CS 5633 Analysis of Algorithms – Spring 09

4/21/09

10. Homework

Due **4/28/09** before class

1. **Transitivity (3 points)**

   Show the transitivity property of the polynomial-time reduction "$\leq$" (fact 3 on slide 17):
   Let Π, Π', Π'' be three decision problems. If $\Pi \leq \Pi'$ and $\Pi' \leq \Pi''$ then $\Pi \leq \Pi''$.

2. **To be or not to be... ... in NP (5 points)**
   Which of the problems below are in NP and which are not? Justify your answers.

   (a) Given a directed acyclic graph G. Topologically sort G.

   (b) Given an undirected graph G. Is G a tree?

   (c) Given an unsorted array of n numbers. What is the median?

   (d) Given a connected directed graph $G = (V, E)$. Is there a closed path that visits every edge exactly once?

   (e) Given n numbers. Is there a subset of these numbers that sums up to 42?

3. **Subgraph isomorphism (5 points)**
   Problem 34.5-1 on page 1017.
   Hint: Show that the problem is in NP, and then show that it is NP-hard. For the NP-hardness you need to pick an NP-hard problem, and polynomially reduce it to the subgraph-isomorphism problem. It would make sense to use a problem that involves a graph and a subgraph.

4. **TSP (5 points)**
   Given that the Traveling Salesperson Problem (TSP) for undirected graphs (which we covered in class) is NP-complete, show that TSP for directed graphs is also NP-complete.

5. $\Pi_1 \leq \Pi_2$ **(7 points)**
   Let Π_1 and Π_2 be decision problems and suppose Π_1 is polynomial time reducible to Π_2, so, $\Pi_1 \leq \Pi_2$. Answer and justify each of the questions below:

   - Does this imply that Π_2 is NP-complete?
   - If Π_2 is polynomially reducible to Π_1, are Π_1 and Π_2 both NP-complete?
   - If Π_1 and Π_2 are NP-complete, is Π_2 polynomially reducible to Π_1?
   - If $\Pi_1 \notin NP$ does this imply that $\Pi_2 \notin NP$?
   - If Π_2 is NP-complete, does this imply that $\Pi_1 \in NP$?
   - If Π_1 is NP-complete, does this imply that $\Pi_2 \in NP$?
   - If Π_1 is NP-complete and $\Pi_2 \in P$, what does this imply?

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Related questions from previous PhD Exams

Just for your information. You **do not** need to solve them for homework credit.

1. This problem is concerned with NP-completeness.

   Consider the following two decision problems.

   VERTEX COVER.
   Instance: An undirected graph $G = (V, E)$, and a positive integer k.
   Decision Problem: Is there a vertex cover of size k? A *vertex cover* is a subset $V' \subseteq V$ such that if $(u, v) \in E$, then $u \in V'$ or $v \in V'$ (or both).

   INDEPENDENT SET.
   Instance: An undirected graph $G = (V, E)$ and a positive integer k.
   Decision Problem: Is there an independent set of size k? An *independent set* is a subset $V' \subseteq V$ such that each edge in E is incident on at most one vertex in V'.

   (a) Define *polynomial-time reducibility.*

   (b) Show that INDEPENDENT SET is polynomial-time reducible to VERTEX COVER.

   (c) Suppose problem P_1 is polynomial-time reducible to problem P_2 ($P_1 \leq_{\mathrm{P}} P_2$). If there is a polynomial algorithm for P_1, what can be implied about P_2? If there is a polynomial algorithm for P_1, what can be implied about P_1?

   (d) Consider the complexity classes P, NP, NP-complete, and NP-hard. If P $\neq$ NP, what would be the subset relationships among these four classes? If P = NP, what would be the subset relationships among these four classes?

   (e) Assume that VERTEX COVER is an NP-complete problem. Use VERTEX COVER to show that INDEPENDENT SET is NP-complete. Specify what steps need to be done, and provide the details of your solution.

2. This problem is concerned with NP-completeness. Let Π_1 and Π_2 be two decision problems for which is known that $\Pi_1 \leq_P \Pi_2$ (i.e., Π_1 is polynomially reducable to Π_2). Briefly state what can be inferred (if anything) in each of the following cases.

   (a) $\Pi_1 \in NP$

   (b) $\Pi_2 \in P$

   (c) Π_1 is NP-hard

   (d) Π_2 is NP-complete