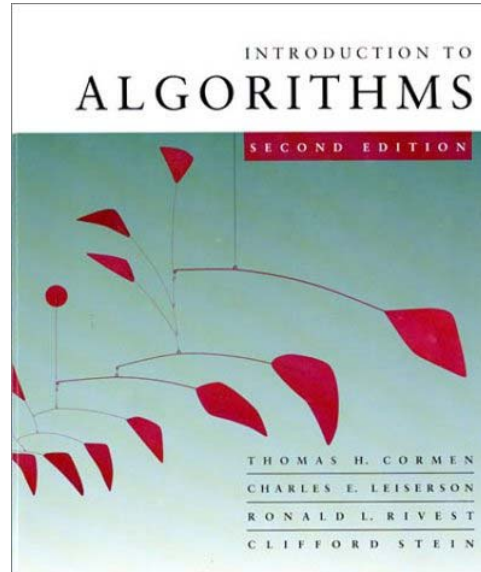




CS 5633 -- Spring 2008



Quicksort

Carola Wenk

Slides courtesy of Charles Leiserson with small changes by Carola Wenk



Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts “in place” (like insertion sort, but not like merge sort).
- Very practical (with tuning).



Divide and conquer

Quicksort an n -element array:

- 1. *Divide:*** Partition the array into two subarrays around a **pivot** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.



- 2. *Conquer:*** Recursively sort the two subarrays.
- 3. *Combine:*** Trivial.

Key: *Linear-time partitioning subroutine.*

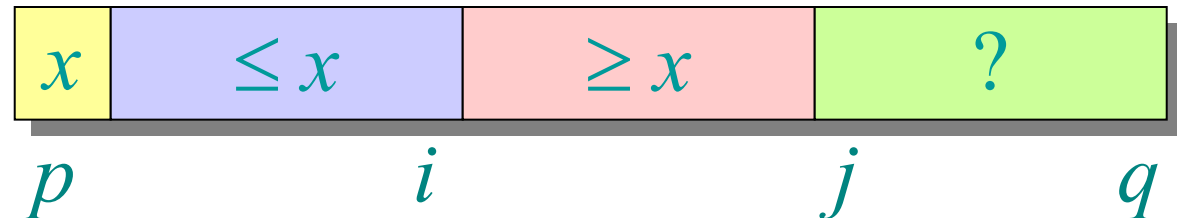


Partitioning subroutine

```
PARTITION( $A, p, q$ )  $\triangleright A[p \dots q]$   
   $x \leftarrow A[p]$   $\triangleright$  pivot =  $A[p]$   
   $i \leftarrow p$   
  for  $j \leftarrow p + 1$  to  $q$   
    do if  $A[j] \leq x$   
      then  $i \leftarrow i + 1$   
          exchange  $A[i] \leftrightarrow A[j]$   
  exchange  $A[p] \leftrightarrow A[i]$   
  return  $i$ 
```

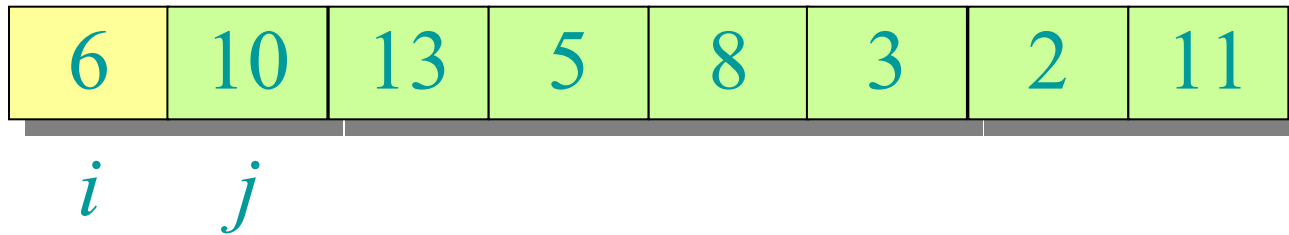
Running time
= $O(n)$ for n
elements.

Invariant:



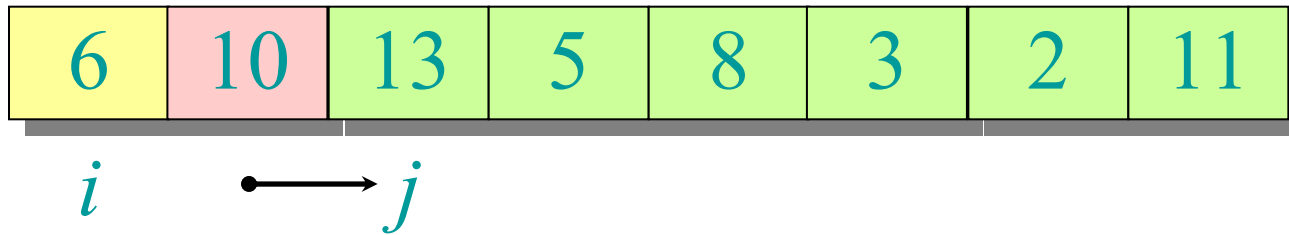


Example of partitioning



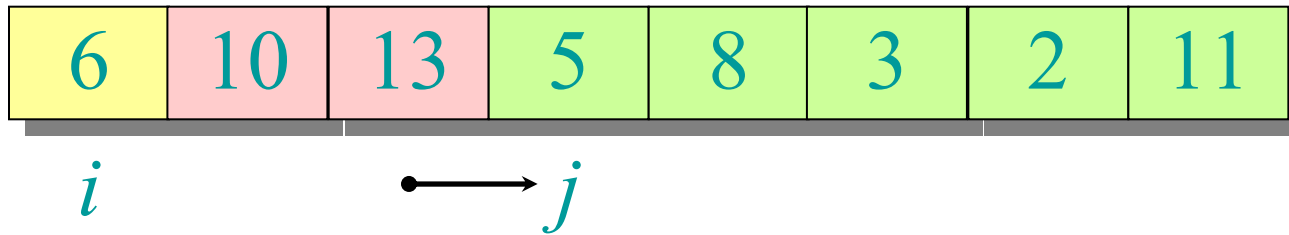


Example of partitioning



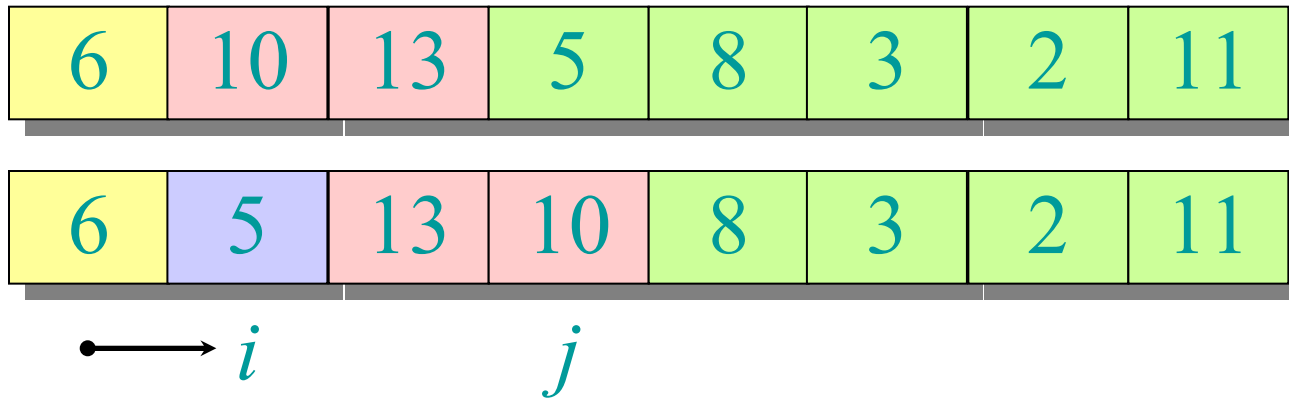


Example of partitioning



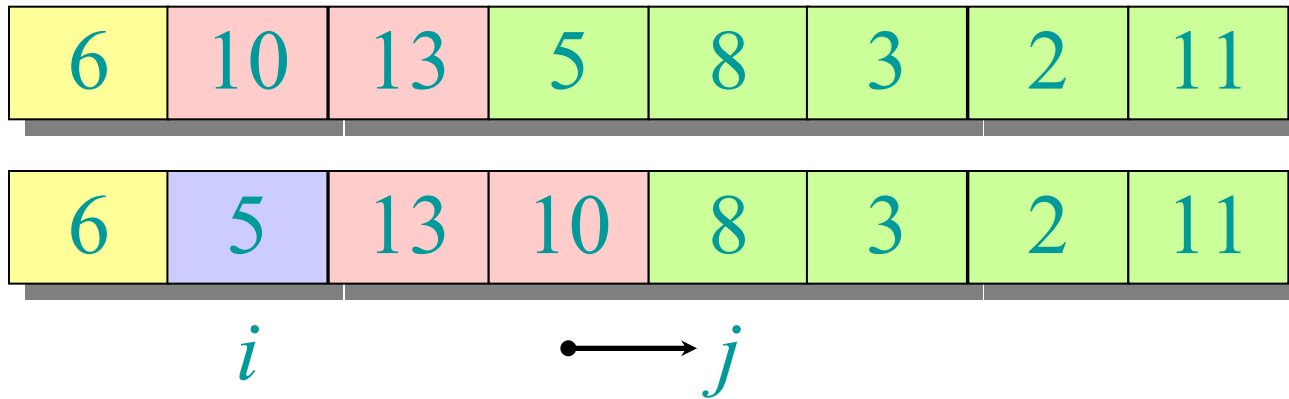


Example of partitioning



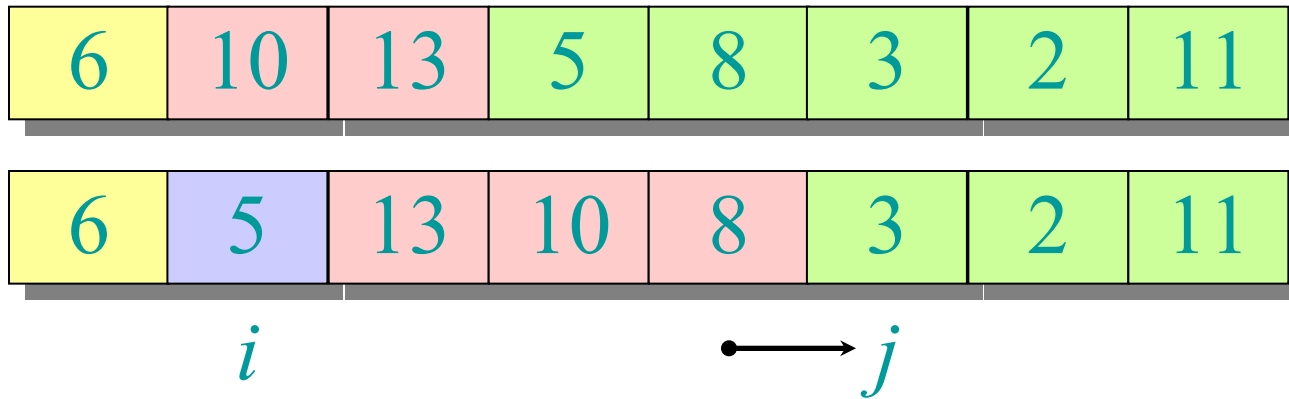


Example of partitioning



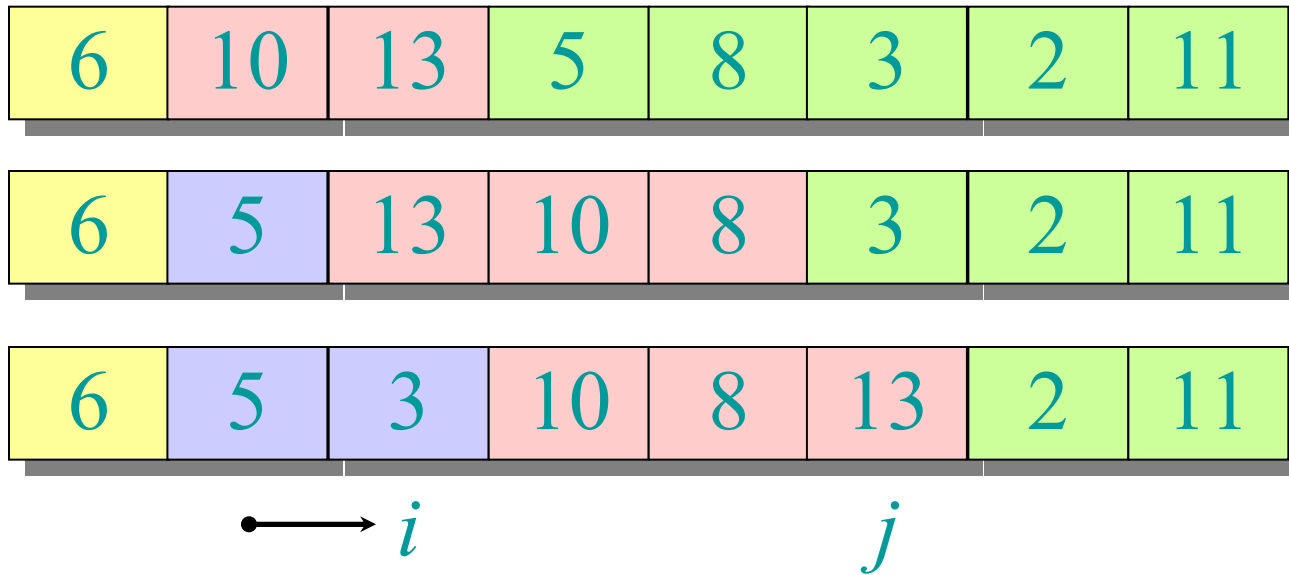


Example of partitioning



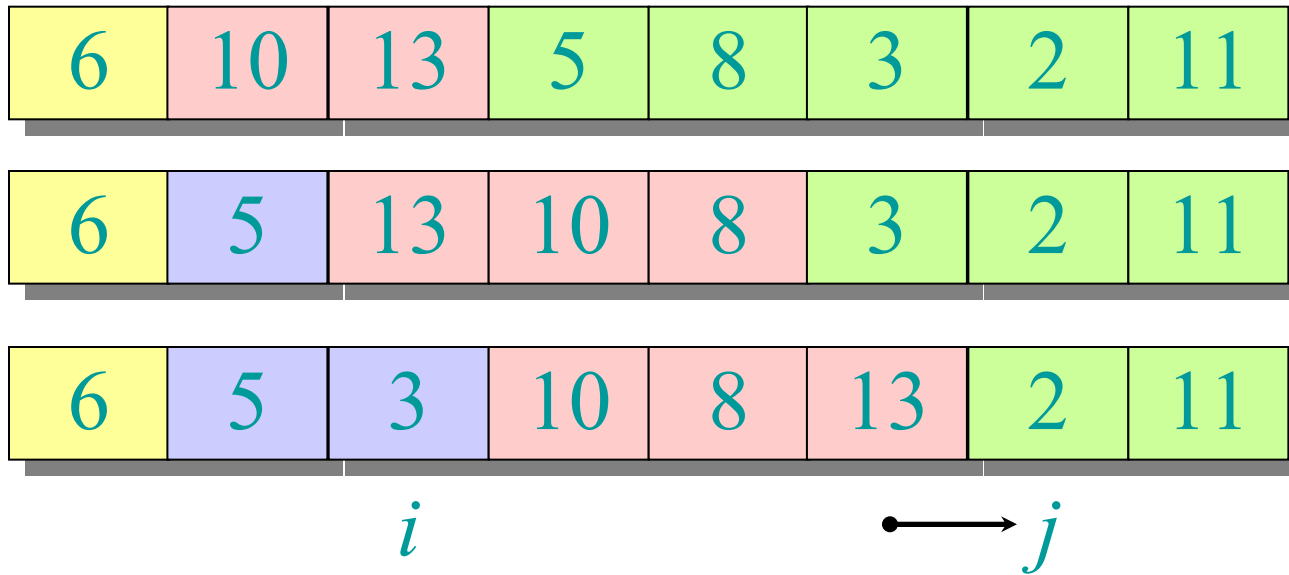


Example of partitioning



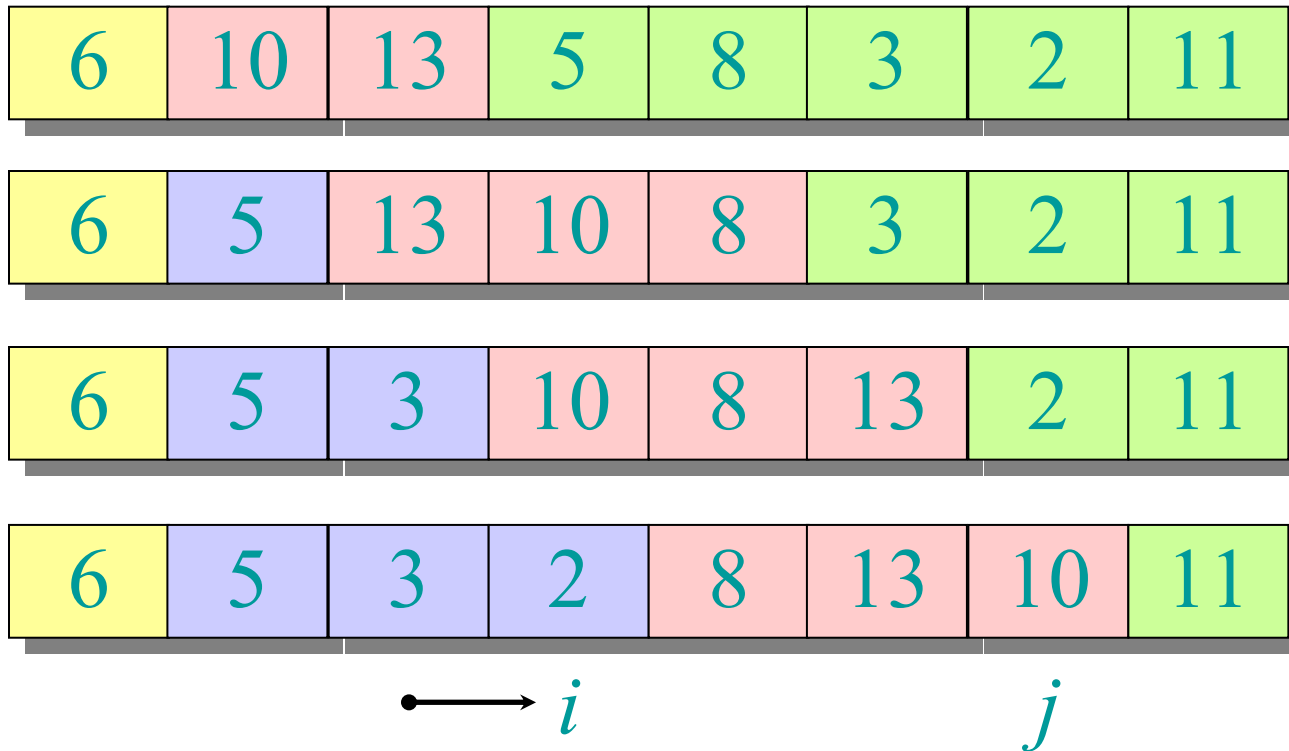


Example of partitioning



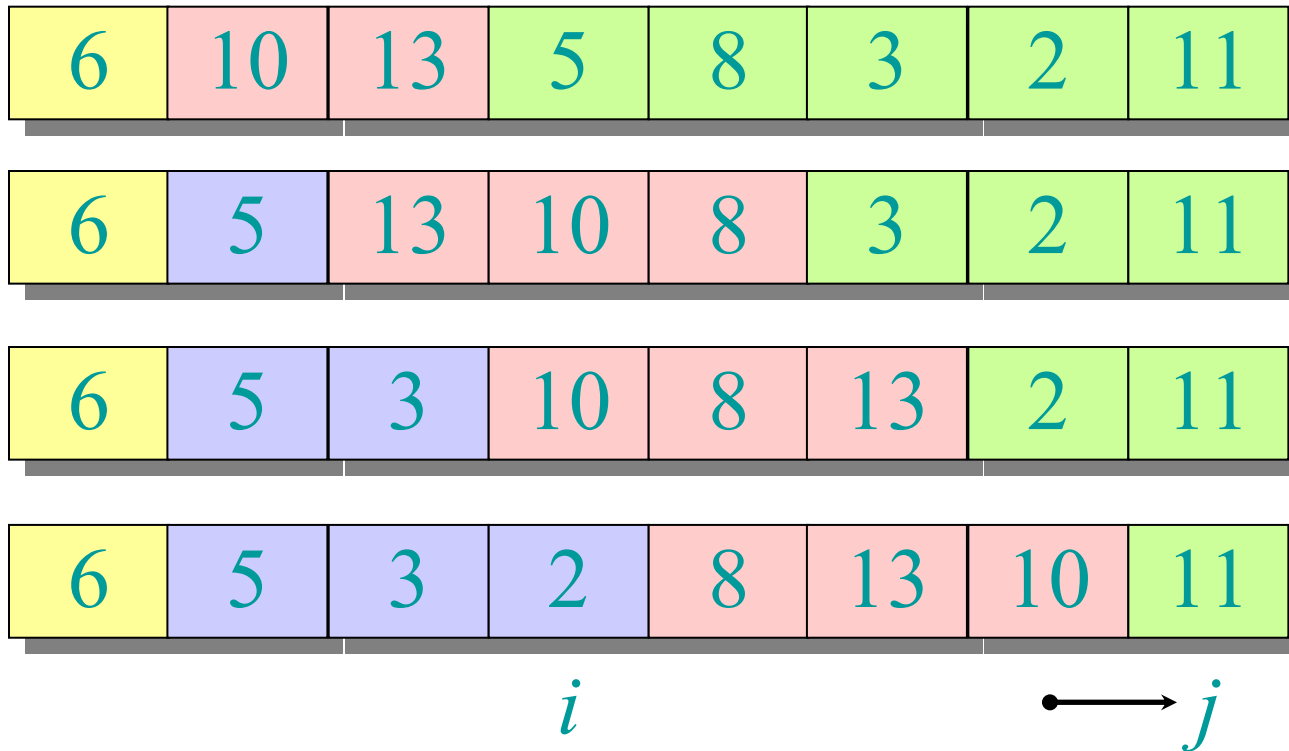


Example of partitioning



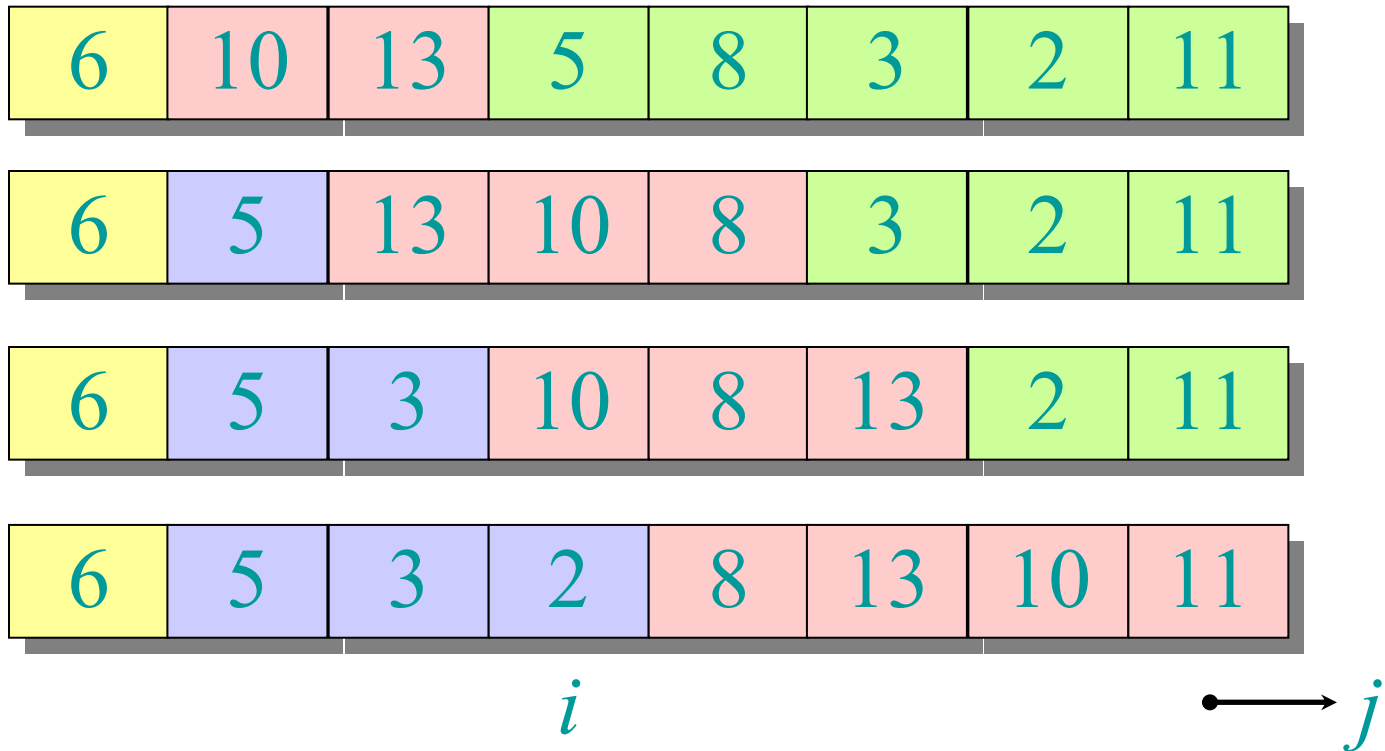


Example of partitioning



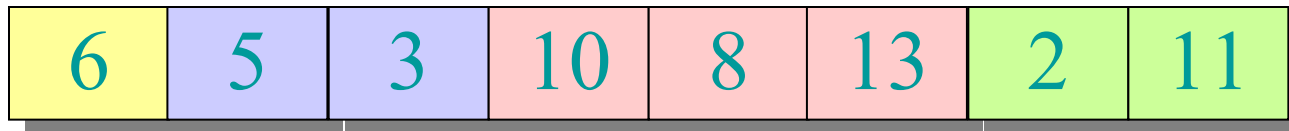
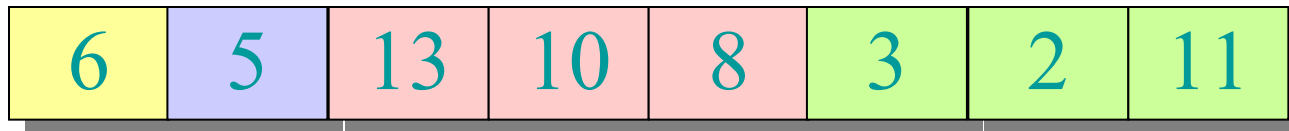


Example of partitioning





Example of partitioning



i



Pseudocode for quicksort

QUICKSORT(A, p, r)

if $p < r$

then $q \leftarrow$ PARTITION(A, p, r)

QUICKSORT($A, p, q-1$)

QUICKSORT($A, q+1, r$)

Initial call: QUICKSORT($A, 1, n$)



Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.



Worst-case of quicksort

```
QUICKSORT( $A, p, r$ )  
  if  $p < r$   
    then  $q \leftarrow$  PARTITION( $A, p, r$ )  
         QUICKSORT( $A, p, q-1$ )  
         QUICKSORT( $A, q+1, r$ )
```

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$\begin{aligned}T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \quad (\textit{arithmetic series})\end{aligned}$$



Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



Worst-case recursion tree

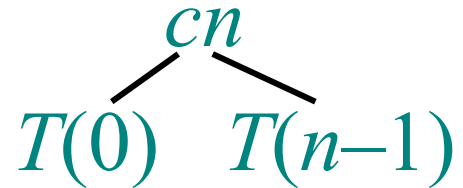
$$T(n) = T(0) + T(n-1) + cn$$

$$T(n)$$



Worst-case recursion tree

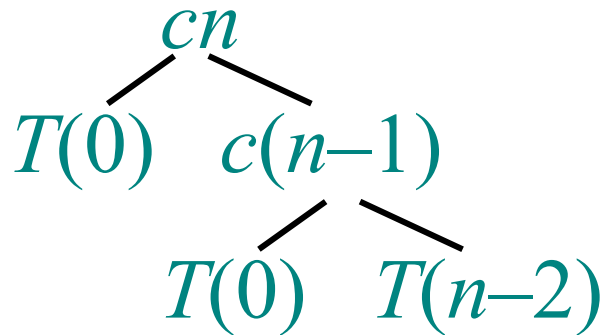
$$T(n) = T(0) + T(n-1) + cn$$





Worst-case recursion tree

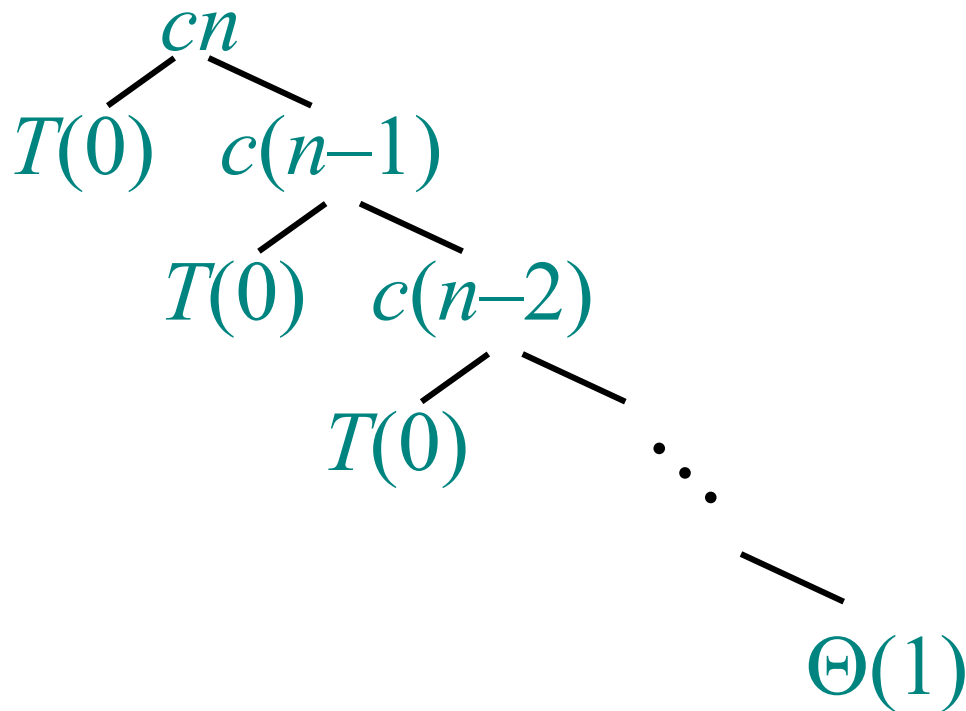
$$T(n) = T(0) + T(n-1) + cn$$





Worst-case recursion tree

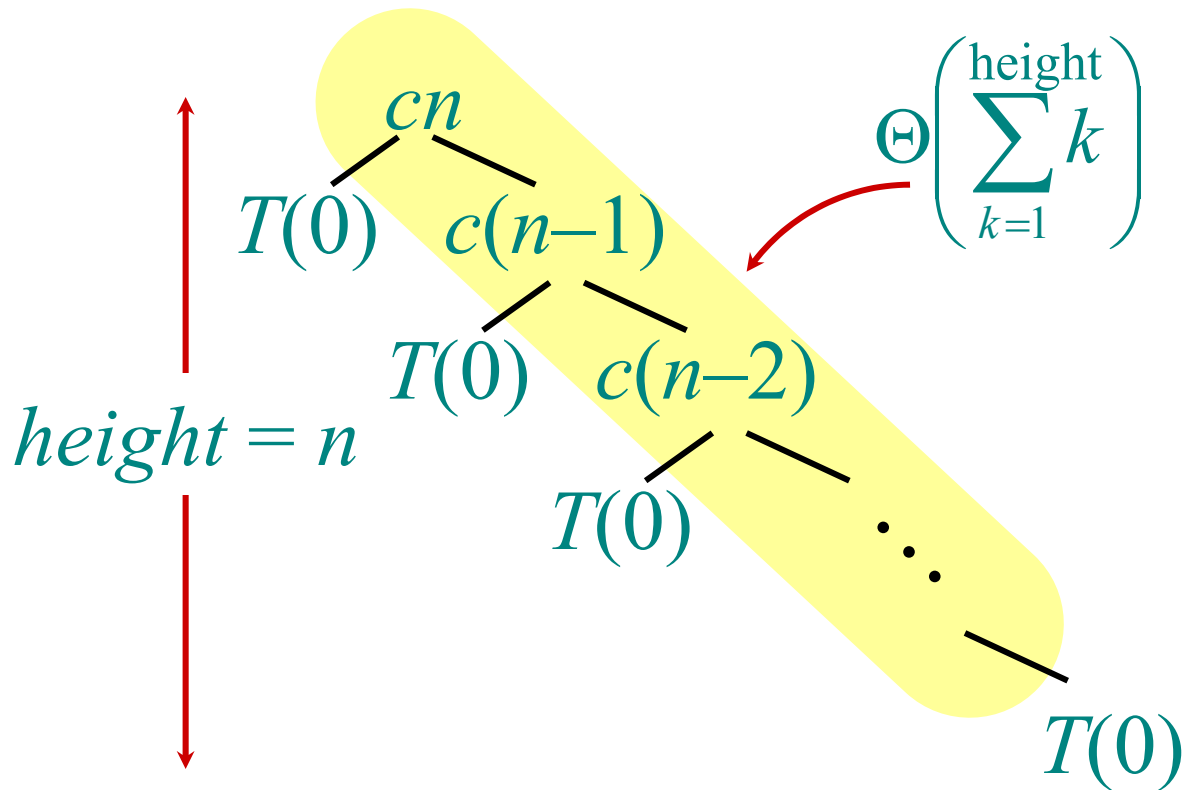
$$T(n) = T(0) + T(n-1) + cn$$





Worst-case recursion tree

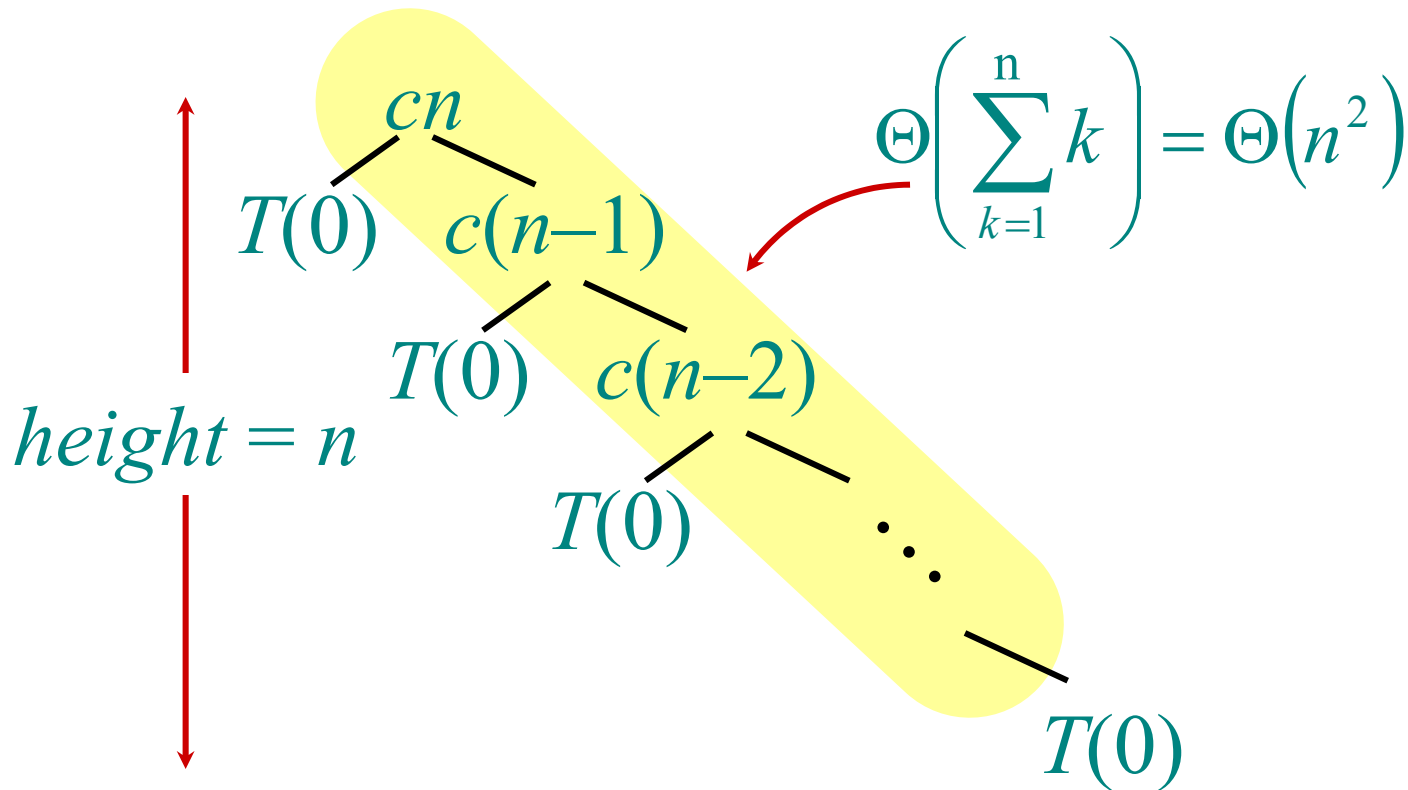
$$T(n) = T(0) + T(n-1) + cn$$





Worst-case recursion tree

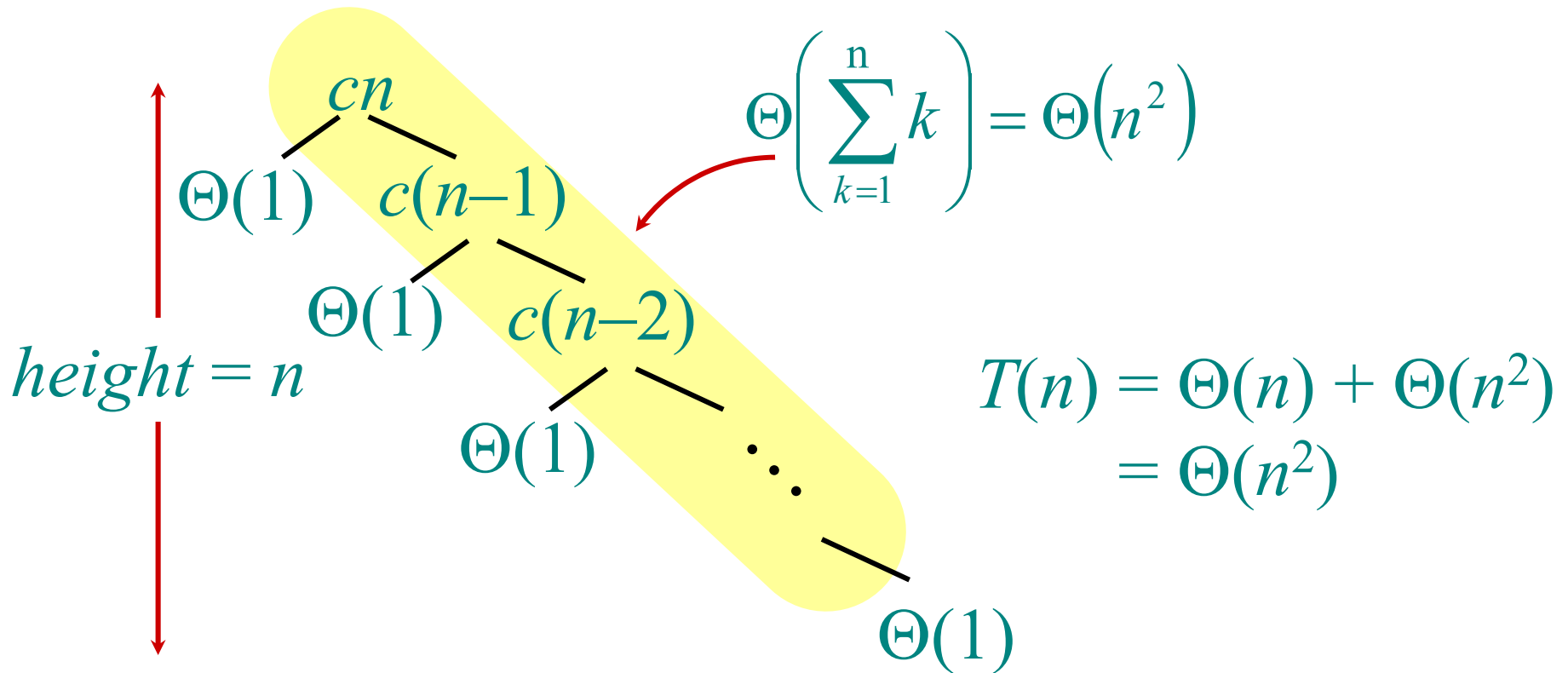
$$T(n) = T(0) + T(n-1) + cn$$





Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$





Best-case analysis

(For intuition only!)

If we're lucky, PARTITION splits the array evenly:

$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \log n) \quad (\text{same as merge sort}) \end{aligned}$$

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

$$T(n) = T\left(\frac{1}{10}n\right) + T\left(\frac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

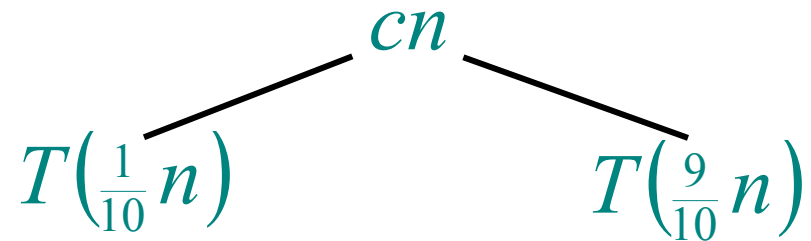


Analysis of “almost-best” case

$$T(n)$$

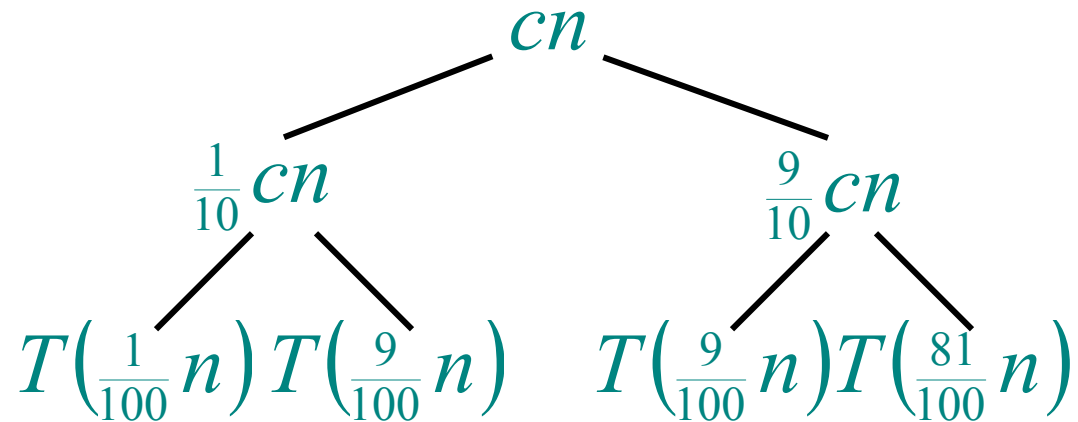


Analysis of “almost-best” case



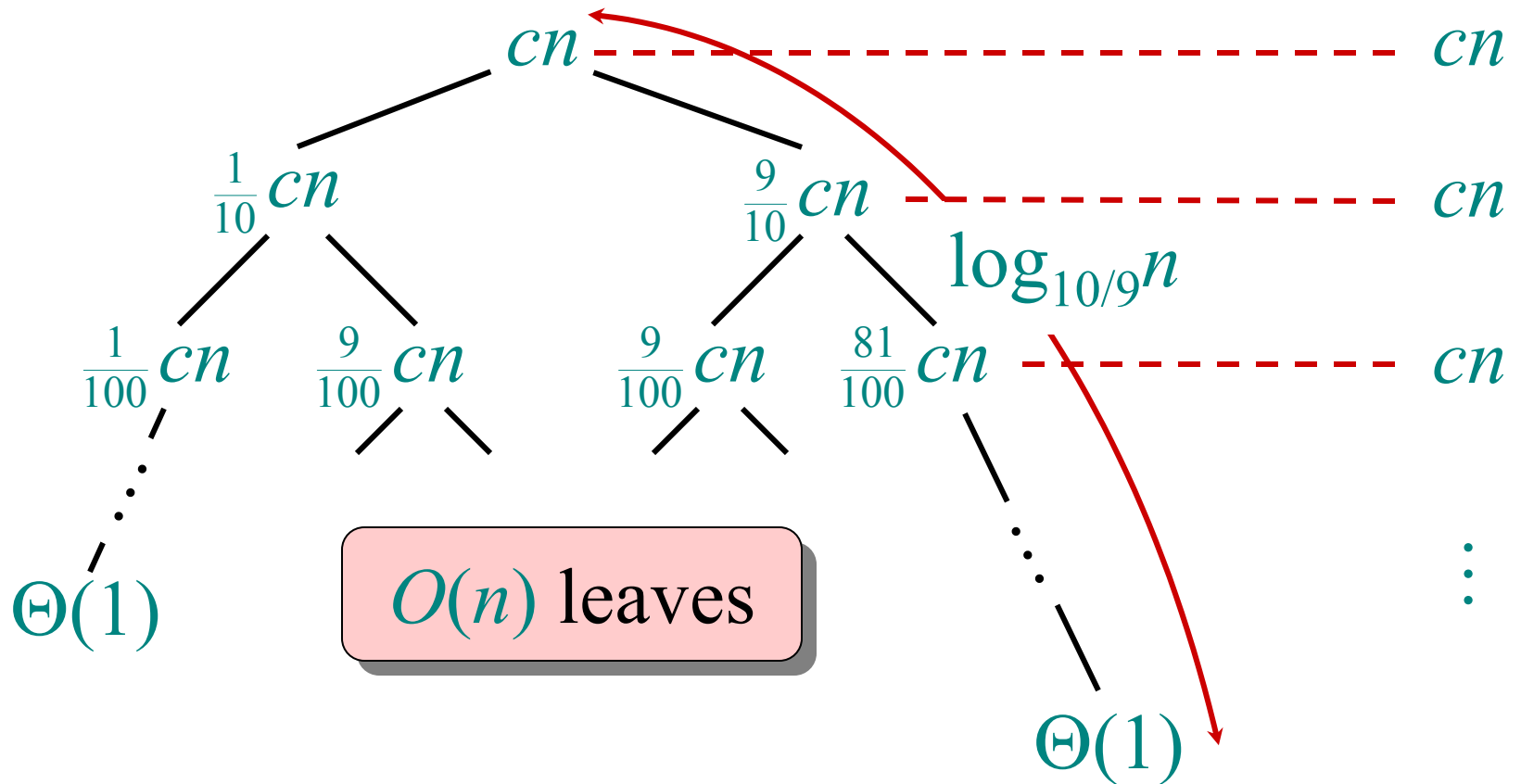


Analysis of “almost-best” case



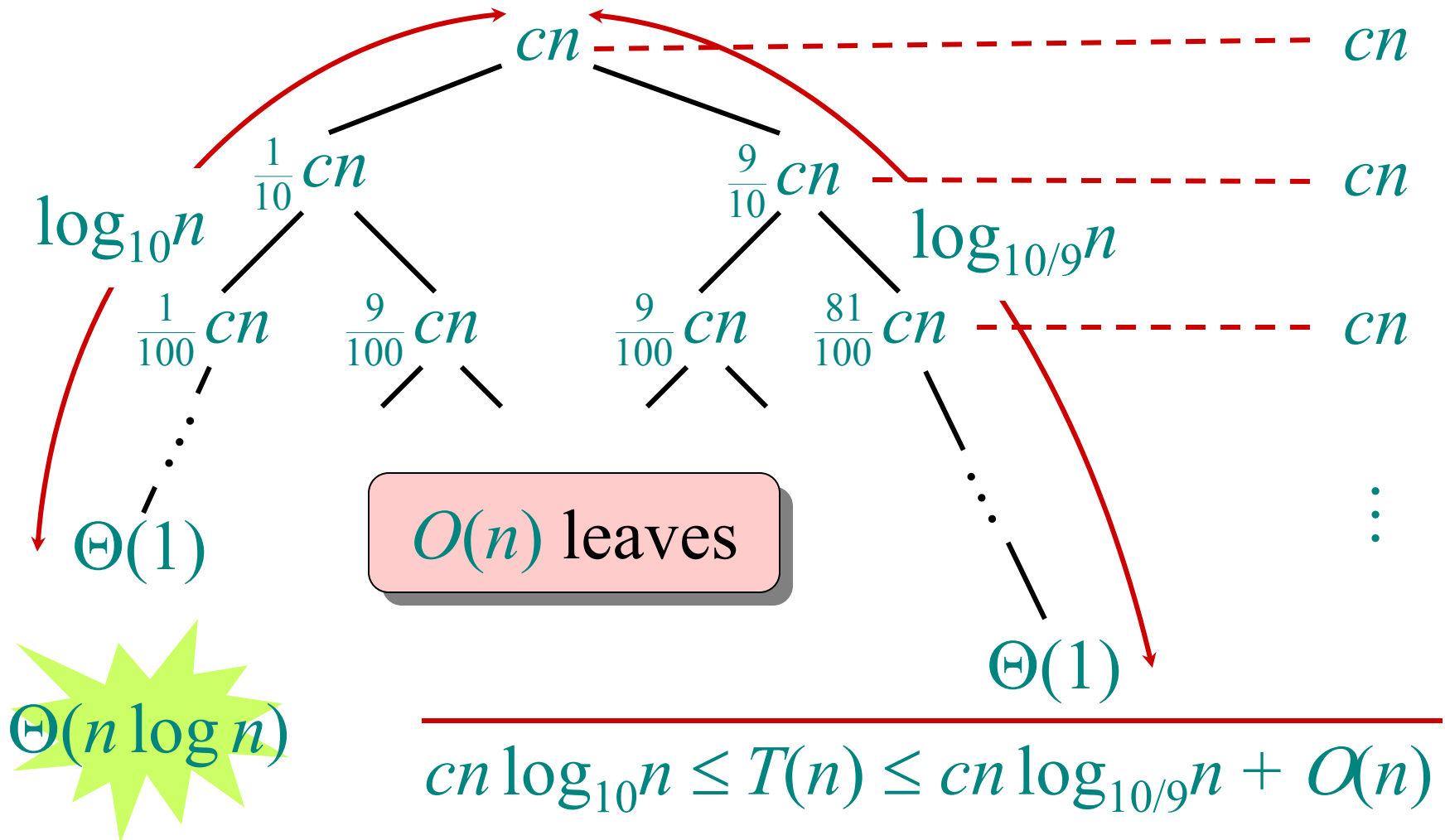


Analysis of “almost-best” case





Analysis of “almost-best” case





Quicksort Runtimes

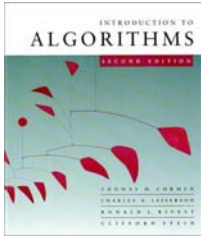
- Best case runtime $T_{\text{best}}(n) \in O(n \log n)$
- Worst case runtime $T_{\text{worst}}(n) \in O(n^2)$
- Worse than mergesort? Why is it called quicksort then?
- Its average runtime $T_{\text{avg}}(n) \in O(n \log n)$
- Better even, the expected runtime of **randomized quicksort** is $O(n \log n)$



Average Runtime

The **average runtime** $T_{\text{avg}}(n)$ for Quicksort is the average runtime over **all possible inputs** of length n .

- What kind of inputs are there?
- How many inputs are there?



Average Runtime

- What kind of inputs are there?
 - Do $[1, 2, \dots, n]$ and $[5, 6, \dots, n+5]$ cause different runtimes of Quicksort?
 - No. Therefore only consider all permutations of $[1, 2, \dots, n]$.
- How many inputs are there?
 - There are $n!$ different permutations of $[1, 2, \dots, n]$



Average Runtime

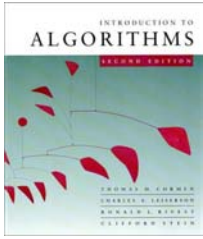
- Therefore, $T_{\text{avg}}(n)$ has to average the runtimes over all $n!$ different input permutations
 - Disadvantage of considering average runtime:
 - There are still worst-case inputs that will have a $O(n^2)$ runtime
 - Are all inputs really equally likely ? That depends on the application
- ⇒ **Better:** Use randomized quicksort



Randomized quicksort

IDEA: Partition around a *random* element.

- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.



Randomized quicksort analysis

Let $T(n)$ = the random variable for the running time of randomized quicksort on an input of size n , assuming random numbers are independent.

For $k = 0, 1, \dots, n-1$, define the *indicator random variable*

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.



Analysis (continued)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \dots \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$
$$= \sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n)).$$



Calculating expectation

$$E[T(n)] = E \left[\sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n)) \right]$$

Take expectations of both sides.



Calculating expectation

$$\begin{aligned} E[T(n)] &= E \left[\sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n)) \right] \\ &= \sum_{k=0}^{n-1} E[X_k (T(k) + T(n-k-1) + \Theta(n))] \end{aligned}$$

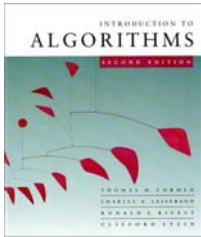
Linearity of expectation.



Calculating expectation

$$\begin{aligned} E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n))\right] \\ &= \sum_{k=0}^{n-1} E[X_k (T(k) + T(n-k-1) + \Theta(n))] \\ &= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)] \end{aligned}$$

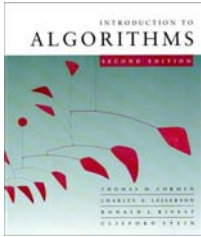
Independence of X_k from other random choices.



Calculating expectation

$$\begin{aligned} E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n))\right] \\ &= \sum_{k=0}^{n-1} E[X_k (T(k) + T(n-k-1) + \Theta(n))] \\ &= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)] \\ &= \frac{1}{n} \sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n} \sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n} \sum_{k=0}^{n-1} \Theta(n) \end{aligned}$$

Linearity of expectation; $E[X_k] = 1/n$.



Calculating expectation

$$\begin{aligned} E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n))\right] \\ &= \sum_{k=0}^{n-1} E[X_k (T(k) + T(n-k-1) + \Theta(n))] \\ &= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)] \\ &= \frac{1}{n} \sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n} \sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n} \sum_{k=0}^{n-1} \Theta(n) \\ &= \frac{2}{n} \sum_{k=0}^{n-1} E[T(k)] + \Theta(n) \end{aligned}$$

Summations have identical terms.



Hairy recurrence

$$E[T(n)] = \frac{2}{n} \sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \leq an \log n$ for constant $a > 0$.

- Choose a large enough so that $an \log n$ dominates $E[T(n)]$ for sufficiently small $n \geq 2$.

Use fact: $\sum_{k=2}^{n-1} k \log k \leq \frac{1}{2} n^2 \log n - \frac{1}{8} n^2$ (exercise).



Substitution method

$$E[T(n)] \leq \frac{2}{n} \sum_{k=2}^{n-1} ak \log k + \Theta(n)$$

Substitute inductive hypothesis.



Substitution method

$$\begin{aligned} E[T(n)] &\leq \frac{2}{n} \sum_{k=2}^{n-1} ak \log k + \Theta(n) \\ &\leq \frac{2a}{n} \left(\frac{1}{2} n^2 \log n - \frac{1}{8} n^2 \right) + \Theta(n) \end{aligned}$$

Use fact.



Substitution method

$$\begin{aligned} E[T(n)] &\leq \frac{2}{n} \sum_{k=2}^{n-1} ak \log k + \Theta(n) \\ &\leq \frac{2a}{n} \left(\frac{1}{2} n^2 \log n - \frac{1}{8} n^2 \right) + \Theta(n) \\ &= an \log n - \left(\frac{an}{4} - \Theta(n) \right) \end{aligned}$$

Express as *desired – residual*.



Substitution method

$$\begin{aligned} E[T(n)] &\leq \frac{2}{n} \sum_{k=2}^{n-1} ak \log k + \Theta(n) \\ &= \frac{2a}{n} \left(\frac{1}{2} n^2 \log n - \frac{1}{8} n^2 \right) + \Theta(n) \\ &= an \log n - \left(\frac{an}{4} - \Theta(n) \right) \\ &\leq an \log n \end{aligned}$$

,

if a is chosen large enough so that $an/4$ dominates the $\Theta(n)$.



Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from *code tuning*.
- Quicksort behaves well even with caching and virtual memory.