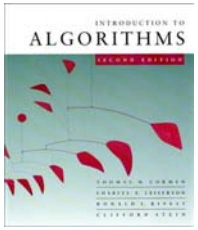
# CS 5633 -- Spring 2008



# *Red-black trees*

**Carola Wenk**

Slides courtesy of Charles Leiserson with small changes by Carola Wenk

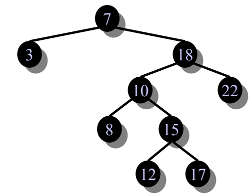


# Search Trees

• A binary search tree is a binary tree. Each node stores a key. The tree fulfills the **binary search tree property**:

For every node $x$ holds:

- $y \leq x$ , for all $y$ in the subtree left of $x$
- $x < y$, for all $y$ in the subtree right of $x$

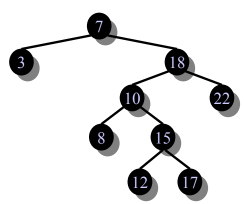





# Search Trees

Different variants of search trees:

- Balanced search trees (guarantee height of *log n* for *n* elements)
- *k*-ary search trees (such as B-trees, 2-3-4-trees)
- Search trees that store the keys only in the leaves, and store additional split-values in the internal nodes



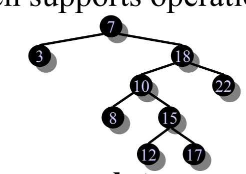



# ADT Dictionary / Dynamic Set

**Abstract data type (ADT) Dictionary (also called Dynamic Set):**

A data structure which supports operations

- Insert
- Delete
- Find




Using **balanced binary search trees** we can implement a dictionary data structure such that each operation takes $O(\log n)$ time.

## Balanced search trees

***Balanced search tree:*** A search-tree data structure for which a height of $O(\log n)$ is guaranteed when implementing a dynamic set of $n$ items.

**Examples:**
- AVL trees
- 2-3 trees
- 2-3-4 trees
- B-trees
- Red-black trees



## Red-black trees

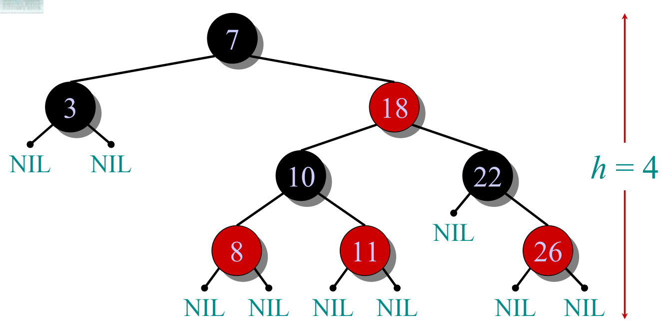
This data structure requires an extra one-bit color field in each node.

***Red-black properties:***

1. Every node is either red or black.
2. The root is black.
3. The leaves (NIL's) are black.
4. If a node is red, then both its children are black.
5. All simple paths from any node $x$, excluding $x$, to a descendant leaf have the same number of black nodes = black-height($x$).



## Example of a red-black tree



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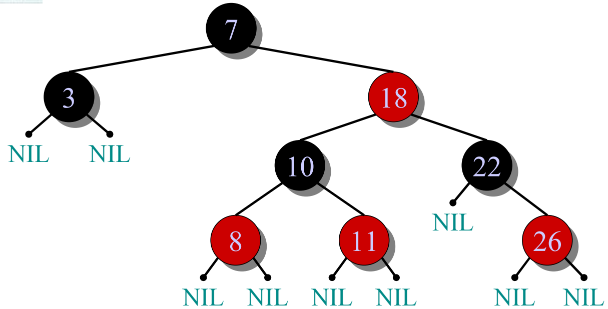


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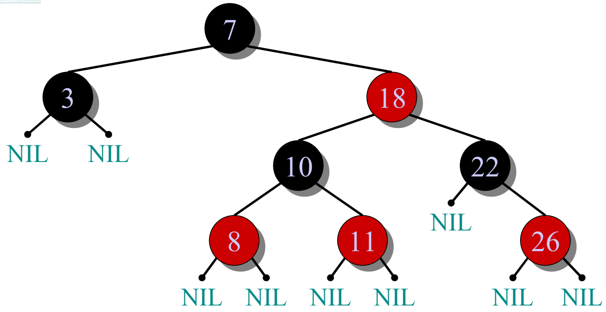
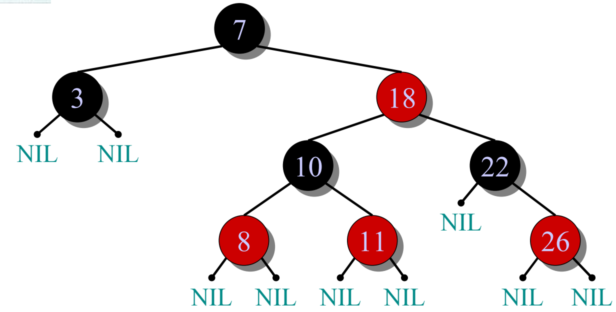



2., 3. The root and leaves (NIL's) are black.



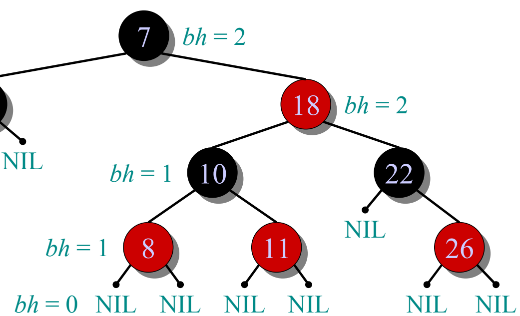
## Example of a red-black tree




4. If a node is red, then both its children are black.



## Example of a red-black tree




5. All simple paths from any node $x$, excluding $x$, to a descendant leaf have the same number of black nodes = *black-height*($x$).



## Height of a red-black tree

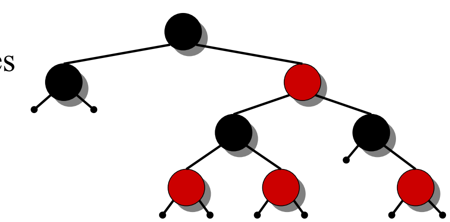
**Theorem.** A red-black tree with $n$ keys has height

$$h \leq 2 \log(n + 1).$$

*Proof.* (The book uses induction. Read carefully.)

**INTUITION:**

- Merge red nodes into their black parents.



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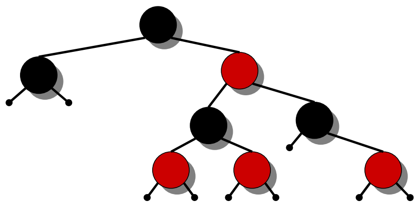



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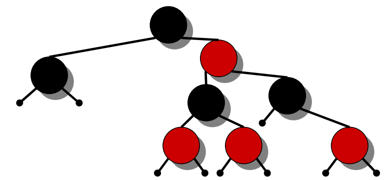



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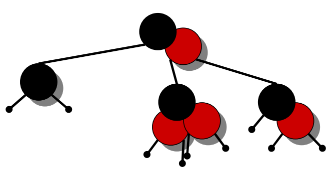



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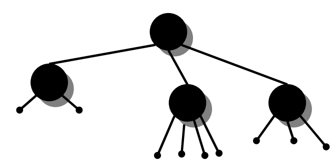

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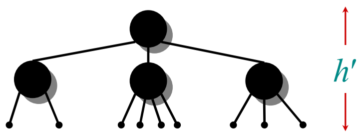
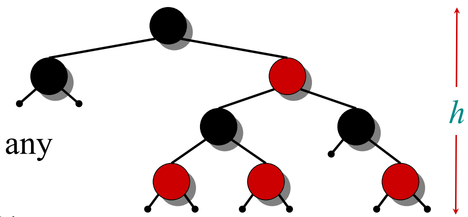


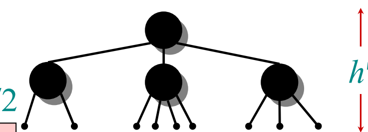

- This process produces a tree in which each node has 2, 3, or 4 children.
- The 2-3-4 tree has uniform depth $h'$ of leaves.



## Proof (continued)



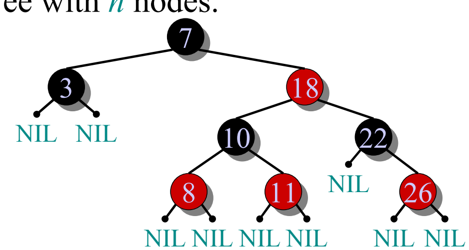

- We have $h' \ge h/2$, since at most half the vertices on any path are red.
- The number of leaves in each tree is $n + 1$
  $\Rightarrow n + 1 \ge 2^{h'}$
  $\Rightarrow \log(n + 1) \ge h' \ge h/2$
  $\Rightarrow h \le 2 \log(n + 1)$. □






## Query operations

**Corollary.** The queries SEARCH, MIN, MAX, SUCCESSOR, and PREDECESSOR all run in $O(\log n)$ time on a red-black tree with $n$ nodes.



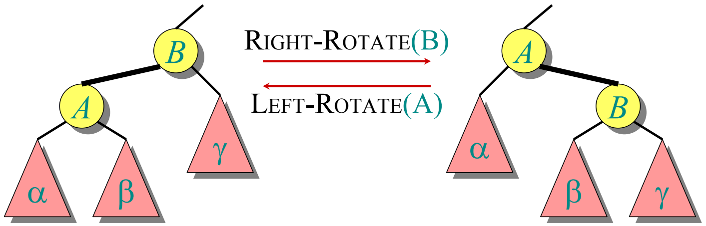



## Modifying operations

The operations INSERT and DELETE cause modifications to the red-black tree:

1. the operation itself,
2. color changes,
3. restructuring the links of the tree via ***"rotations"***.

# Rotations




- Rotations maintain the inorder ordering of keys:
  $a \in \alpha, b \in \beta, c \in \gamma \Rightarrow a \le A \le b \le B \le c$.
- Rotations maintain the binary search tree property
- A rotation can be performed in $O(1)$ time.



# Red-black trees

This data structure requires an extra one-bit color field in each node.

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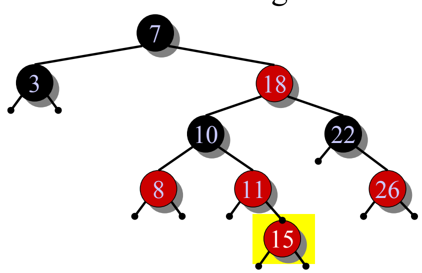
# Insertion into a red-black tree

**IDEA:** Insert $x$ in tree. Color $x$ red. Only red-black property 4 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

**Example:**

- Insert $x$ =15.



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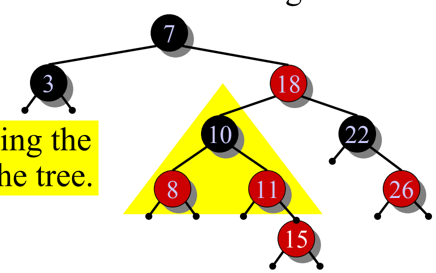
- Insert $x$ =15.
- Recolor, moving the violation up the tree.



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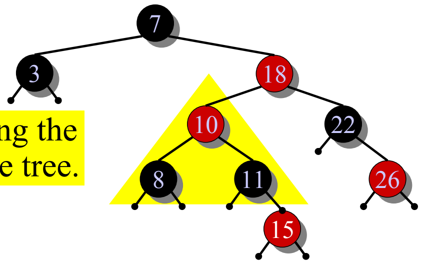




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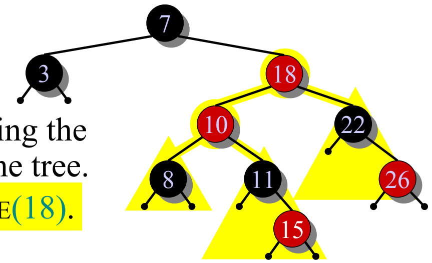
- RIGHT-ROTATE(18).



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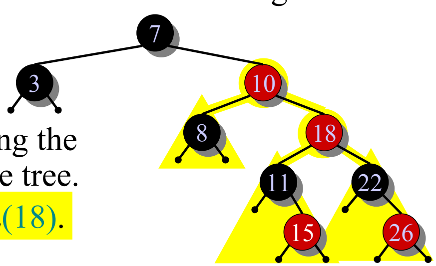




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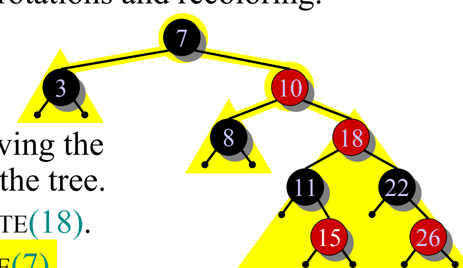
- LEFT-ROTATE(7)



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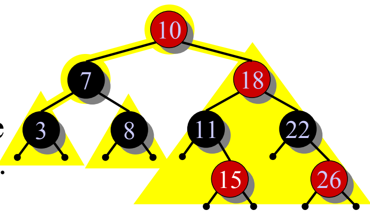




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**Example:**

- Insert *x* =15.
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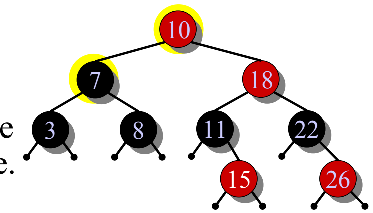
- RIGHT-ROTATE(18).
- LEFT-ROTATE(7) and recolor.



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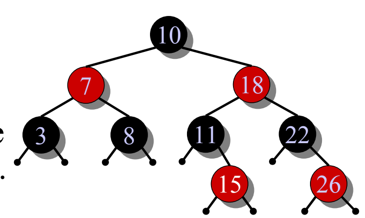





# Pseudocode

```
RB-INSERT(T, x)
    TREE-INSERT(T, x)
    color[x] ← RED        ◁ only RB property 4 can be violated
    while x ≠ root[T] and color[p[x]] = RED
        do if p[x] = left[p[p[x]]]
            then y ← right[p[p[x]]]        ◁ y = aunt/uncle of x
                 if color[y] = RED
                  then ⟨Case 1⟩
                  else  if x = right[p[x]]
                           then ⟨Case 2⟩   ◁ Case 2 falls into Case 3
                         ⟨Case 3⟩
            else ⟨"then" clause with "left" and "right" swapped⟩
    color[root[T]] ← BLACK
```

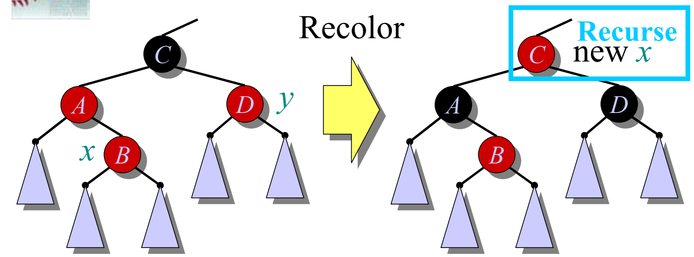
# Graphical notation

Let △ denote a subtree with a black root.

All △'s have the same black-height.



# Case 1




(Or, $A$'s children are swapped.)

$p[x] = left[p[p[x]]]$

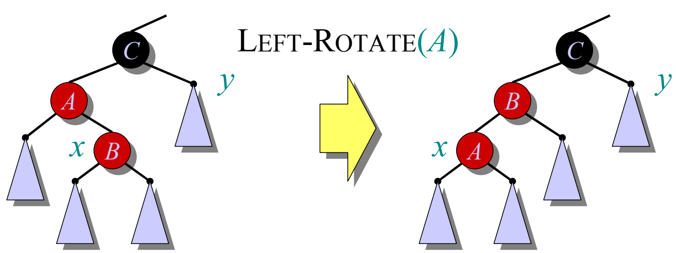
$y = right[p[p[x]]]$

$color[y] =$ RED

Push $C$'s black onto $A$ and $D$, and recurse, since $C$'s parent may be red.



# Case 2




$p[x] = left[p[p[x]]]$

$y = right[p[p[x]]]$

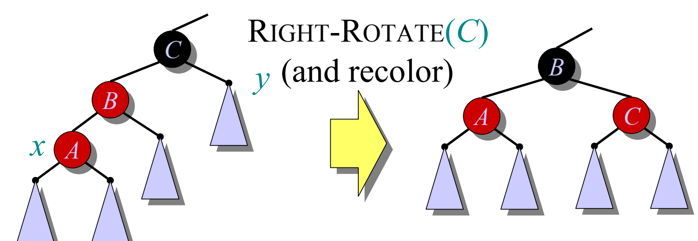
$color[y] =$ BLACK

$x = right[p[x]]$

Transform to Case 3.



# Case 3




$p[x] = left[p[p[x]]]$

$y = right[p[p[x]]]$

$color[y] =$ BLACK

$x = left[p[x]]$

Done! No more violations of RB property 4 are possible.

# Analysis

- Go up the tree performing Case 1, which only recolors nodes.
- If Case 2 or Case 3 occurs, perform 1 or 2 rotations, and terminate.

**Running time:** $O(\log n)$ with $O(1)$ rotations.

RB-DELETE — same asymptotic running time and number of rotations as RB-INSERT (see textbook).

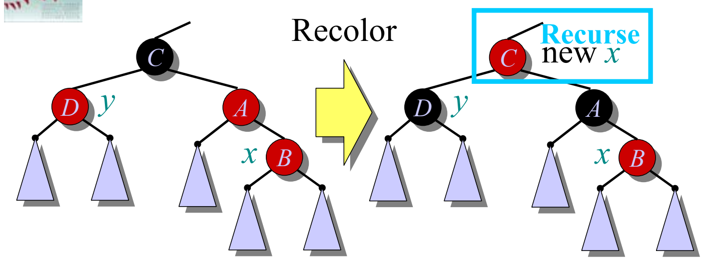


# Pseudocode (part II)

```
else ⟨"then" clause with "left" and "right" swapped⟩
◁ p[x] = right[p[p[x]]
then y ← left[p[p[x]]            ◁ y = aunt/uncle of x
     if color[y] = RED
      then ⟨Case 1'⟩
      else  if x = left[p[x]]
              then ⟨Case 2'⟩    ◁ Case 2' falls into Case 3'
             ⟨Case 3'⟩
color[root[T]] ← BLACK
```



# Case 1'

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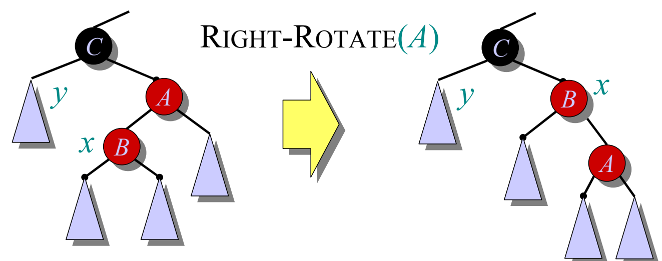
# Case 2'




$p[x] = right[p[p[x]]]$

$y = left[p[p[x]]]$

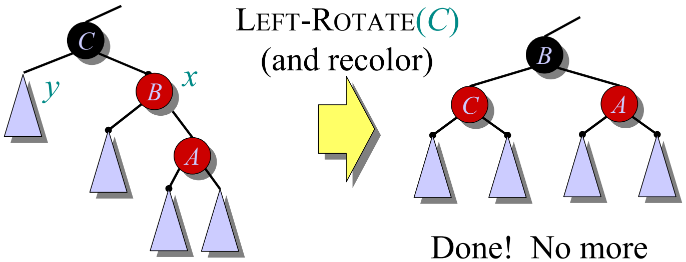
$color[y]$ = BLACK

$x = left[p[x]]$

Transform to Case 3'.

# Case 3’

LEFT-ROTATE($C$)
(and recolor)

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$color[y] = \text{BLACK}$
$x = right[p[x]]$

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