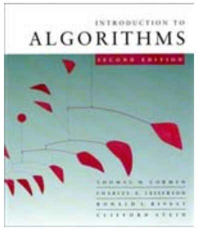
CS 5633 -- Spring 2009



Recurrences and Divide & Conquer

Carola Wenk

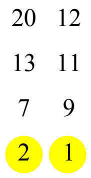
Slides courtesy of Charles Leiserson with small changes by Carola Wenk

Merge sort

```
MERGE-SORT $A[1 \,.\,.\, n]$
    1. If $n = 1$, done.
    2. Recursively sort $A[\,1 \,.\,.\, \lceil n/2 \rceil\,]$
       and $A[\,\lceil n/2 \rceil{+}1 \,.\,.\, n\,]$ .
    3. "Merge" the 2 sorted lists.
```

Key subroutine: MERGE

Merging two sorted arrays



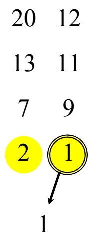
20 12

13 11

7 9

2 1

Merging two sorted arrays



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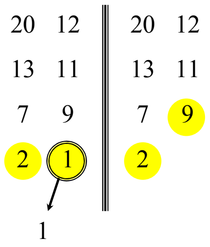

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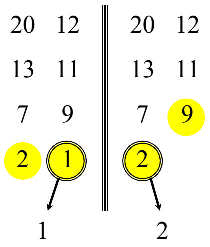
Merging two sorted arrays



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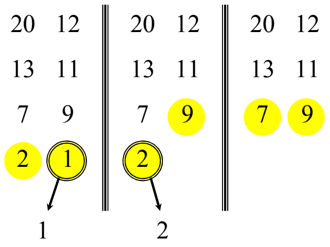

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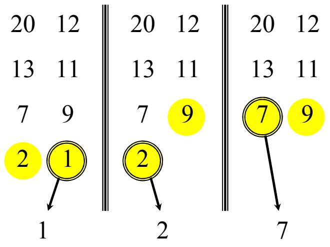

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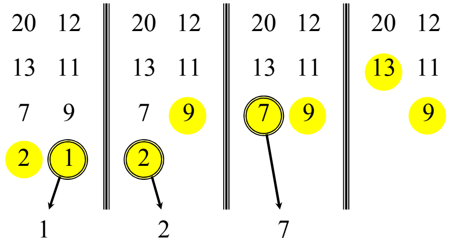

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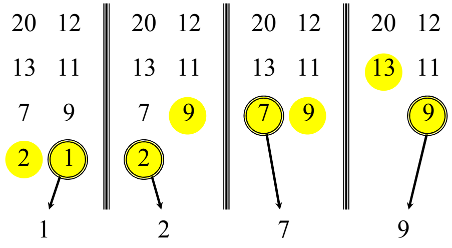

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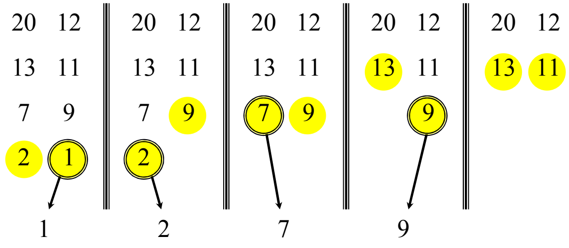

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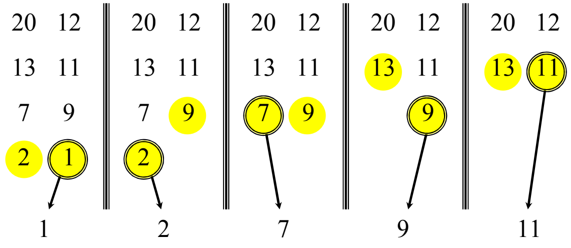

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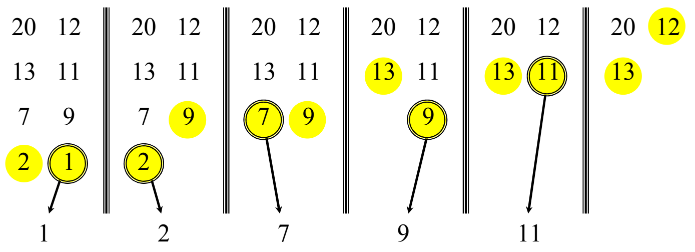

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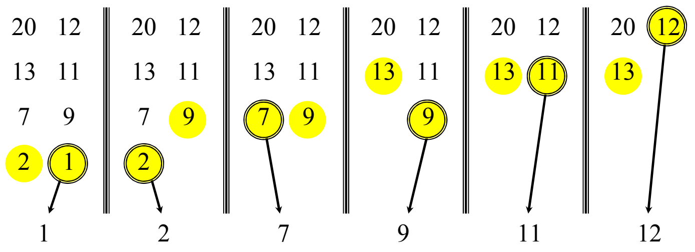

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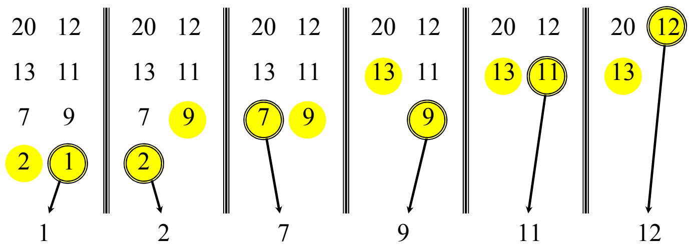

Time $dn = \Theta(n)$ to merge a total of n elements (linear time).

Analyzing merge sort

$T(n)$	MERGE-SORT $A[1\,..\,n]$
d_0	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1\,..\lceil n/2\rceil\,]$ and $A[\lceil n/2\rceil+1\,..\,n\,]$.
dn	***3. “Merge”*** the 2 sorted lists

Sloppiness: Should be $T(\lceil n/2\rceil) + T(\lfloor n/2\rfloor)$, but it turns out not to matter asymptotically.

Recurrence for merge sort

$$T(n) = \begin{cases} d_0 \text{ if } n = 1; \\ 2T(n/2) + dn \text{ if } n > 1. \end{cases}$$

- Later we shall often omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.

- But what does $T(n)$ solve to? I.e., is it $O(n)$ or $O(n^2)$ or $O(n^3)$ or …?

The divide-and-conquer design paradigm

1. ***Divide*** the problem (instance) into subproblems of sizes that are fractions of the original problem size
2. ***Conquer*** the subproblems by solving them recursively.
3. ***Combine*** subproblem solutions.

Example: merge sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.

$$T(n) = \underbrace{2}_{\#\ subproblems}\, T(\underbrace{n/2}_{subproblem\ size}) + \underbrace{\Theta(n)}_{work\ dividing\ and\ combining}$$

Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3 5 7 8 9 12 15

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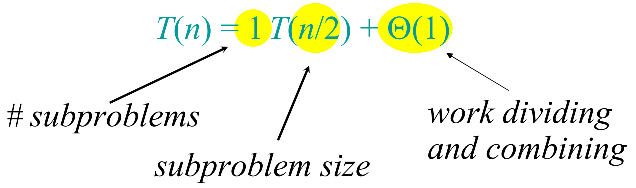
Recurrence for binary search



$$T(n) = 1\,T(n/2) + \Theta(1)$$

# subproblems

subproblem size

work dividing and combining

Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1; \\ 2T(n/2) + \Theta(\mathrm{n}) & \text{if } n > 1. \end{cases}$$

- How do we solve $T(n)$? I.e., how do we find out if it is $O(n)$ or $O(n^2)$ or …?

Recursion tree

Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

Recursion tree

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$T(n)$

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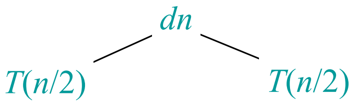

dn

$T(n/2)$ $T(n/2)$

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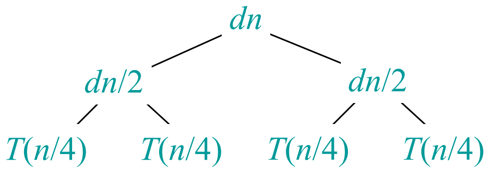
Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.



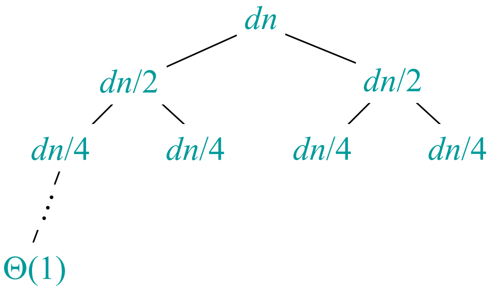
dn

$dn/2$ $dn/2$

$T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$

Recursion tree

Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.



dn

$dn/2$ $dn/2$

$dn/4$ $dn/4$ $dn/4$ $dn/4$

$\Theta(1)$

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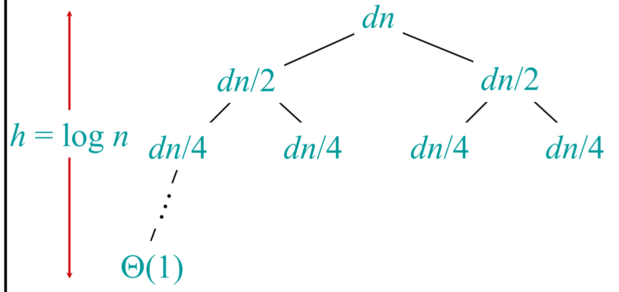

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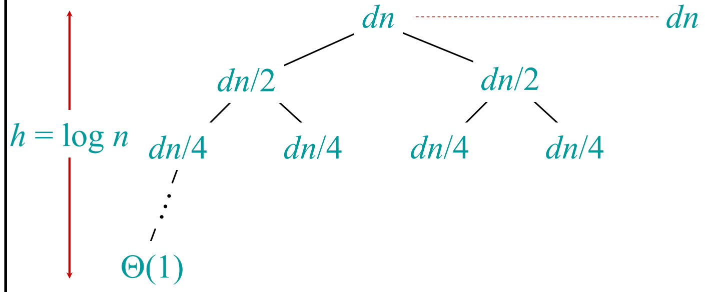

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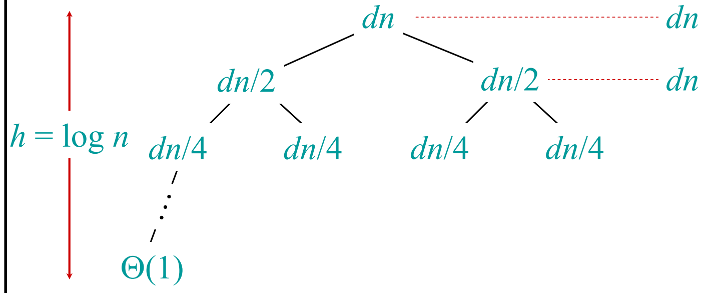

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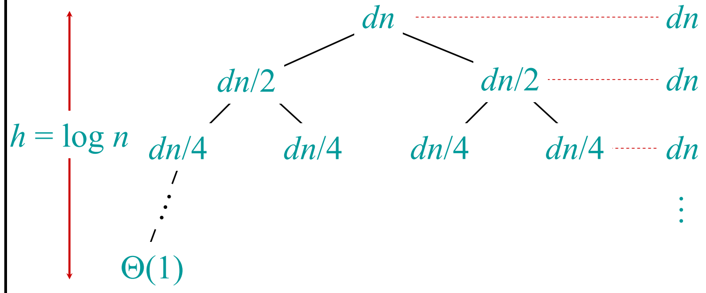

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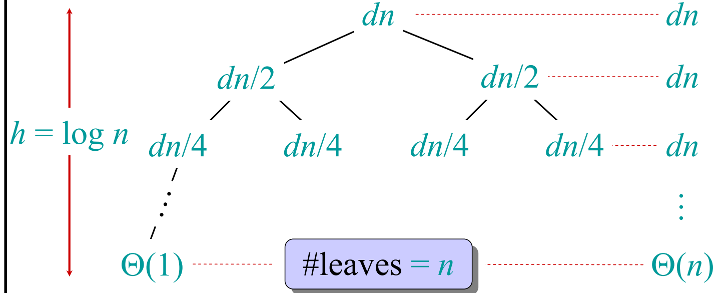

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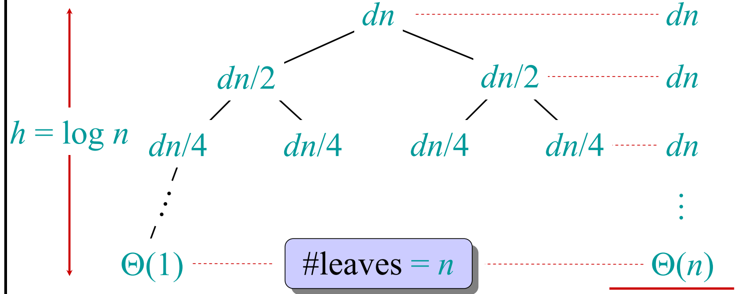

Total $\Theta(n \log n)$

Conclusions

- Merge sort runs in $\Theta(n \log n)$ time.
- $\Theta(n \log n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so. (Why not earlier?)

Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- It is good for generating **guesses** of what the runtime could be.

But: Need to **verify** that the guess is right.
→ Induction (substitution method)

Substitution method

The most general method to solve a recurrence (prove O and Ω separately):

1. ***Guess*** the form of the solution:
   (e.g. using recursion trees, or expansion)
2. ***Verify*** by induction (inductive step).
3. ***Solve*** for O-constants n_0 and c (base case of induction)