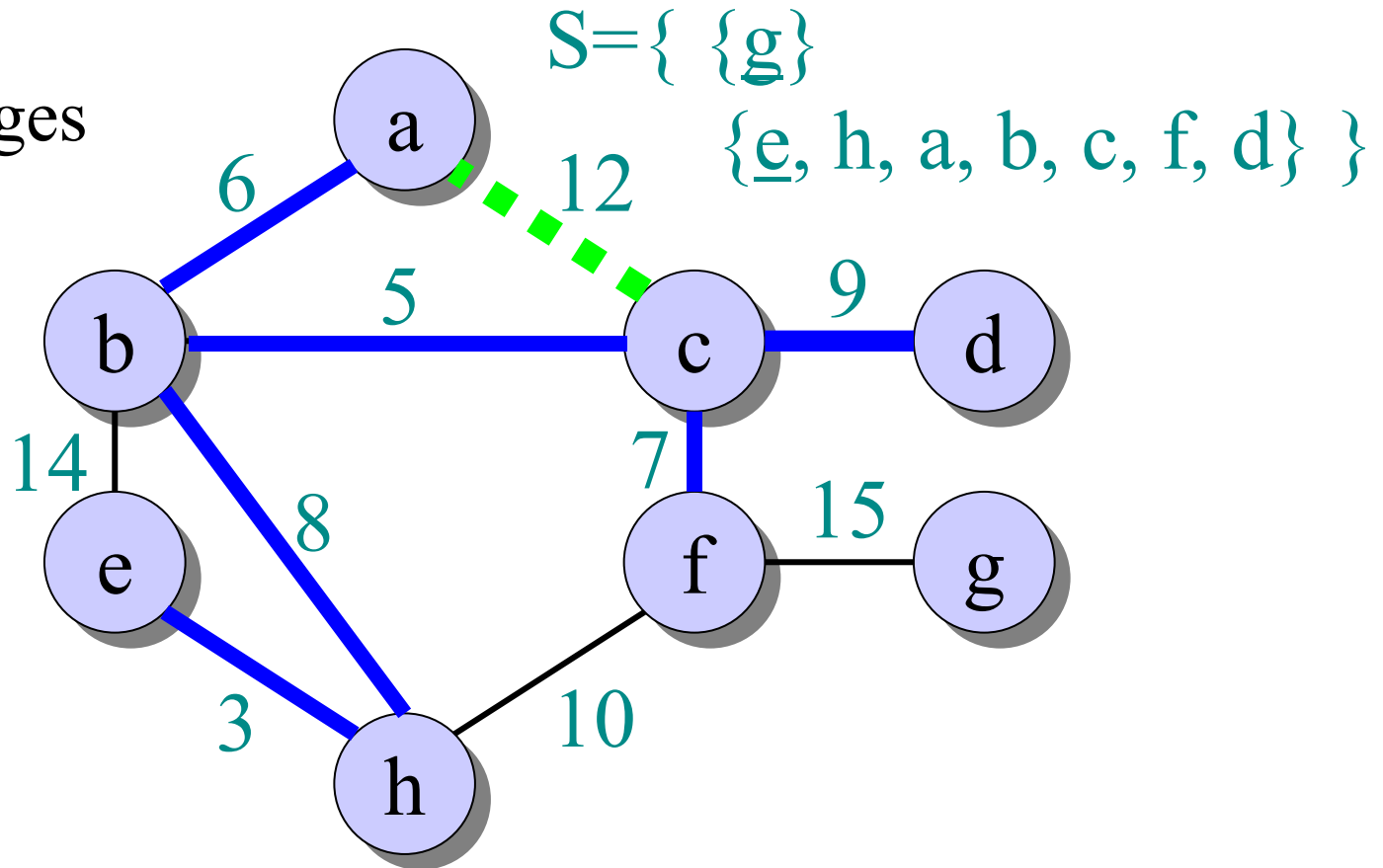


Skip edge 10 as it would cause a cycle.



Example of Kruskal's algorithm

— MST edges
a set repr.

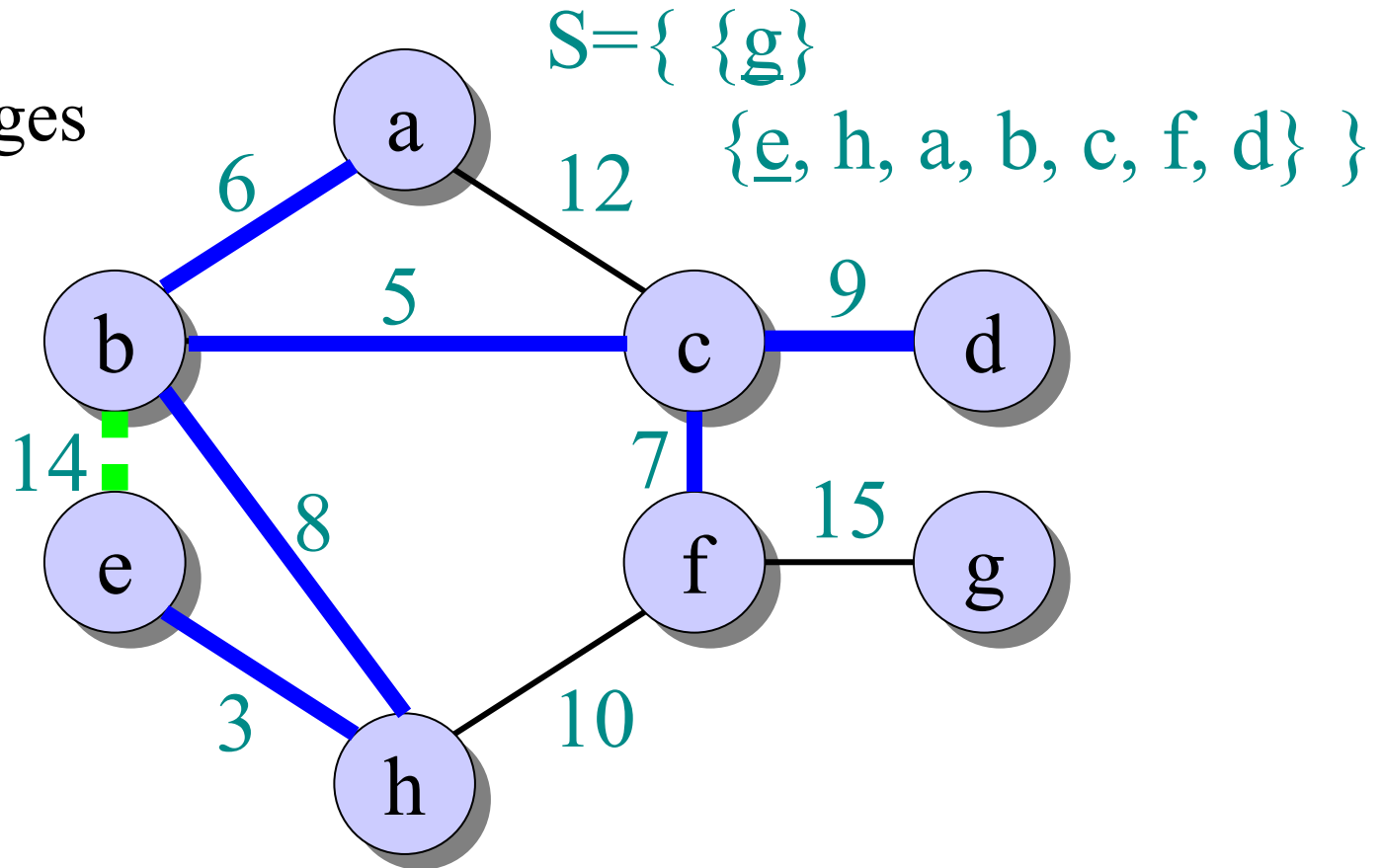


Skip edge 12 as it would cause a cycle.



Example of Kruskal's algorithm

— MST edges
a set repr.



Skip edge 14 as it would cause a cycle.

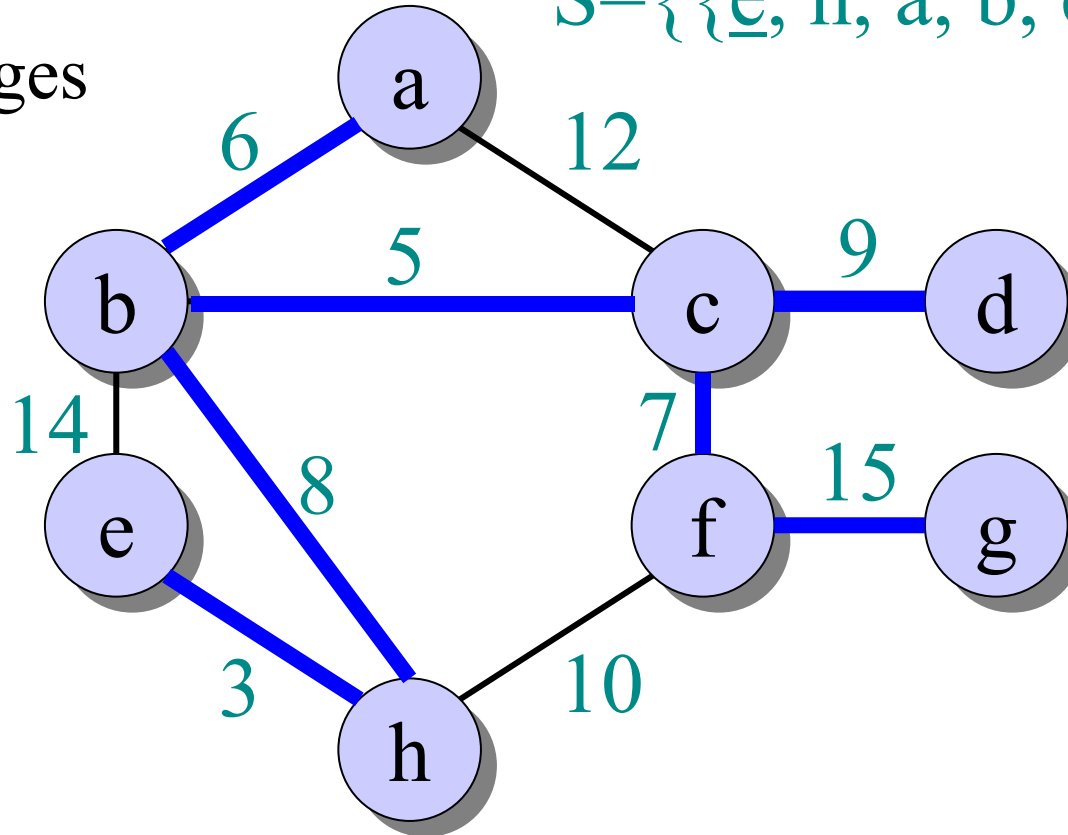


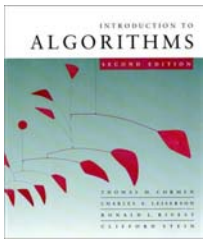
Example of Kruskal's algorithm

— MST edges

a set repr.

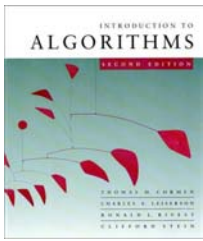
$S = \{ \{ \underline{e}, h, a, b, c, f, d, g \} \}$





Disjoint-set data structure (Union-Find)

- Maintains a dynamic collection of *pairwise-disjoint* sets $\mathbf{S} = \{S_1, S_2, \dots, S_r\}$.
- Each set S_i has one element distinguished as the **representative** element.
- Supports operations:
 - $O(1)$ • MAKE-SET(x): adds new set $\{x\}$ to \mathbf{S}
 - $O(\alpha(n))$ • UNION(x, y): replaces sets S_x, S_y with $S_x \cup S_y$
 - $O(\alpha(n))$ • FIND-SET(x): returns the representative of the set S_x containing element x
- $1 < \alpha(n) < \log^*(n) < \log(\log(n)) < \log(n)$



Kruskal's algorithm

IDEA: Repeatedly pick edge with smallest weight as long as it does not form a cycle.

$S \leftarrow \emptyset$ \triangleright S will contain all MST edges

$O(|V|)$ for each $v \in V$ do MAKE-SET(v)

$O(|E|\log|E|)$ Sort edges of E in non-decreasing order according to w

$O(|E|)$ For each $(u,v) \in E$ taken in this order do

$O(\alpha(|V|))$ $\left\{ \begin{array}{l} \text{if FIND-SET}(u) \neq \text{FIND-SET}(v) \triangleright u,v \text{ in different trees} \\ \quad S \leftarrow S \cup \{(u,v)\} \\ \quad \text{UNION}(u,v) \triangleright \text{Edge } (u,v) \text{ connects the two trees} \end{array} \right.$

Runtime: $O(|V| + |E|\log|E| + |E|\alpha(|V|)) = O(|E|\log|E|)$



MST algorithms

- Prim's algorithm:
 - Maintains one tree
 - Runs in time $O(|E| \log |V|)$, with binary heaps.
- Kruskal's algorithm:
 - Maintains a forest and uses the disjoint-set data structure
 - Runs in time $O(|E| \log |E|)$
- Best to date: Randomized algorithm by Karger, Klein, Tarjan [1993]. Runs in expected time $O(|V| + |E|)$