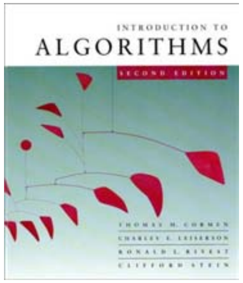
CS 5633 -- Spring 2005



Minimum Spanning Trees

Carola Wenk

Slides courtesy of Charles Leiserson with changes and additions by Carola Wenk

Graphs (review)

Definition. A ***directed graph (digraph)*** $G = (V, E)$ is an ordered pair consisting of

- a set V of ***vertices*** (singular: ***vertex***),
- a set $E \subseteq V \times V$ of ***edges***.

In an ***undirected graph*** $G = (V, E)$, the edge set E consists of *unordered* pairs of vertices.

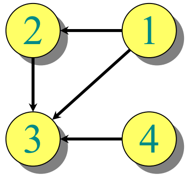
In either case, we have $|E| = O(|V|^2)$. Moreover, if G is connected, then $|E| \geq |V| - 1$.

(Review CLRS, Appendix B.4 and B.5.)

Adjacency-matrix representation

The ***adjacency matrix*** of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1 \,.\,.\, n, 1 \,.\,.\, n]$ given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in \text{E}, \\ 0 & \text{if } (i, j) \notin \text{E}. \end{cases}$$

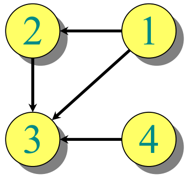


A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

$\Theta(|V|^2)$ storage $\Rightarrow$ ***dense*** representation.

Adjacency-list representation

An ***adjacency list*** of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to v.



$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3\}$

For undirected graphs, $|Adj[v]| = degree(v)$.

For digraphs, $|Adj[v]| = out\text{-}degree(v)$.

Adjacency-list representation

Handshaking Lemma:

- For undirected graphs:

  $\sum_{v \in V} degree(v) = 2|E|$

- For digraphs:

  $\sum_{v \in V} in\text{-}degree(v) + \sum_{v \in V} out\text{-}degree(v) = 2 | E |$

⇒ adjacency lists use $\Theta(|V| + |E|)$ storage

⇒ a ***sparse*** representation

Minimum spanning trees

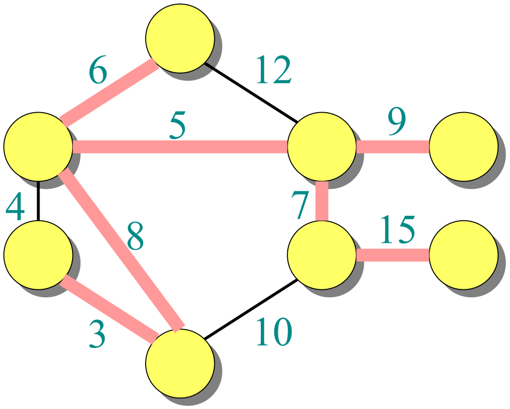
Input: A connected, undirected graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$.

- For simplicity, assume that all edge weights are distinct. (CLRS covers the general case.)

Output: A ***spanning tree*** T — a tree that connects all vertices — of minimum weight:

$$w(T) = \sum_{(u,v) \in T} w(u,v) .$$

Example of MST



Hallmark for "greedy" algorithms

Greedy-choice property

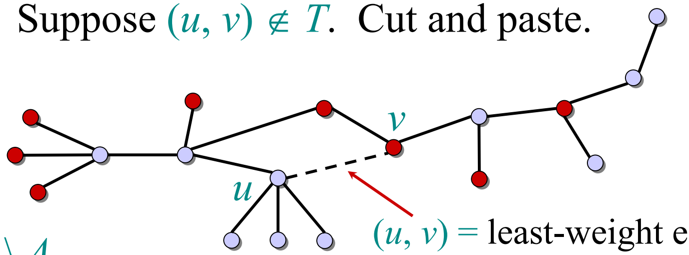
A locally optimal choice is globally optimal.

Theorem. Let T be the MST of $G = (V, E)$, and let $A \subseteq V$. Suppose that $(u, v) \in E$ is the least-weight edge connecting A to $V \setminus A$. Then, $(u, v) \in T$.

Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.

T:



○ ∈ A

● ∈ $V \setminus A$

(u, v) = least-weight edge connecting A to $V \setminus A$

Proof of theorem

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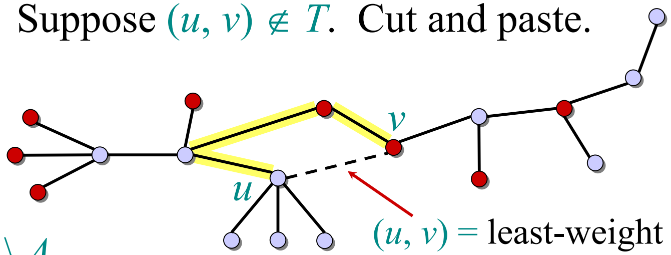

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(u, v) = least-weight edge connecting A to $V \setminus A$

Consider the unique simple path from u to v in T.

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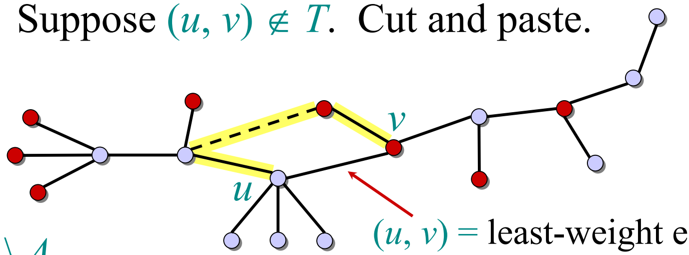

○ ∈ A

● ∈ $V \setminus A$

(u, v) = least-weight edge connecting A to $V \setminus A$

Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V \setminus A$.

Proof of theorem

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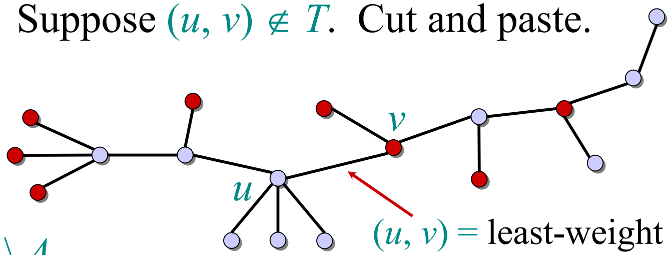


○ ∈ A

● ∈ $V \setminus A$

(u, v) = least-weight edge connecting A to $V \setminus A$

Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V \setminus A$.

A lighter-weight spanning tree than T results. □

Prim's algorithm

IDEA: Maintain $V \setminus A$ as a priority queue Q. Key each vertex in Q with the weight of the least-weight edge connecting it to a vertex in A.

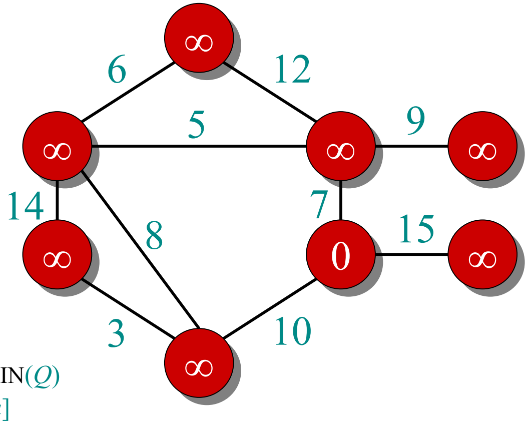
```
$Q \leftarrow V$
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
while $Q \neq \emptyset$
    do $u \leftarrow$ EXTRACT-MIN($Q$)
        for each $v \in Adj[u]$
            do if $v \in Q$ and $w(u, v) < key[v]$
                then $key[v] \leftarrow w(u, v)$      ⊳ DECREASE-KEY
                    $\pi[v] \leftarrow u$
```

At the end, $\{(v, \pi[v])\}$ forms the MST.

Example of Prim's algorithm

○ $\in A$

● $\in V \setminus A$



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Example of Prim's algorithm

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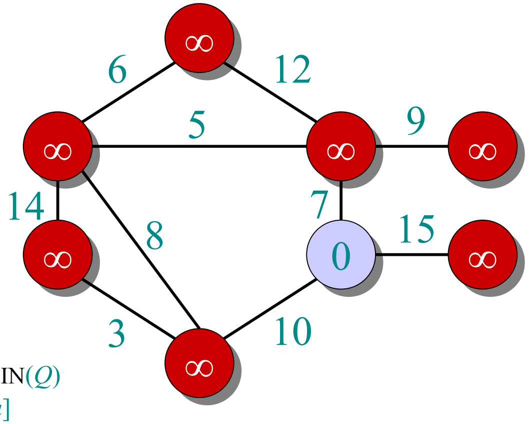

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Example of Prim's algorithm

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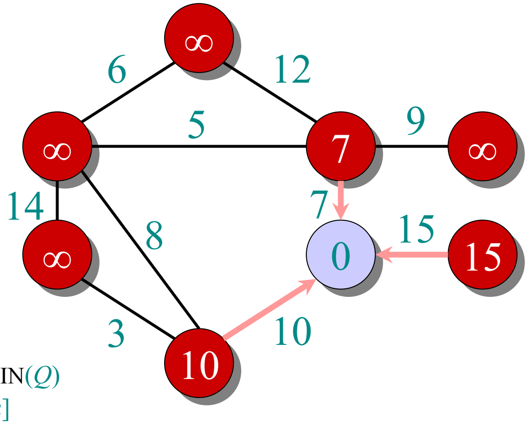

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Example of Prim's algorithm

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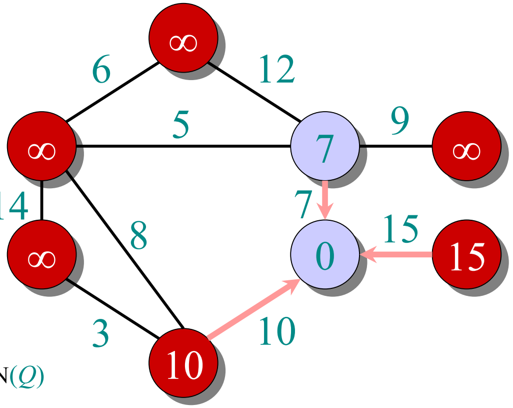

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Example of Prim's algorithm

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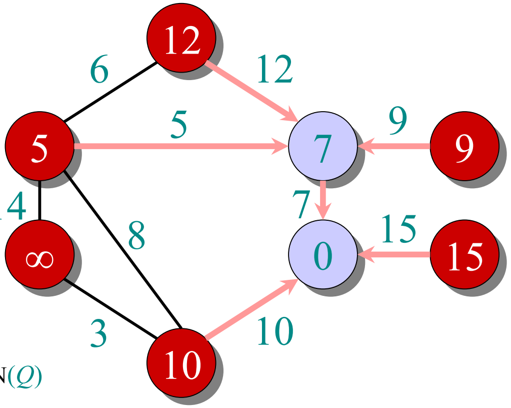

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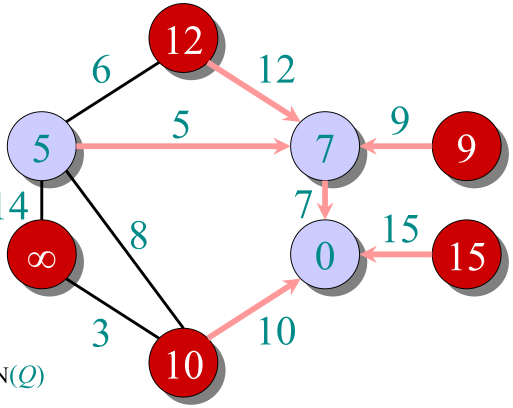

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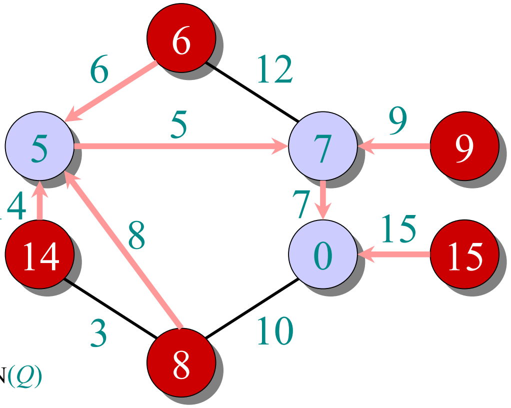

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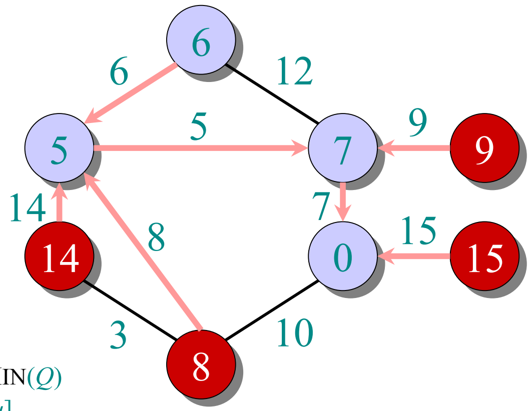
Example of Prim's algorithm

○ ∈ *A*

● ∈ *V* \ *A*



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Example of Prim's algorithm

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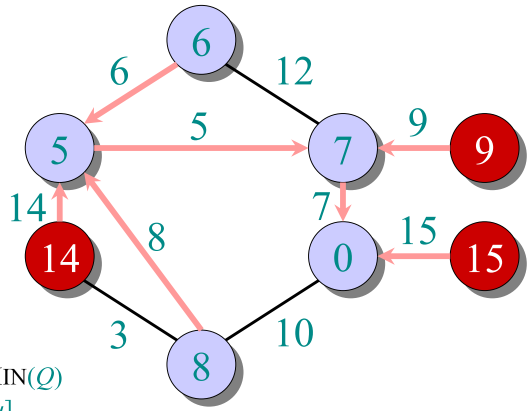
● ∈ *V* \ *A*



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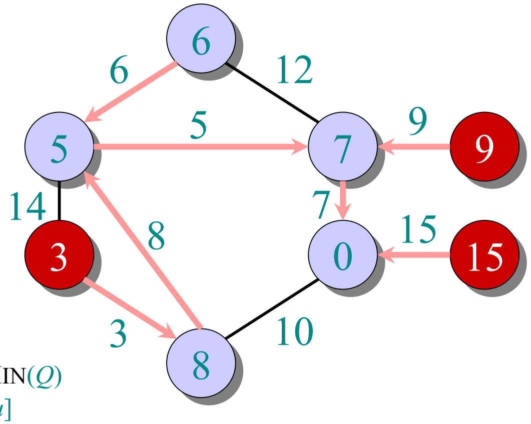

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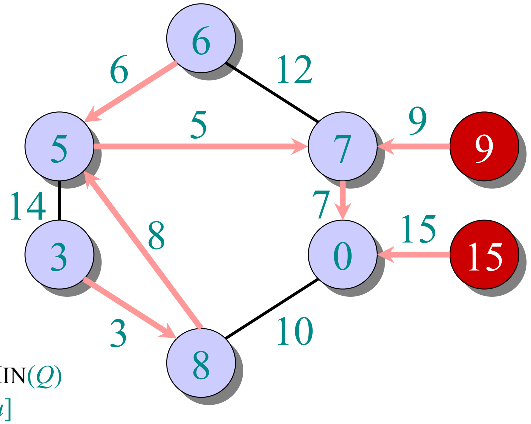

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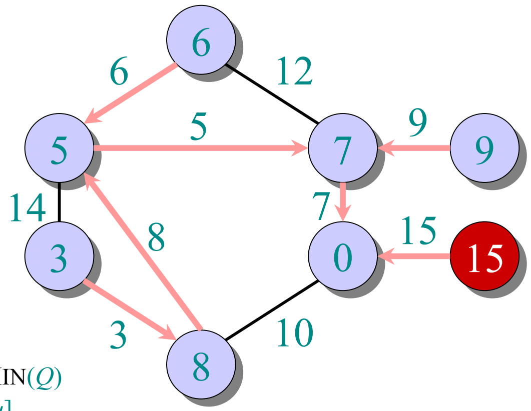
Example of Prim's algorithm

○ $\in A$

● $\in V \setminus A$



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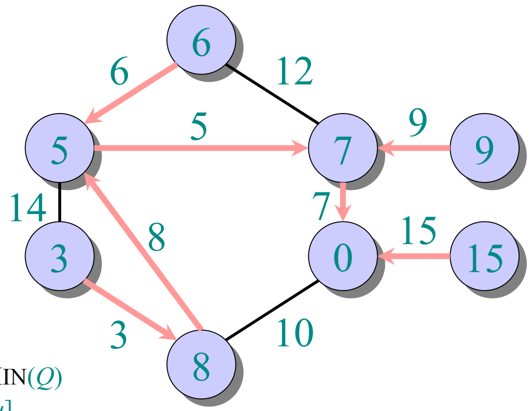
Example of Prim's algorithm

○ $\in A$

● $\in V \setminus A$



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```

Analysis of Prim

$\Theta(|V|)$ total

```
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$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
while $Q \neq \emptyset$
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                then $key[v] \leftarrow w(u, v)$
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```

$|V|$ times

$degree(u)$ times

Handshaking Lemma ⇒ $\Theta(|E|)$ implicit DECREASE-KEY's.

Time = $\Theta(|V|) \cdot T_{\text{EXTRACT-MIN}} + \Theta(|E|) \cdot T_{\text{DECREASE-KEY}}$

Analysis of Prim (continued)

Time = $\Theta(|V|) \cdot T_{\text{EXTRACT-MIN}} + \Theta(|E|) \cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(\|V\|)$	$O(1)$	$O(\|V\|^2)$
binary heap	$O(\log \|V\|)$	$O(\log \|V\|)$	$\boxed{O(\|E\| \log \|V\|)}$
Fibonacci heap	$O(\log \|V\|)$ amortized	$O(1)$ amortized	$O(\|E\| + \|V\| \log \|V$ worst case

Kruskal's algorithm

IDEA (again greedy):
Repeatedly pick edge with smallest weight as long as it does not form a cycle.

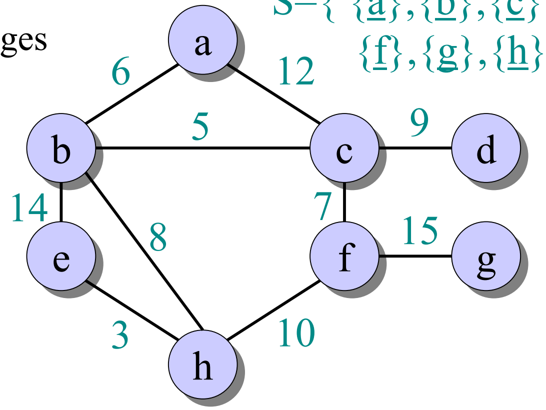
- The algorithm creates a set of trees (a **forest**)
- During the algorithm the added edges merge the trees together, such that in the end only one tree remains

- The correctness of this greedy strategy is not obvious and needs to be proven. (Proof skipped here.)

Example of Kruskal's algorithm

MST edges

a set repr.



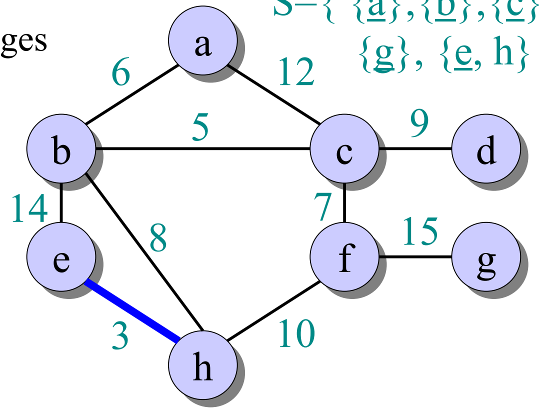
S={ {a},{b},{c},{d},{e}, {f},{g},{h} }

Every node is a single tree.

Example of Kruskal's algorithm

MST edges

a set repr.



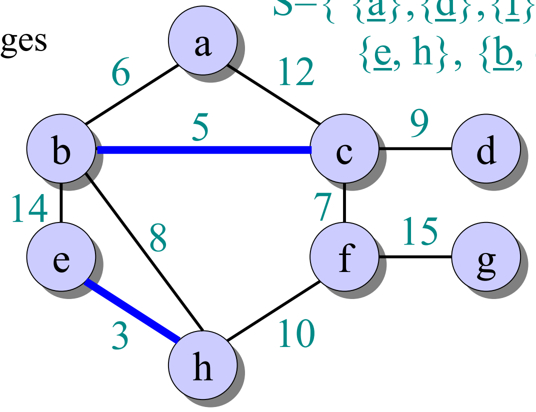
S={ {a},{b},{c},{d},{f}, {g}, {e, h} }

Edge 3 merged two singleton trees.

Example of Kruskal's algorithm

MST edges

a set repr.



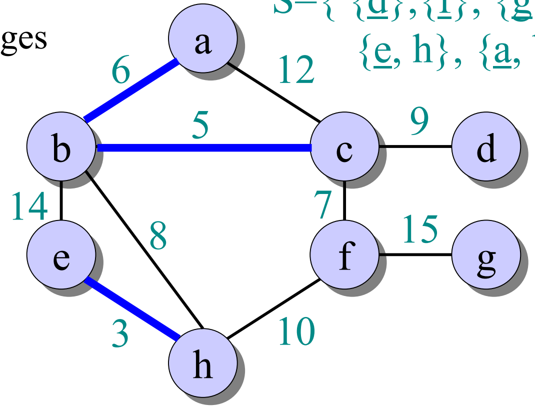
S={ {a},{d},{f}, {g}, {e, h}, {b, c} }

Example of Kruskal's algorithm

▬ MST edges

a set repr.

S={ {d},{f}, {g}
{e, h}, {a, b, c} }



Example of Kruskal's algorithm

▬ MST edges

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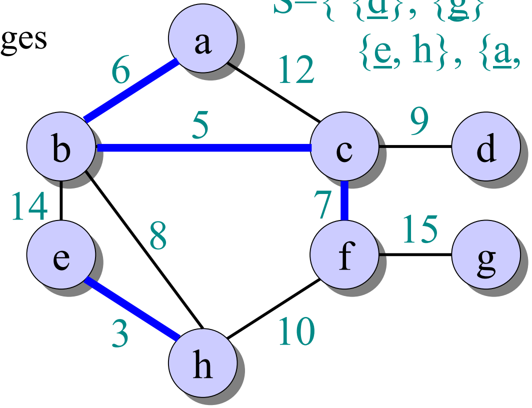
S={ {d}, {g}
{e, h}, {a, b, c, f} }



Example of Kruskal's algorithm

▬ MST edges

a set repr.

S={ {d}, {g}
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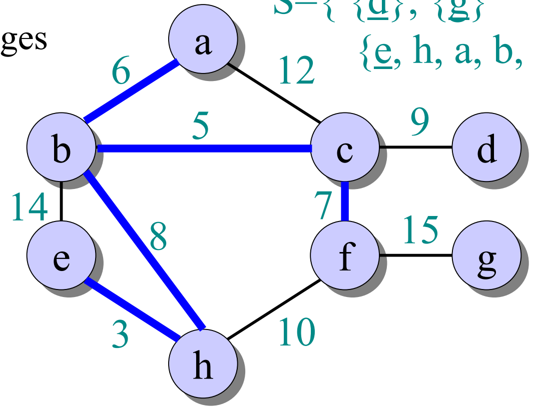


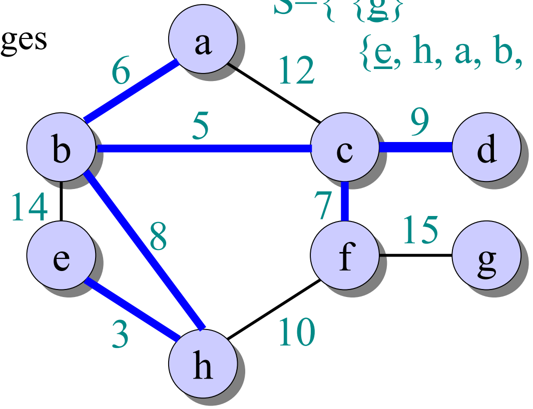
Edge 8 merged the two bigger trees.

Example of Kruskal's algorithm

▬ MST edges

a set repr.

S={ {g}
{e, h, a, b, c, f, d} }



Example of Kruskal's algorithm

▬ MST edges

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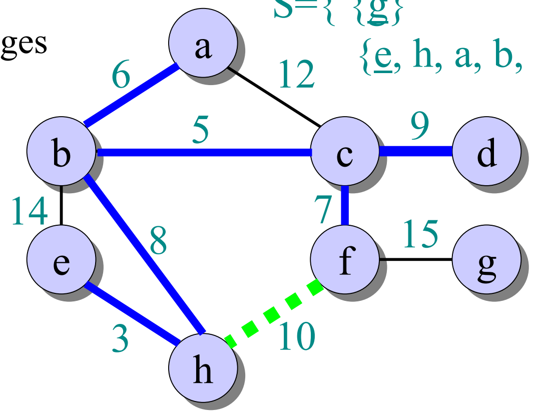


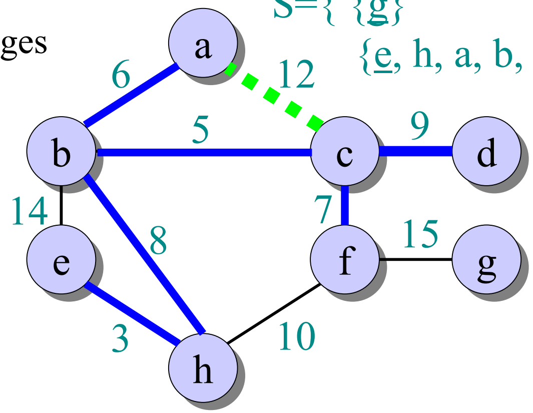
Skip edge 10 as it would cause a cycle.

Example of Kruskal's algorithm

▬ MST edges

a set repr.

S={ {g}
{e, h, a, b, c, f, d} }



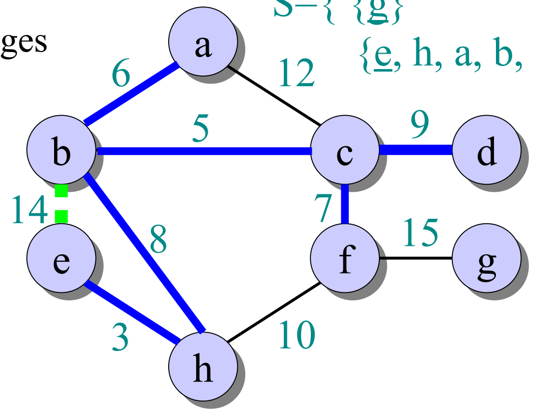
Skip edge 12 as it would cause a cycle.

Example of Kruskal's algorithm

▬ MST edges

a set repr.

S={ {g}
{e, h, a, b, c, f, d} }



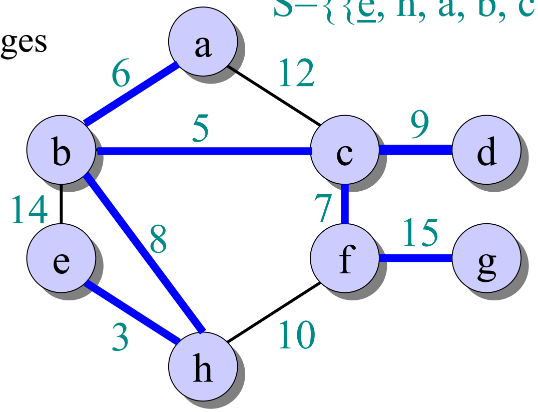
Skip edge 14 as it would cause a cycle.

Example of Kruskal's algorithm

▬ MST edges

a set repr.

S={{e, h, a, b, c, f, d, g}



Disjoint-set data structure (Union-Find)

- Maintains a dynamic collection of *pairwise-disjoint* sets $\mathsf{S} = \{S_1, S_2, \ldots, S_r\}$.
- Each set S_i has one element distinguished as the **representative** element.
- Supports operations:
  - $O(1)$ • MAKE-SET(x): adds new set $\{x\}$ to S
  - $O(\alpha(n))$ • UNION(x, y): replaces sets S_x, S_y with $S_x \cup S_y$
  - $O(\alpha(n))$ • FIND-SET(x): returns the representative of the set S_x containing element x
- $1 < \alpha(n) < \log^*(n) < \log(\log(n)) < \log(n)$

Kruskal's algorithm

IDEA: Repeatedly pick edge with smallest weight as long as it does not form a cycle.

```
                 $S \leftarrow \emptyset$   ▷ $S$ will contain all MST edges
$O(|V|)$         for each $v \in V$ do MAKE-SET($v$)
$O(|E|\log|E|)$  Sort edges of $E$ in non-decreasing order according to $w$
$O(|E|)$         For each $(u,v) \in E$ taken in this order do
$O(\alpha(|V|))$ {  if FIND-SET($u$) $\neq$ FIND-SET($v$)   ▷ $u,v$ in different trees
                       $A \leftarrow A \cup \{(u,v)\}$
                       UNION($u,v$)   ▷ Edge $(u,v)$ connects the two trees
```

Runtime: $O(|V|+|E|\log|E|+|E|\alpha(|V|)) = O(|E| \log |E|)$

MST algorithms

- Prim's algorithm:
  - Maintains one tree
  - Runs in time $O(|E| \log |V|)$, with binary heaps.
- Kruskal's algorithm:
  - Maintains a forest and uses the disjoint-set data structure
  - Runs in time $O(|E| \log |E|)$
- Best to date: Randomized algorithm by Karger, Klein, Tarjan [1993]. Runs in expected time $O(|V| + |E|)$