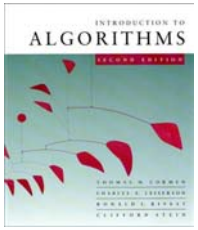
CS 5633 -- Spring 2005



Quicksort

Carola Wenk

Slides courtesy of Charles Leiserson with small changes by Carola Wenk

Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts "in place" (like insertion sort, but not like merge sort).
- Very practical (with tuning).

Divide and conquer

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\le x \le$ elements in upper subarray.

2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

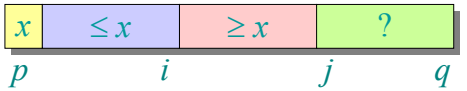
Key: *Linear-time partitioning subroutine.*

Partitioning subroutine

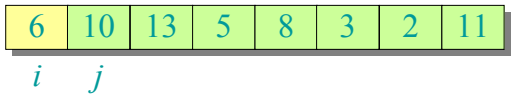
```
PARTITION(A, p, q)   ▷ A[p . . q]
  x ← A[p]           ▷ pivot = A[p]
  i ← p
  for j ← p + 1 to q
     do if A[j] ≤ x
           then  i ← i + 1
                 exchange A[i] ↔ A[j]
  exchange A[p] ↔ A[i]
  return i
```

Running time $= O(n)$ for n elements.

Invariant:



Example of partitioning



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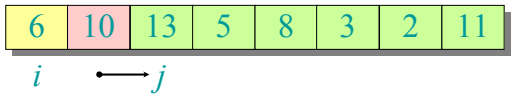

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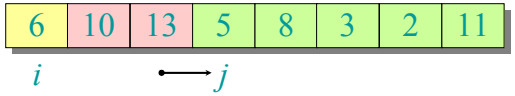

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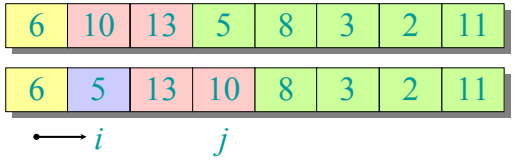

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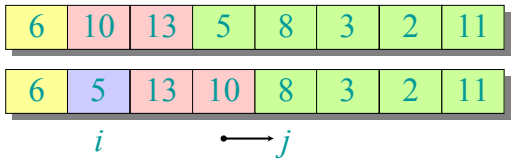
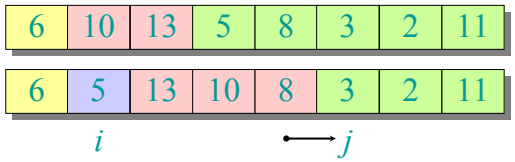
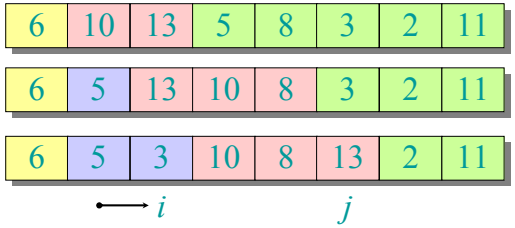


Example of partitioning



Example of partitioning



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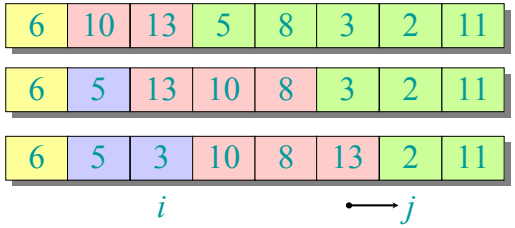

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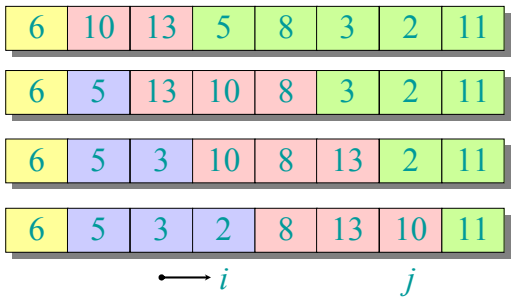

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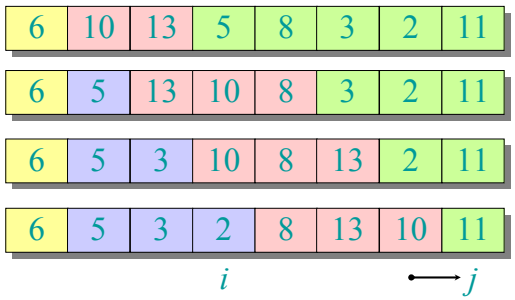

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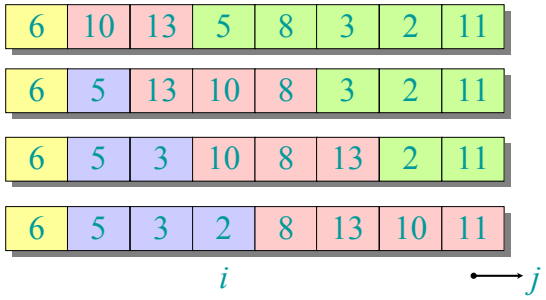

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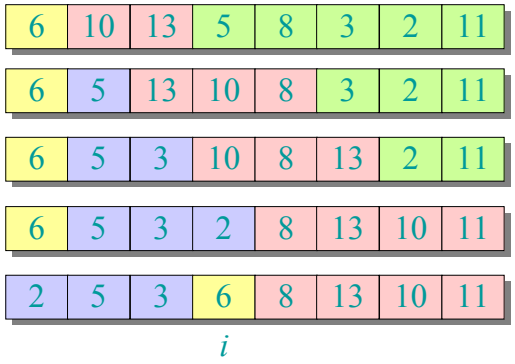

Pseudocode for quicksort

```
QUICKSORT(A, p, r)
  if p < r
    then q ← PARTITION(A, p, r)
         QUICKSORT(A, p, q–1)
         QUICKSORT(A, q+1, r)
```

Initial call: QUICKSORT(*A*, 1, *n*)

Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.

Worst-case of quicksort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \end{aligned}$$

(arithmetic series)

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

Worst-case recursion tree

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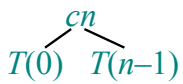
$T(n)$

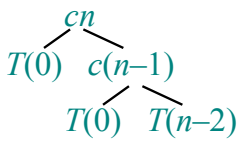
Worst-case recursion tree

$$T(n) = T(0) + T(n–1) + cn$$



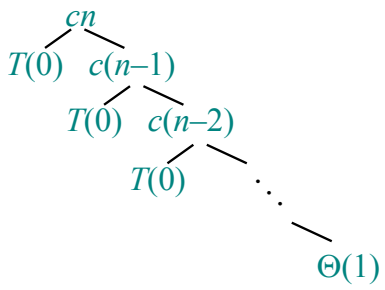
Worst-case recursion tree

$$T(n) = T(0) + T(n–1) + cn$$



Worst-case recursion tree

$$T(n) = T(0) + T(n–1) + cn$$



Worst-case recursion tree

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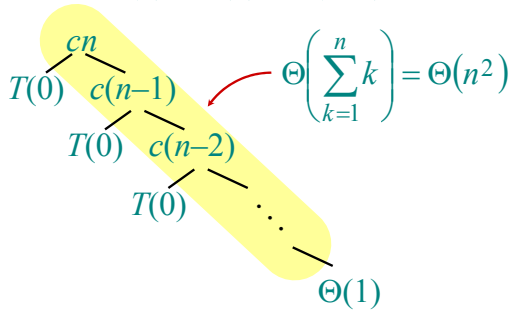
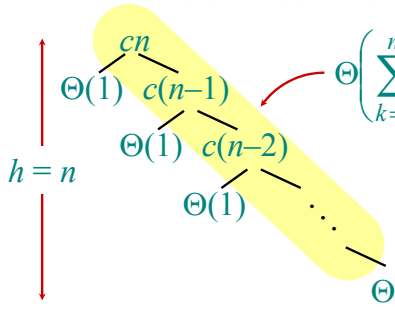


Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



$$T(n) = \Theta(n) + \Theta(n^2)$$
$$= \Theta(n^2)$$

Best-case analysis

(For intuition only!)

If we're lucky, PARTITION splits the array evenly:

$$T(n) = 2T(n/2) + \Theta(n)$$
$$= \Theta(n \log n)$$ (same as merge sort)

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

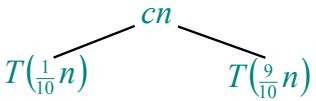
$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

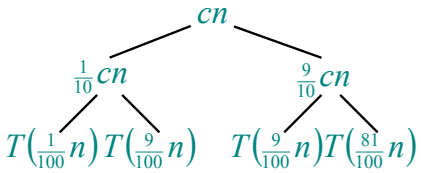
Analysis of "almost-best" case

$T(n)$

Analysis of "almost-best" case



Analysis of "almost-best" case



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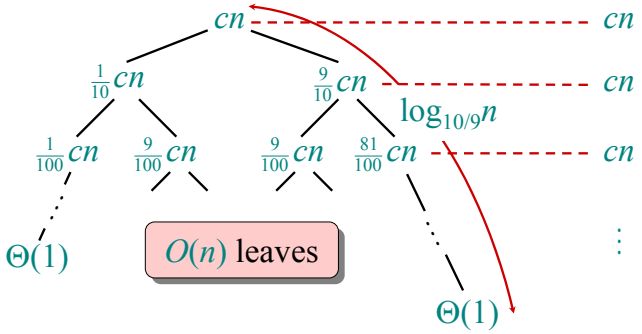

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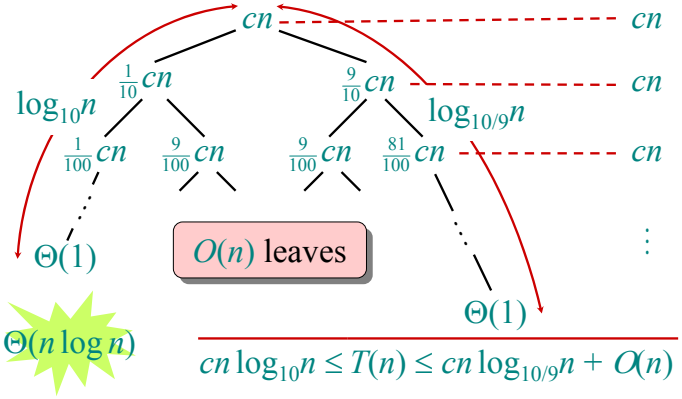

$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$

Randomized quicksort

IDEA: Partition around a ***random*** element.

- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.

Randomized quicksort analysis

Let $T(n)$ = the random variable for the running time of randomized quicksort on an input of size n, assuming random numbers are independent.

For $k = 0, 1, \ldots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.

Analysis (continued)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad\vdots \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right).$$

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Take expectations of both sides.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$
$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Linearity of expectation.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$
$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$
$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\left[T(k) + T(n-k-1) + \Theta(n)\right]$$

Independence of X_k from other random choices.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$
$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$
$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\left[T(k) + T(n-k-1) + \Theta(n)\right]$$
$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$
$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$
$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\left[T(k) + T(n-k-1) + \Theta(n)\right]$$
$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$
$$= \frac{2}{n}\sum_{k=0}^{n-1} E[T(k)] + \Theta(n)$$

Summations have identical terms.

Hairy recurrence

$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \le a\,n \log n$ for constant $a > 0$.

- Choose a large enough so that $a\,n \log n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

Use fact: $\sum_{k=2}^{n-1} k \log k \le \frac{1}{2} n^2 \log n - \frac{1}{8} n^2$ (exercise).

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \log k + \Theta(n)$$

Substitute inductive hypothesis.

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \log k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2} n^2 \log n - \frac{1}{8} n^2\right) + \Theta(n)$$

Use fact.

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \log k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2} n^2 \log n - \frac{1}{8} n^2\right) + \Theta(n)$$

$$= an \log n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as ***desired – residual***.

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \log k + \Theta(n)$$

$$= \frac{2a}{n}\left(\frac{1}{2}n^2 \log n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an \log n - \left(\frac{an}{4} - \Theta(n)\right)$$

$$\le an \log n \quad ,$$

if a is chosen large enough so that $an/4$ dominates the $\Theta(n)$.

Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.