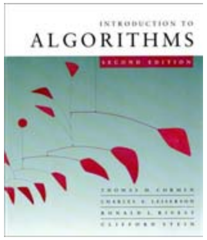
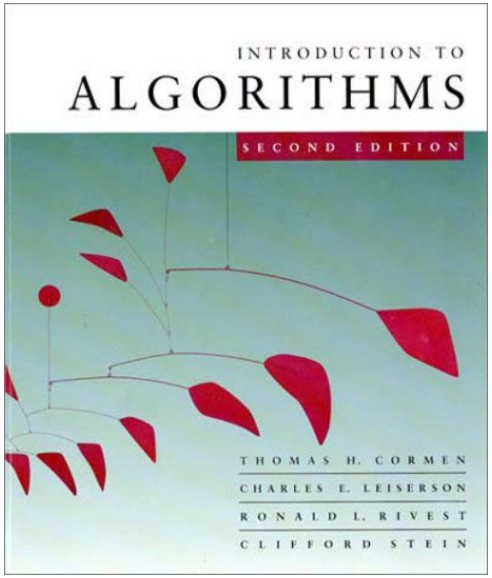


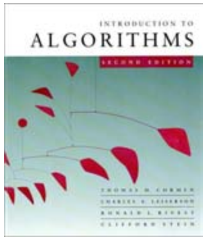
CS 5633 -- Spring 2004



Computational Geometry

Carola Wenk

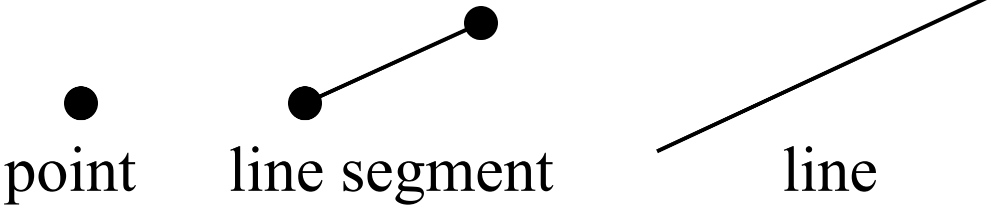
Slides courtesy of Charles Leiserson with small changes by Carola Wenk



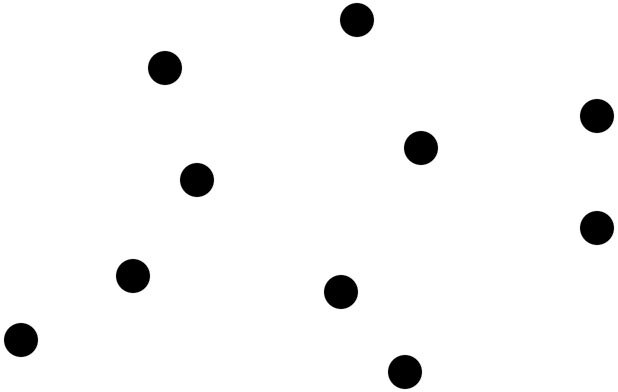
Computational geometry

Algorithms for solving “geometric problems” in 2D and higher.

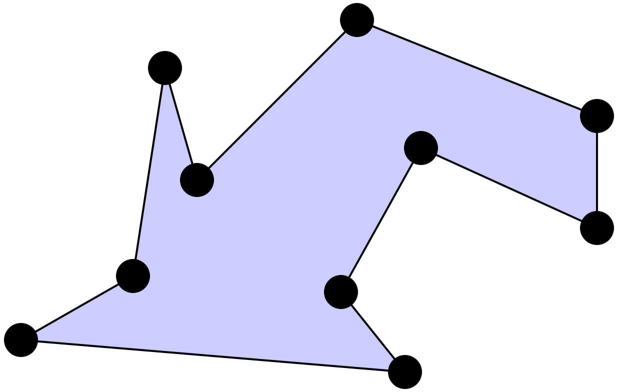
Fundamental objects:



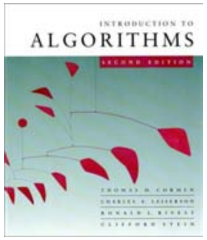
Basic structures:



point set



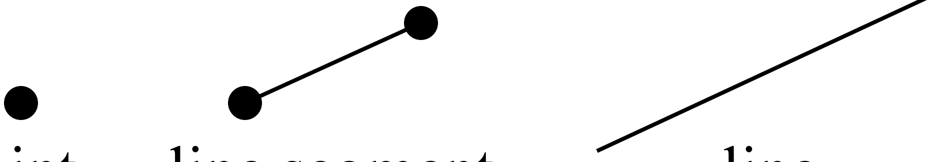
polygon



Computational geometry

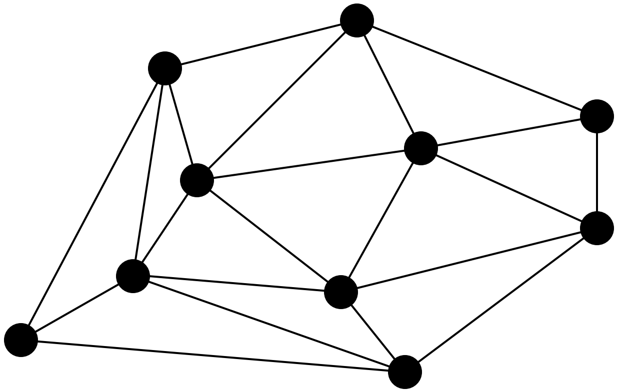
Algorithms for solving "geometric problems" in 2D and higher.

Fundamental objects:

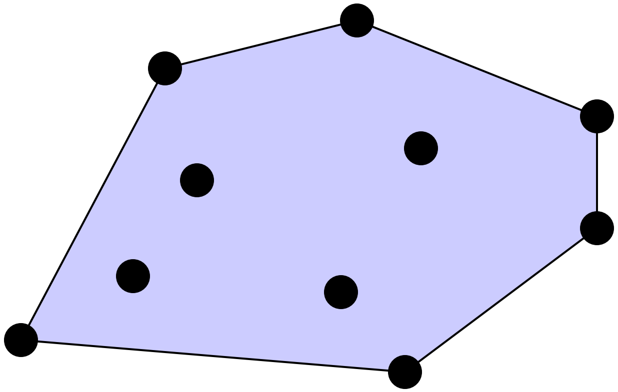


point line segment line

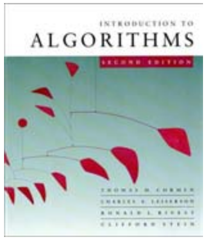
Basic structures:



triangulation



convex hull



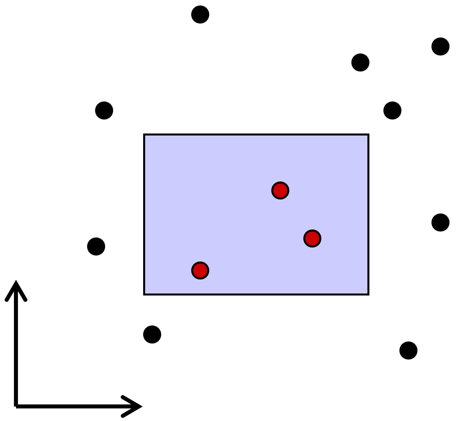
Orthogonal range searching

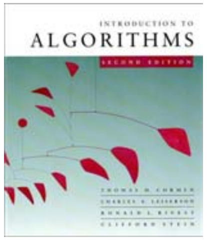
Input: n points in d dimensions

- E.g., representing a database of n records each with d numeric fields

Query: Axis-aligned ***box*** (in 2D, a rectangle)

- Report on the points inside the box:
  - Are there any points?
  - How many are there?
  - List the points.





Orthogonal range searching

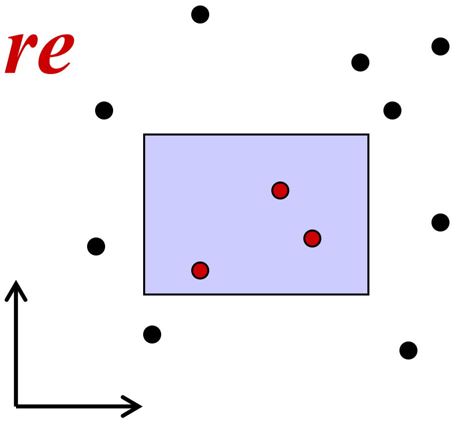
Input: n points in d dimensions

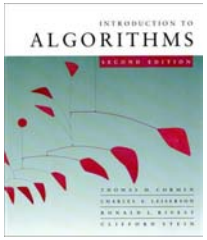
Query: Axis-aligned ***box*** (in 2D, a rectangle)

- Report on the points inside the box

Goal: Preprocess points into a data structure to support fast queries

- Primary goal: ***Static data structure***
- In 1D, we will also obtain a dynamic data structure supporting insert and delete





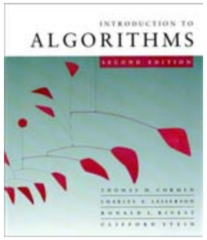
1D range searching

In 1D, the query is an interval:

First solution:

- Sort the points and store them in an array
  - Solve query by binary search on endpoints.
  - Obtain a static structure that can list k answers in a query in $O(k + \log n)$ time.

Goal: Obtain a dynamic structure that can list k answers in a query in $O(k + \log n)$ time.

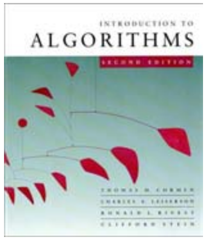


1D range searching

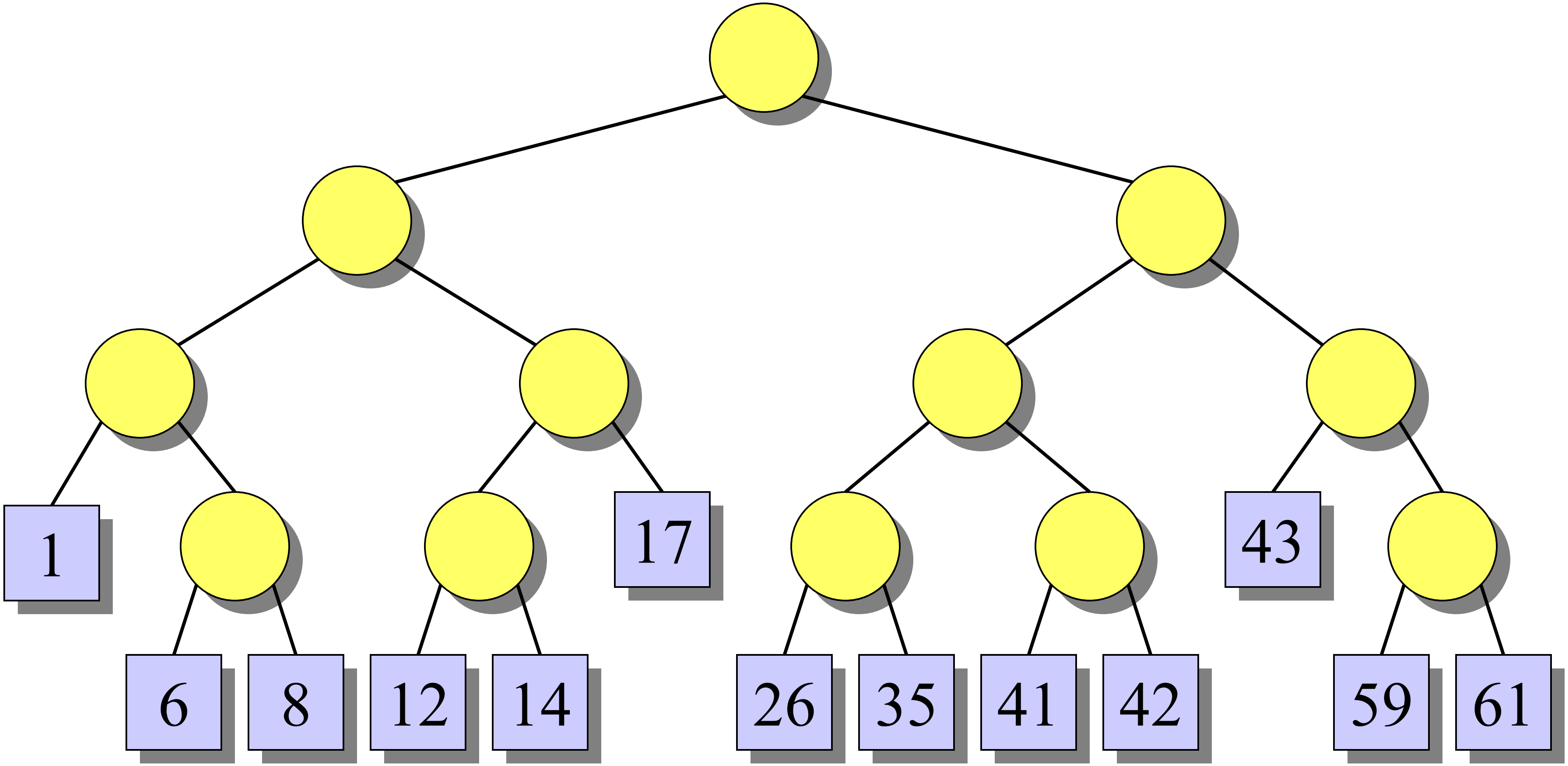
In 1D, the query is an interval:

New solution that extends to higher dimensions:

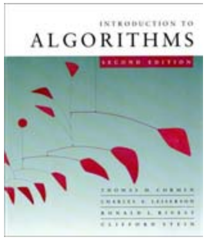
- Balanced binary search tree
    - New organization principle: Store points in the *leaves* of the tree.
    - Internal nodes store copies of the leaves to satisfy binary search property:
        - Node x stores in $key[x]$ the maximum key of any leaf in the left subtree of x.



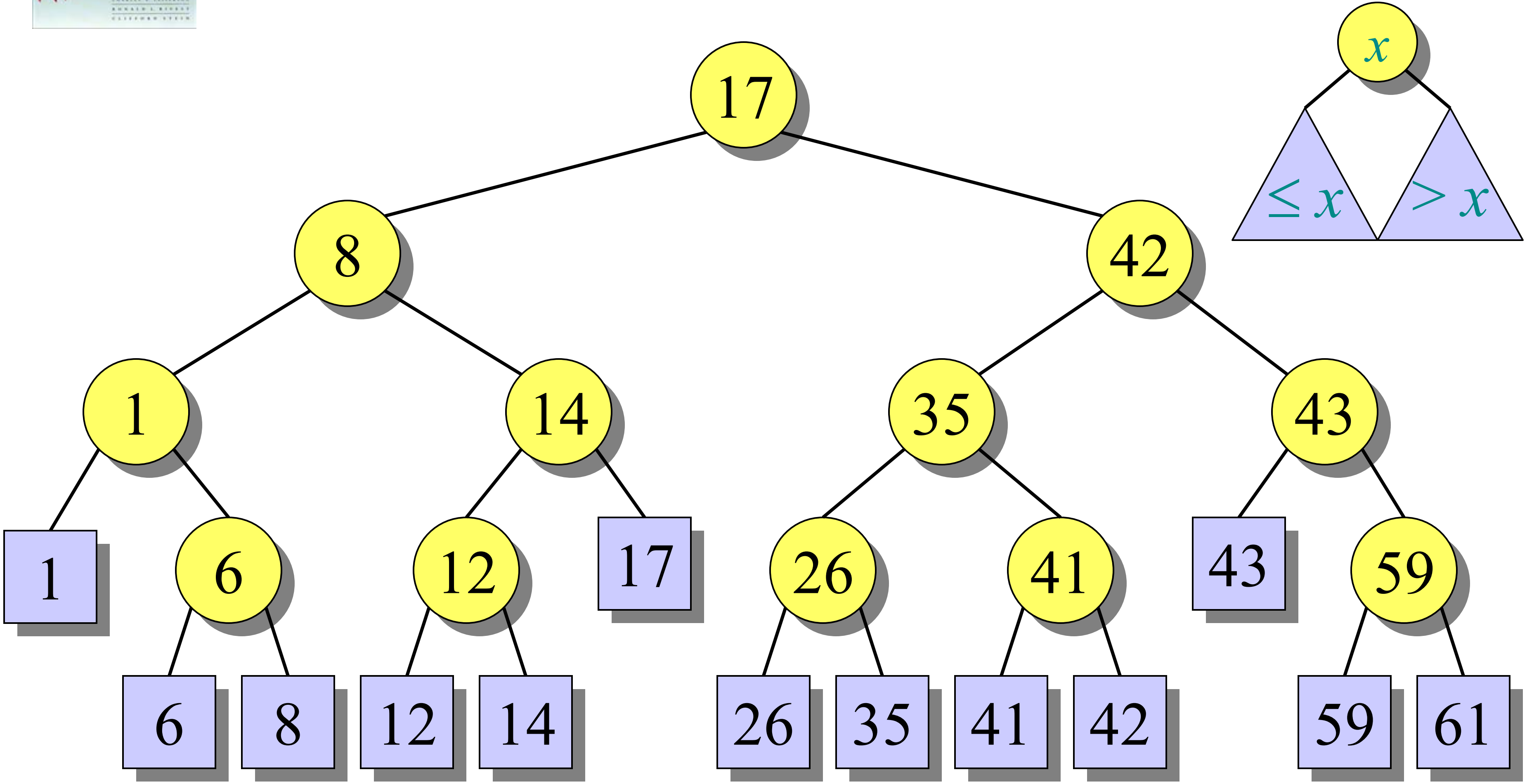
Example of a 1D range tree



key[*x*] is the maximum key of any leaf in the left subtree of *x*.



Example of a 1D range tree



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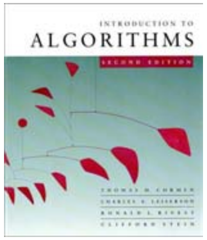
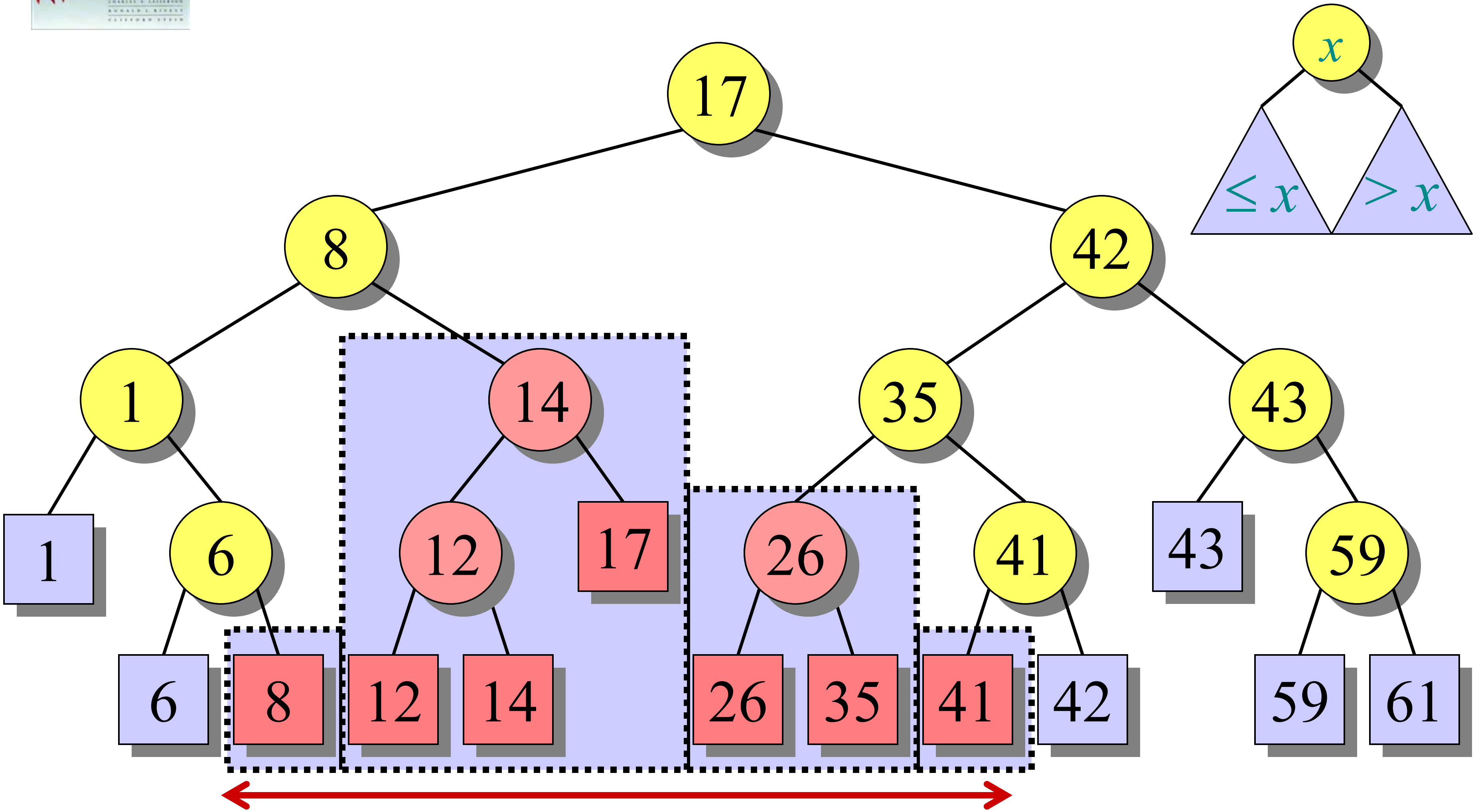
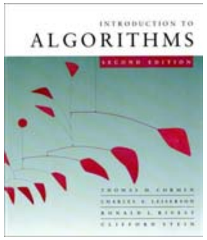


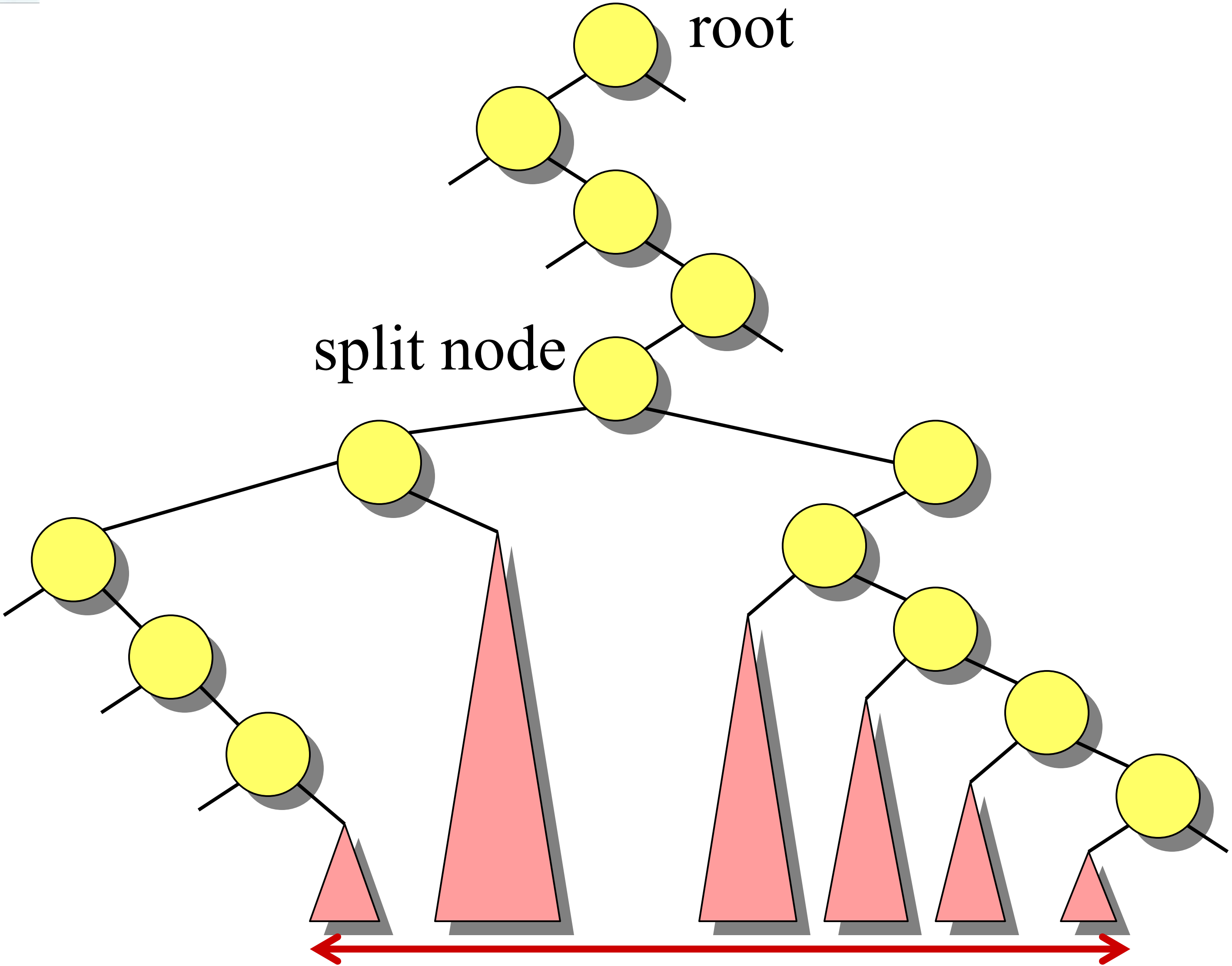
Example of a 1D range query

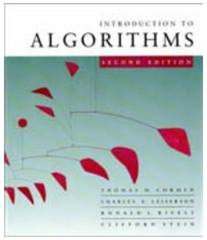


RANGE-QUERY([7, 41])



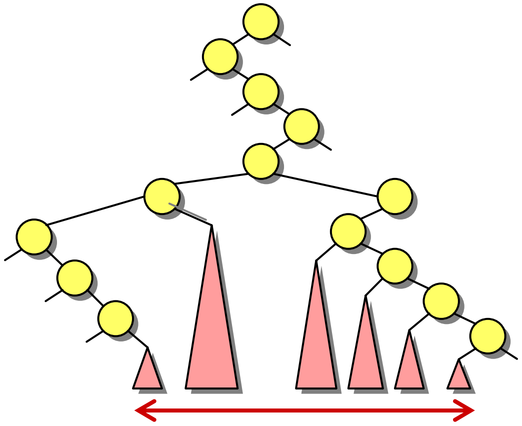
General 1D range query

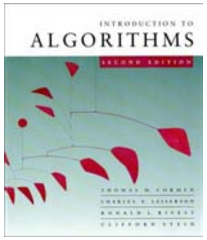




Pseudocode, part 1: Find the split node

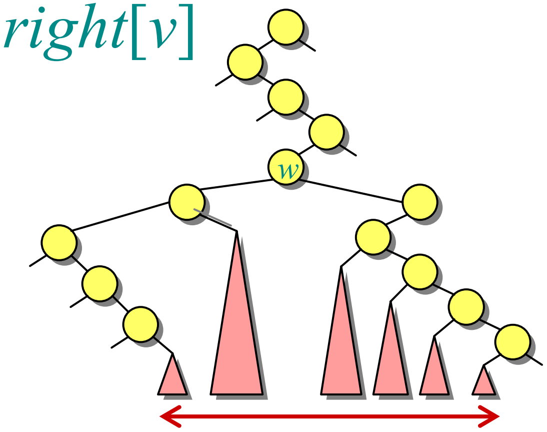
```
1D-RANGE-QUERY($T$, $[x_1, x_2]$)
    $w \leftarrow \mathrm{root}[T]$
    while $w$ is not a leaf and ($x_2 \leq key[w]$ or $key[w] < x_1$)
        do if $x_2 \leq key[w]$
            then $w \leftarrow left[w]$
            else $w \leftarrow right[w]$
    // $w$ is now the split node
    [traverse left and right from $w$ and report relevant subtrees]
```

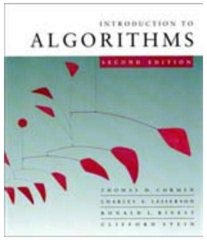




Pseudocode, part 2: Traverse left and right from split node

```
1D-RANGE-QUERY($T$, $[x_1, x_2]$)
    [find the split node]
    // $w$ is now the split node
    if $w$ is a leaf
     then output the leaf $w$ if $x_1 \le key[w] \le x_2$
     else $v \leftarrow left[w]$                                  // Left traversal
          while $v$ is not a leaf
            do if $x_1 \le key[v]$
                 then output the subtree rooted at $right[v]$
                      $v \leftarrow left[v]$
                 else $v \leftarrow right[v]$
          output the leaf $v$ if $x_1 \le key[v] \le x_2$
          [symmetrically for right traversal]
```



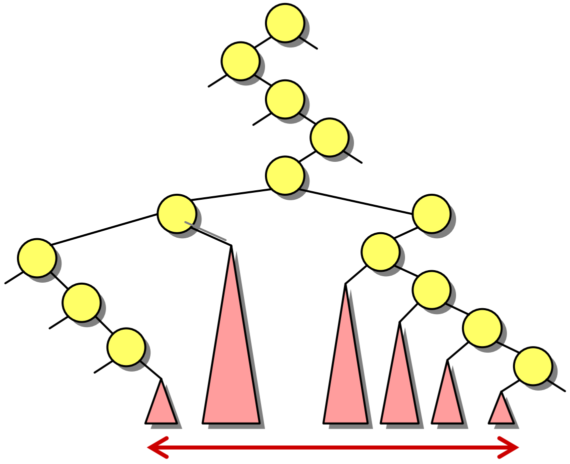


Analysis of 1D-RANGE-QUERY

Query time: Answer to range query represented by O(log n) subtrees found in O(log n) time.

Thus:

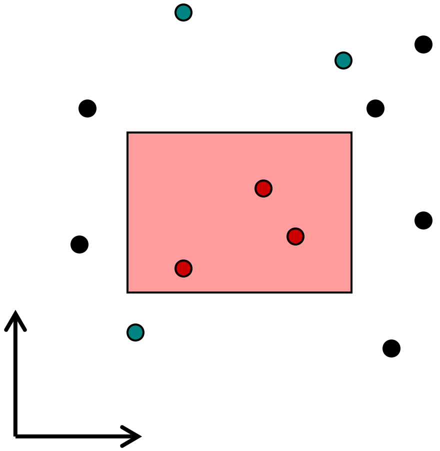
- Can test for points in interval in O(log n) time.
- Can report the first k points in interval in O(k + log n) time.
- Can count points in interval in O(log n) time (exercise)

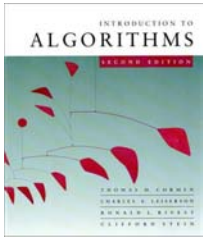


Space: O(n)

Preprocessing time: O(n log n)

2D range trees



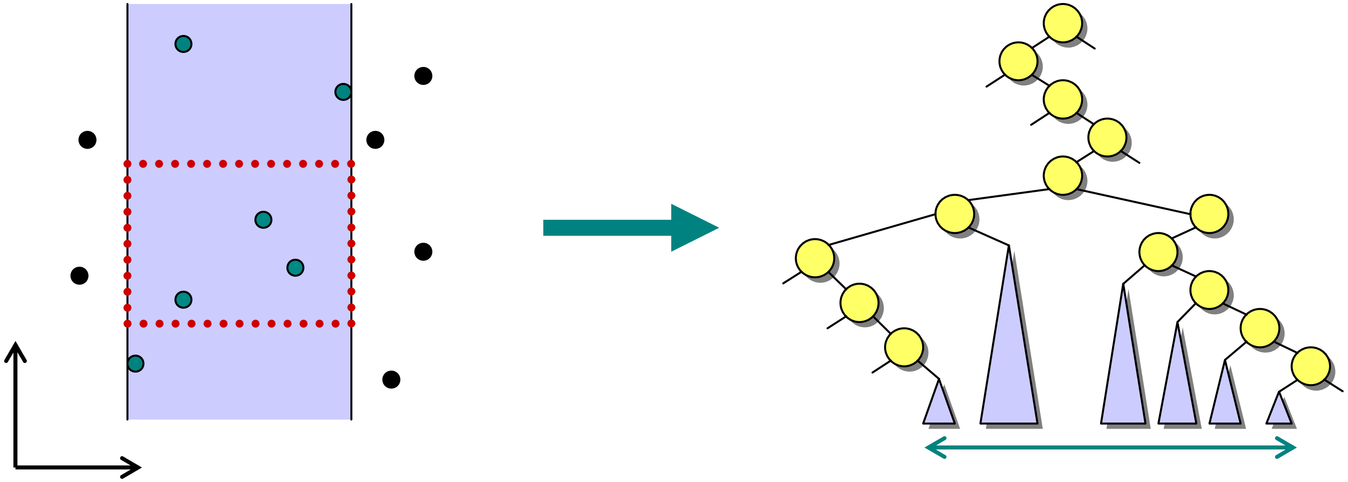


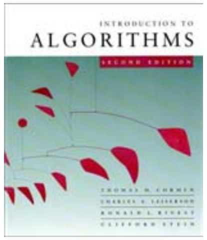
2D range trees

Store a ***primary*** 1D range tree for all the points based on *x*-coordinate.

Thus in O(log *n*) time we can find O(log *n*) subtrees representing the points with proper *x*-coordinate.

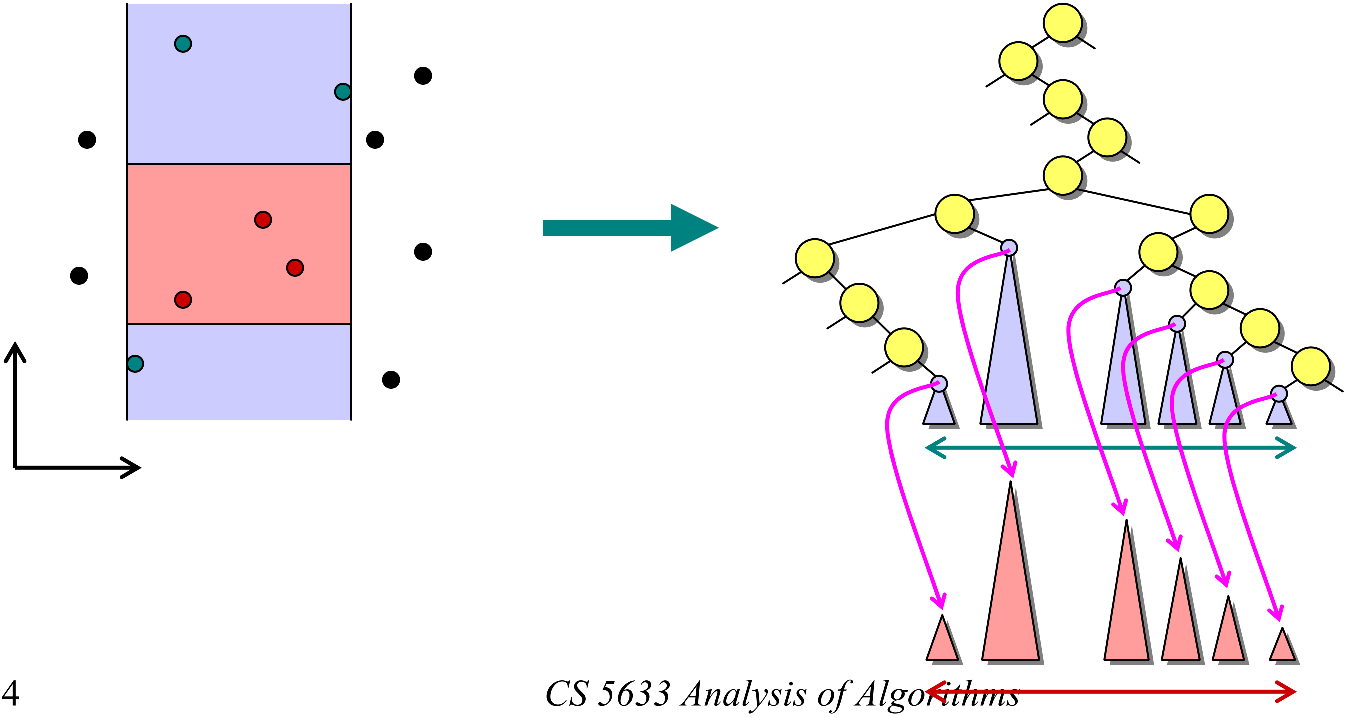
How to restrict to points with proper *y*-coordinate?

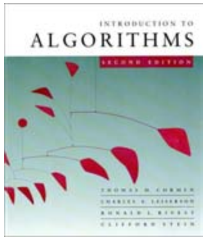




2D range trees

Idea: In primary 1D range tree of *x*-coordinate, every node stores a ***secondary*** 1D range tree based on *y*-coordinate for all points in the subtree of the node. Recursively search within each.



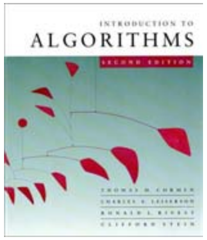


Analysis of 2D range trees

Query time: In $O(\log^2 n) = O((\log n)^2)$ time, we can represent answer to range query by $O(\log^2 n)$ subtrees. Total cost for reporting k points: $O(k + (\log n)^2)$.

Space: The secondary trees at each level of the primary tree together store a copy of the points. Also, each point is present in each secondary tree along the path from the leaf to the root. Either way, we obtain that the space is $O(n \log n)$.

Preprocessing time: $O(n \log n)$

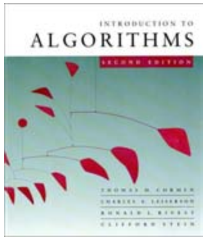


d-dimensional range trees

Each node of the secondary y-structure stores a tertiary z-structure representing the points in the subtree rooted at the node, etc.

Query time: $O(k + \log^d n)$ to report k points.
Space: $O(n \log^{d-1} n)$
Preprocessing time: $O(n \log^{d-1} n)$

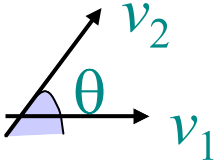
Best data structure to date:
Query time: $O(k + \log^{d-1} n)$ to report k points.
Space: $O(n (\log n / \log \log n)^{d-1})$
Preprocessing time: $O(n \log^{d-1} n)$



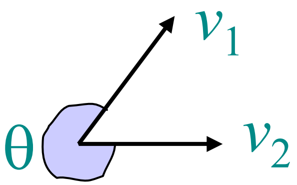
Primitive operations: Crossproduct

Given two vectors $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$, is their counterclockwise angle θ

- ***convex*** ($< 180°$),
- ***reflex*** ($> 180°$), or
- borderline (0 or $180°$)?



convex



reflex

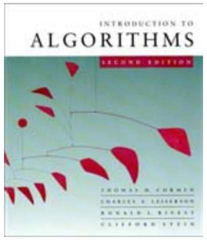
Crossproduct $v_1 \times v_2 = x_1\, y_2 - y_1\, x_2$

$$= |v_1|\,|v_2| \sin\theta\ .$$

Thus, $\operatorname{sign}(v_1 \times v_2) = \operatorname{sign}(\sin\theta)$ > 0 if θ convex,

< 0 if θ reflex,

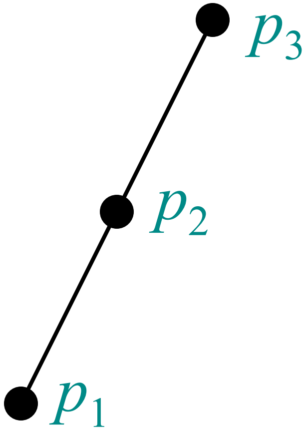
$= 0$ if θ borderline.



Primitive operations: Orientation test

Given three points p_1, p_2, p_3 are they

- in ***clockwise (cw) order***,
- in ***counterclockwise (ccw) order***, or
- ***collinear***?



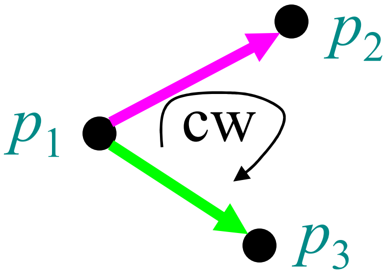
collinear

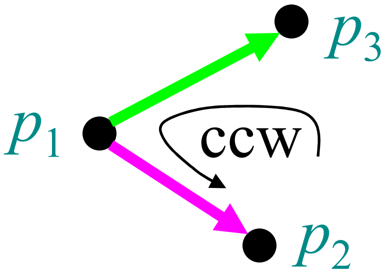
$$(p_2 - p_1) \times (p_3 - p_1)$$

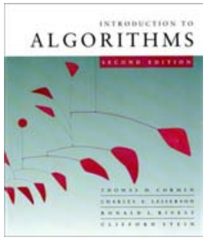
> 0 if ccw

< 0 if cw

$= 0$ if collinear



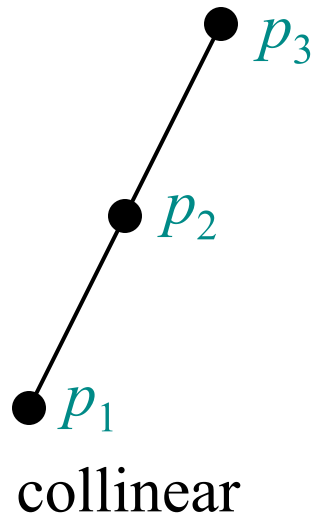




Primitive operations: Sidedness test

Given three points p_1, p_2, p_3 are they

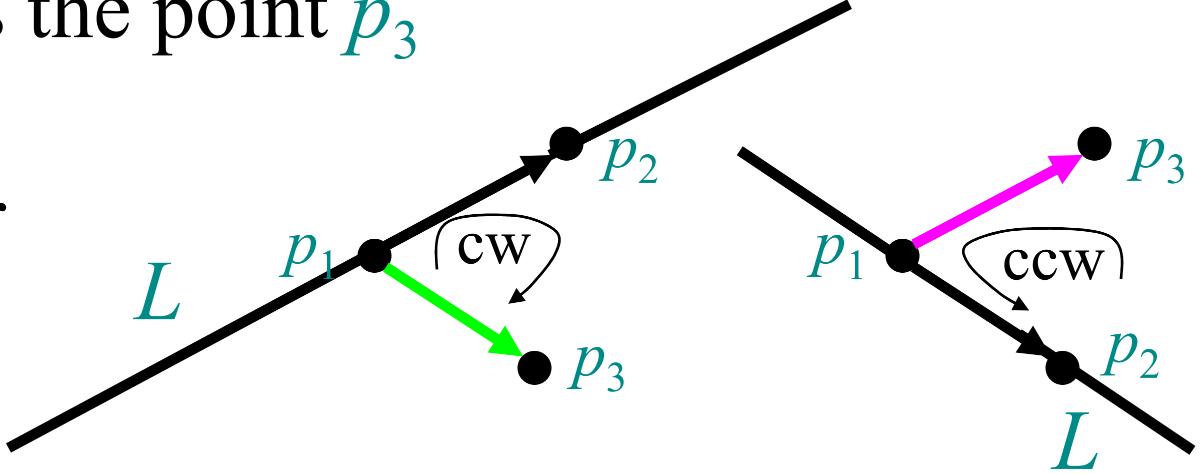
- in ***clockwise (cw) order***,
- in ***counterclockwise (ccw) order***, or
- ***collinear***?

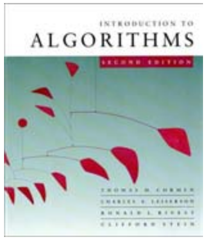


Let L be the oriented line from p_1 to p_2.

Equivalently, is the point p_3

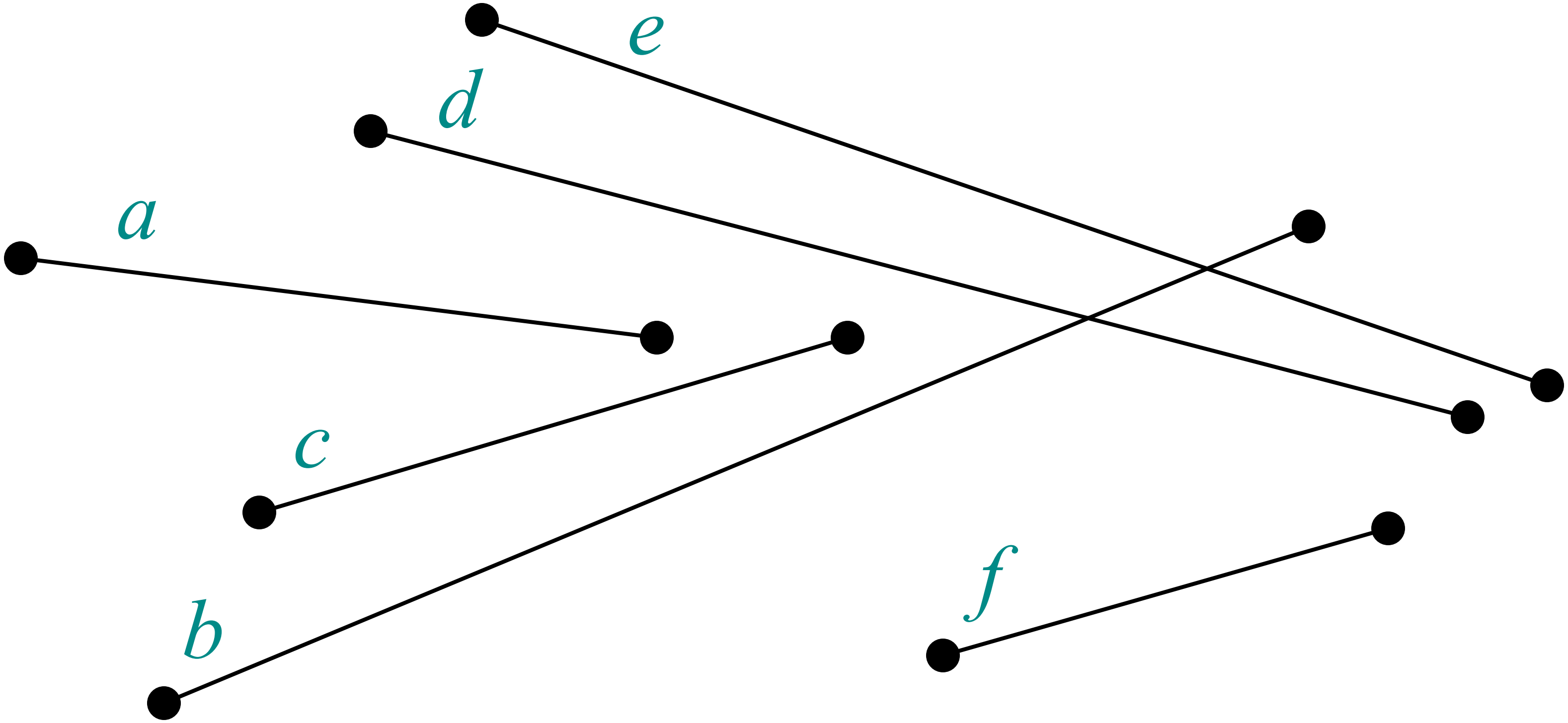
- ***right*** of L,
- ***left*** of L, or
- ***on*** L?

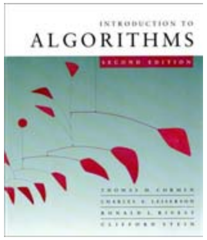




Line-segment intersection

Given n line segments, does any pair intersect?
Obvious algorithm: $O(n^2)$.





Sweep-line algorithm

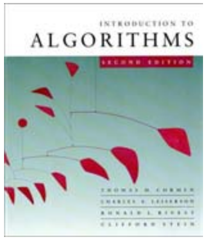
- Sweep a vertical line from left to right (conceptually replacing x-coordinate with time).
- Maintain dynamic set S of segments that intersect the sweep line, ordered by y-coordinate of intersection.

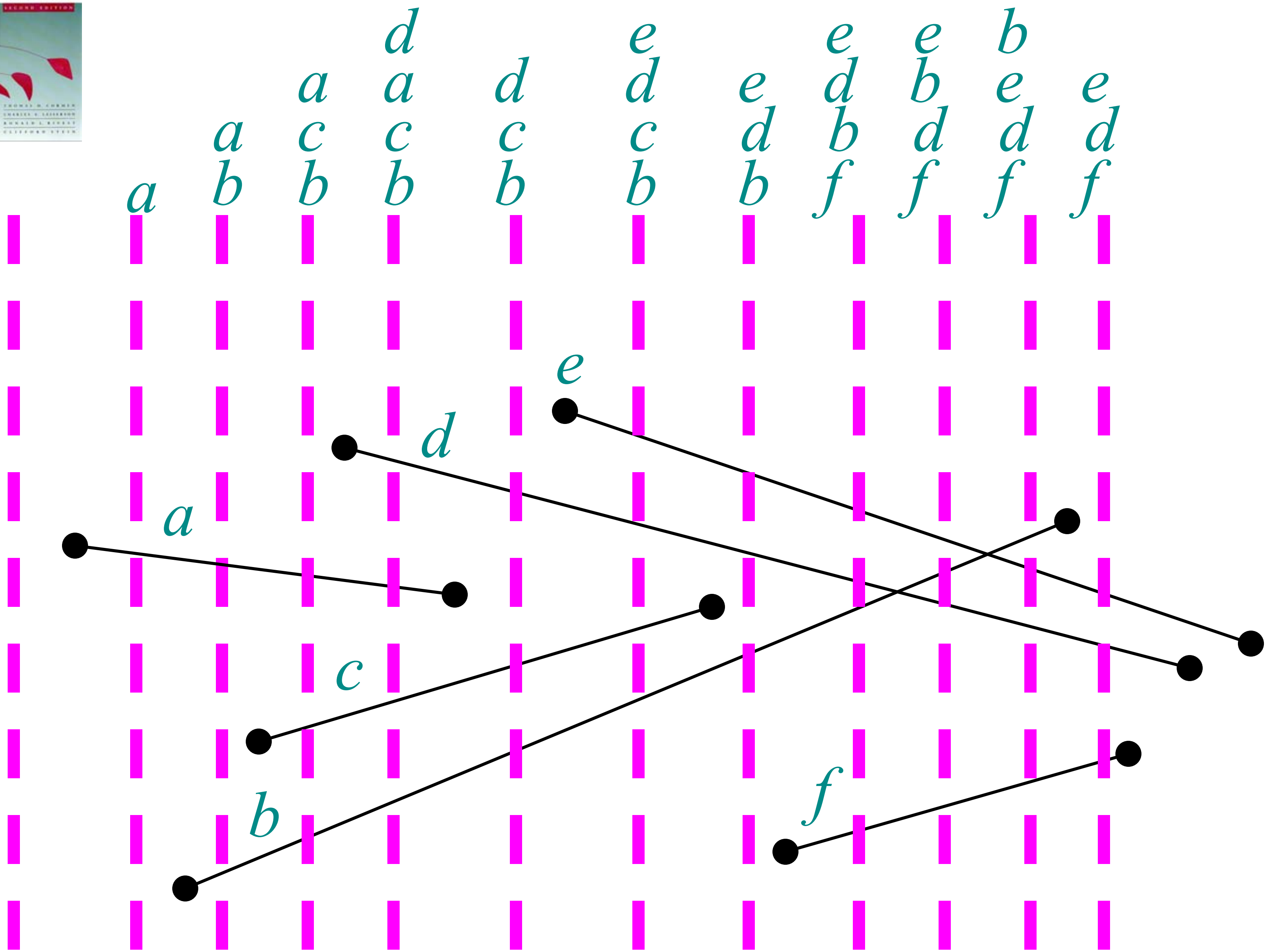
sweep-line status

- Order changes when
  - new segment is encountered,
  - existing segment finishes, or
  - two segments cross

segment endpoints

- Key ***event points*** are therefore segment endpoints.


INTRODUCTION TO
ALGORITHMS
SECOND EDITION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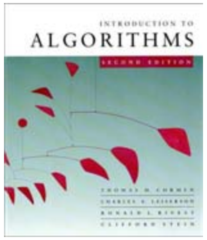
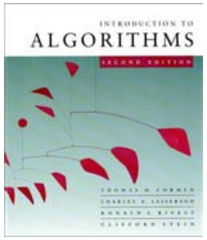
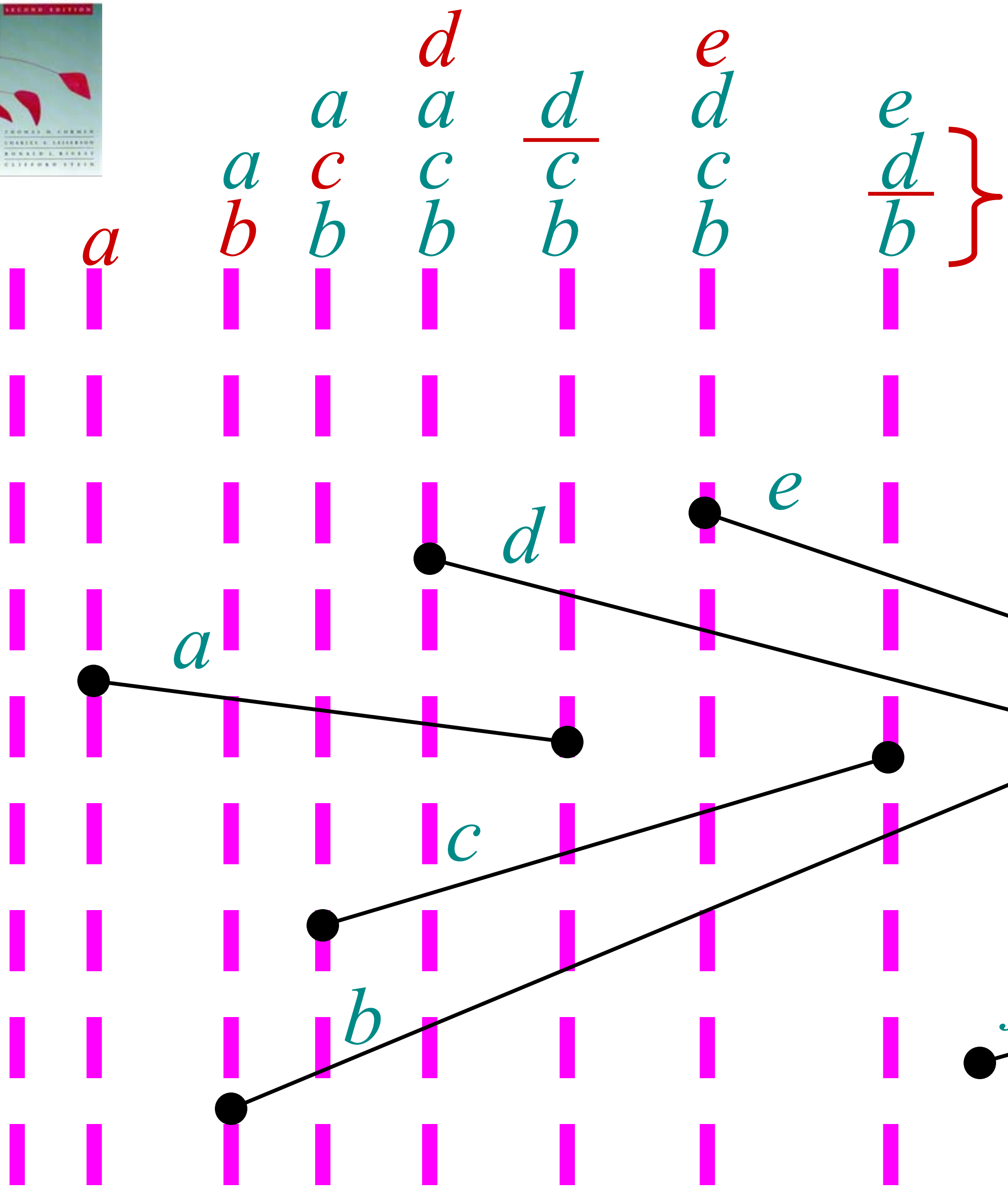


Sweep-line algorithm

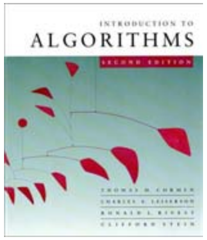
Process event points in order by sorting segment endpoints by x-coordinate and looping through:

- For a left endpoint of segment s:
  - Add segment s to dynamic set S.
  - Check for intersection between s and its neighbors in S.
- For a right endpoint of segment s:
  - Remove segment s from dynamic set S.
  - Check for intersection between the neighbors of s in S.



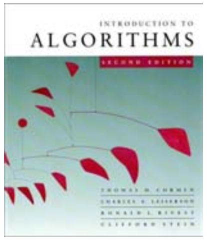


intersection found



Analysis

Use balanced search tree to store dynamic set S.



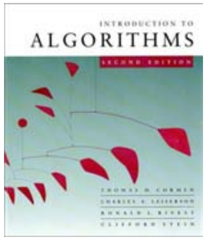
Sweep-line algorithm

$O(n)$ Process event points in order by sorting segment endpoints by x-coordinate and looping through:

$O(\log n)$
- For a left endpoint of segment s:
  - Add segment s to dynamic set S.
  - Check for intersection between s and its neighbors in S.

$O(\log n)$
- For a right endpoint of segment s:
  - Remove segment s from dynamic set S.
  - Check for intersection between the neighbors of s in S.

$O(n \log n)$

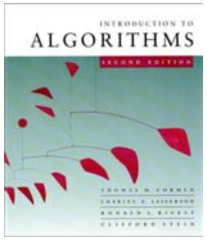


Analysis

Use balanced search tree to store dynamic set S.

Total running time: $O(n \log n)$.

Note that the algorithm stops after finding the first intersection point. If we want to report all intersection points, the algorithm can be extended to run in $O((n+k) \log n)$ time, where k is the number of intersections.



Correctness

Theorem: If there is an intersection, the algorithm finds it.

Proof: Let X be the leftmost intersection point. Assume for simplicity that

- only two segments s_1, s_2 pass through X, and
- no two points have the same x-coordinate.

At some point before we reach X,
s_1 and s_2 become consecutive in the order of S.
Either initially consecutive when s_1 or s_2 inserted,
or became consecutive when another deleted. □