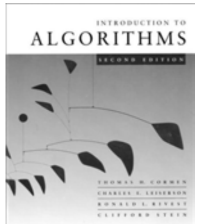
CS 5633 -- Spring 2008



Dynamic Programming

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Slides courtesy of Charles Leiserson with small changes by Carola Wenk

Dynamic programming

- Algorithm design technique (like divide and conquer)
- Is a technique for solving problems that have
  - overlapping subproblems
  - and, when used for optimization, have an optimal substructure property
- **Idea:** Do not repeatedly solve the same subproblems, but solve them only once and store the solutions in a **dynamic programming table**

Example: Fibonacci numbers

- F(0)=0; F(1)=1; F(n)=F(n-1)+F(n-2) for $n \geq 2$
- Implement this recursion naively:

F(n) → F(n-1), F(n-2)

F(n-1) → F(n-2), F(n-3)

F(n-2) → F(n-3), F(n-4)

> Solve same subproblems many times !
>
> Runtime is exponential in n.

- Store 1D DP-table and fill bottom-up in *O(n)* time:

F:

0	1	1	2	3	5	8				

Longest Common Subsequence

Example: *Longest Common Subsequence (LCS)*

- Given two sequences $x[1 \,..\, m]$ and $y[1 \,..\, n]$, find a longest subsequence common to them both.

"a" *not* "the"

x: A B C B D A B

y: B D C A B A

BCBA = LCS(x, y)

functional notation, but not a function

Brute-force LCS algorithm

Check every subsequence of $x[1\,..\,m]$ to see if it is also a subsequence of $y[1\,..\,n]$.

Analysis

- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).
- Hence, the runtime would be exponential !

Towards a better algorithm

Two-Step Approach:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|s|$.

Strategy: Consider ***prefixes*** of x and y.

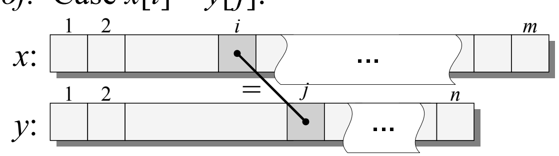
- Define $c[i, j] = |\text{LCS}(x[1\,..\,i], y[1\,..\,j])|$.
- Then, $c[m, n] = |\text{LCS}(x, y)|$.

Recursive formulation

Theorem.

$$c[i,j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

Proof. Case $x[i] = y[j]$:



Let $z[1\,..\,k] = \text{LCS}(x[1\,..\,i], y[1\,..\,j])$, where $c[i, j] = k$. Then, $z[k] = x[i]$, or else z could be extended. Thus, $z[1\,..\,k-1]$ is CS of $x[1\,..\,i-1]$ and $y[1\,..\,j-1]$.

Proof (continued)

Claim: $z[1\,..\,k-1] = \text{LCS}(x[1\,..\,i-1], y[1\,..\,j-1])$.

Suppose w is a longer CS of $x[1\,..\,i-1]$ and $y[1\,..\,j-1]$, that is, $|w| > k-1$. Then, ***cut and paste***: $w \,||\, z[k]$ (w concatenated with $z[k]$) is a common subsequence of $x[1\,..\,i]$ and $y[1\,..\,j]$ with $|w \,||\, z[k]| > k$. Contradiction, proving the claim.

Thus, $c[i-1, j-1] = k-1$, which implies that $c[i, j] = c[i-1, j-1] + 1$.

Other cases are similar. □

Dynamic-programming hallmark #1

Optimal substructure
An optimal solution to a problem (instance) contains optimal solutions to subproblems.

➡ ***Recurrence***

If $z = \text{LCS}(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y.

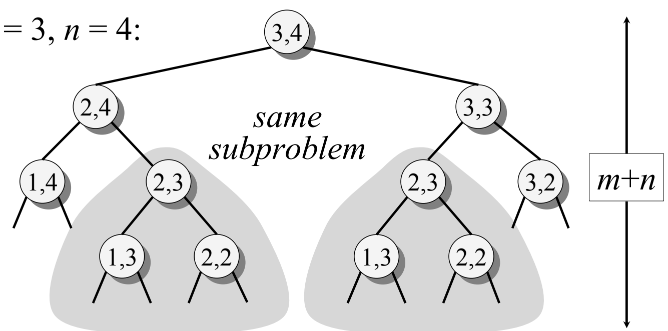
Recursive algorithm for LCS

LCS(x, y, i, j)
  if $x[i] = y[j]$
    then $c[i, j] \leftarrow \text{LCS}(x, y, i-1, j-1) + 1$
    else $c[i, j] \leftarrow \max\{\text{LCS}(x, y, i-1, j), \text{LCS}(x, y, i, j-1)\}$

Worst-case: $x[i] \neq y[j]$, in which case the algorithm evaluates two subproblems, each with only one parameter decremented.

Recursion tree

$m = 3$, $n = 4$:



Height = $m + n \Rightarrow$ work potentially exponential, but we're solving subproblems already solved!

Dynamic-programming hallmark #2

Overlapping subproblems
A recursive solution contains a "small" number of distinct subproblems repeated many times.

The number of distinct LCS subproblems for two strings of lengths m and n is only mn.

Dynamic-programming

There are two variants of dynamic programming:

1. Memoization
2. Bottom-up dynamic programming (often referred to as "dynamic programming")

Memoization algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

for all i, j**:** $c[i,0]=0$ **and** $c[0,j]=0$
LCS(x, y, i, j)
  if $c[i, j]$ = NIL
    then if $x[i] = y[j]$
      then $c[i, j] \leftarrow$ LCS($x, y, i-1, j-1$) + 1
      else $c[i, j] \leftarrow \max\{$LCS($x, y, i-1, j$), LCS($x, y, i, j-1$)$\}$
same as before

Time = $\Theta(mn)$ = constant work per table entry.
Space = $\Theta(mn)$.

Memoization

LCS(x,y,7,6)
(6,6) (7,5)
(5,5) (6,4)
(4,5) (5,4) (5,3)
⋮ ⋮ ⋮

		1	2	3	4	5	6	7
	x:	A	B	C	B	D	A	B
y:	0	0	0	0	0	0	0	0
1 B	0	0	1	nil	nil	nil	nil	nil
2 D	0	0	1	nil	nil	nil	nil	nil
3 C	0	0	2	nil	nil	nil	nil	nil
4 A	0	1	nil	nil	nil	nil	nil	nil
5 B	0	nil	nil	nil	nil	nil	nil	nil
6 A	0	nil	nil	nil	nil	nil	nil	nil

Bottom-up dynamic-programming algorithm

IDEA:
Compute the table bottom-up.

Time = $\Theta(mn)$.

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

Bottom-up dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct LCS by back-tracing.

Space = $\Theta(mn)$.

Exercise: $O(\min\{m, n\})$.

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4