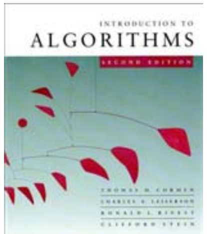
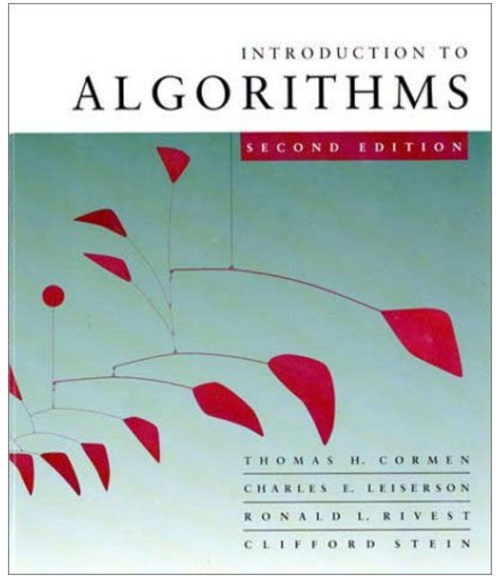


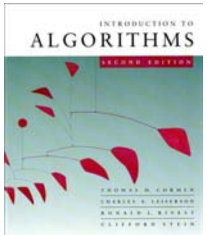
CS 5633 -- Spring 2004



Recurrences and Divide & Conquer

Carola Wenk

Slides courtesy of Charles Leiserson with small changes by Carola Wenk

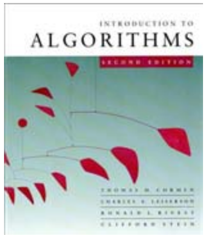


Merge sort

MERGE-SORT $A[1\,.\,.\,n]$

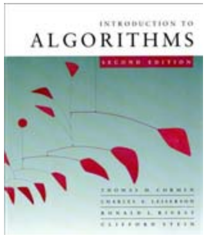
1. If $n = 1$, done.
2. Recursively sort $A[\,1\,.\,.\lceil n/2\rceil\,]$ and $A[\,\lceil n/2\rceil+1\,.\,.\,n\,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: **MERGE**



Merging two sorted arrays

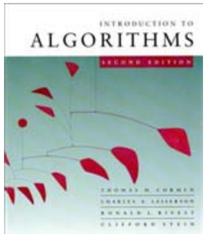
20	12
13	11
7	9
2	1



Merging two sorted arrays

20	12
13	11
7	9
2	1

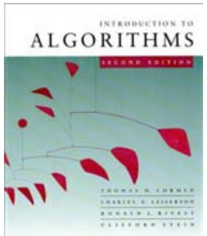
1



Merging two sorted arrays

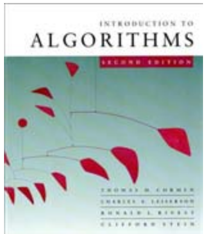
20	12	‖	20	12
13	11	‖	13	11
7	9	‖	7	9
2	1	‖	2	

1

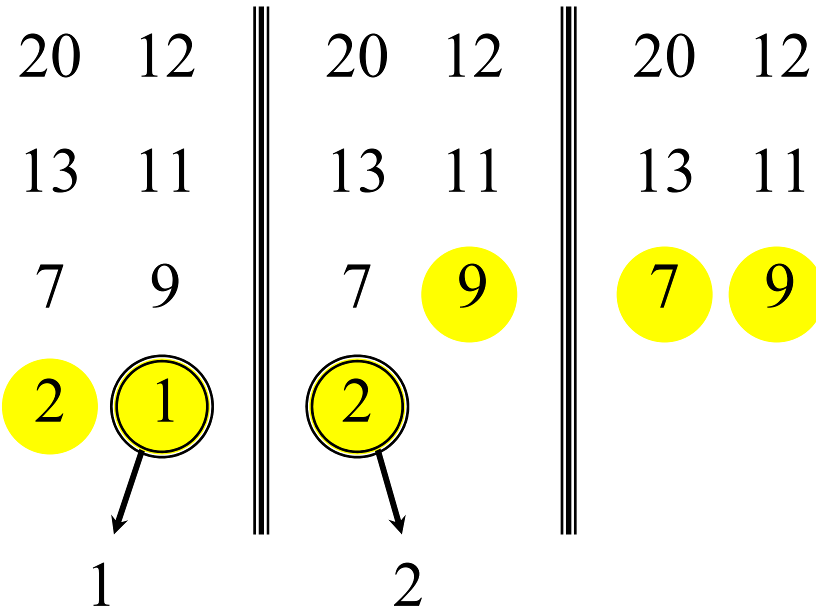


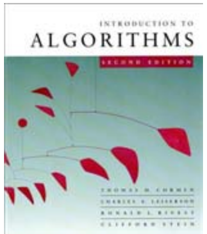
Merging two sorted arrays

20	12	‖	20	12
13	11	‖	13	11
7	9	‖	7	9
2	1	‖	2	
1			2	

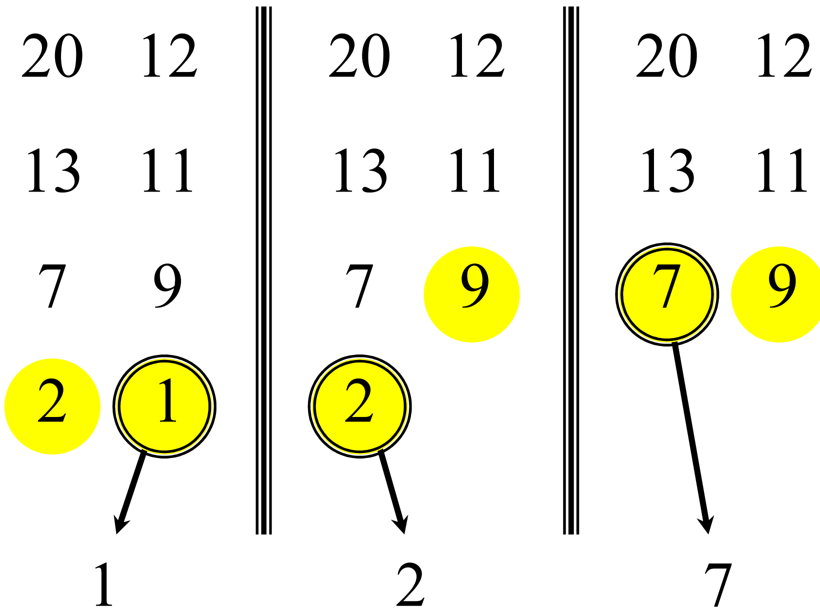


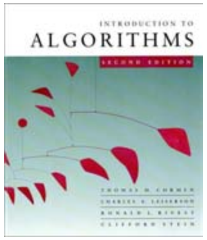
Merging two sorted arrays





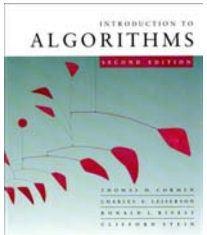
Merging two sorted arrays





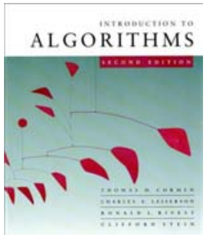
Merging two sorted arrays

20	12	20	12	20	12	20	12
13	11	13	11	13	11	13	11
7	9	7	9	7	9		9
2	1	2					
1		2		7			

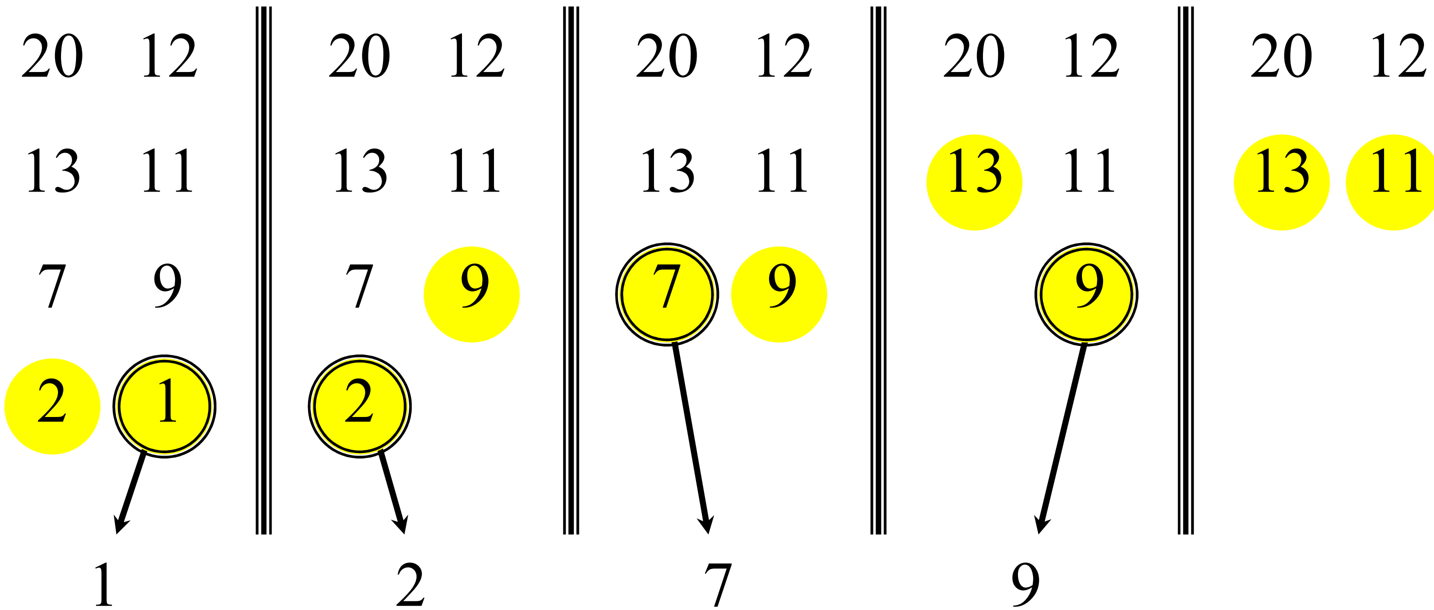


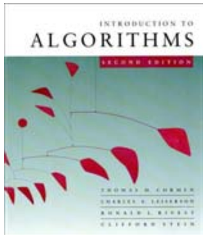
Merging two sorted arrays

20	12	20	12	20	12	20	12
13	11	13	11	13	11	13	11
7	9	7	9	7	9		9
2	1	2					
1		2		7		9	

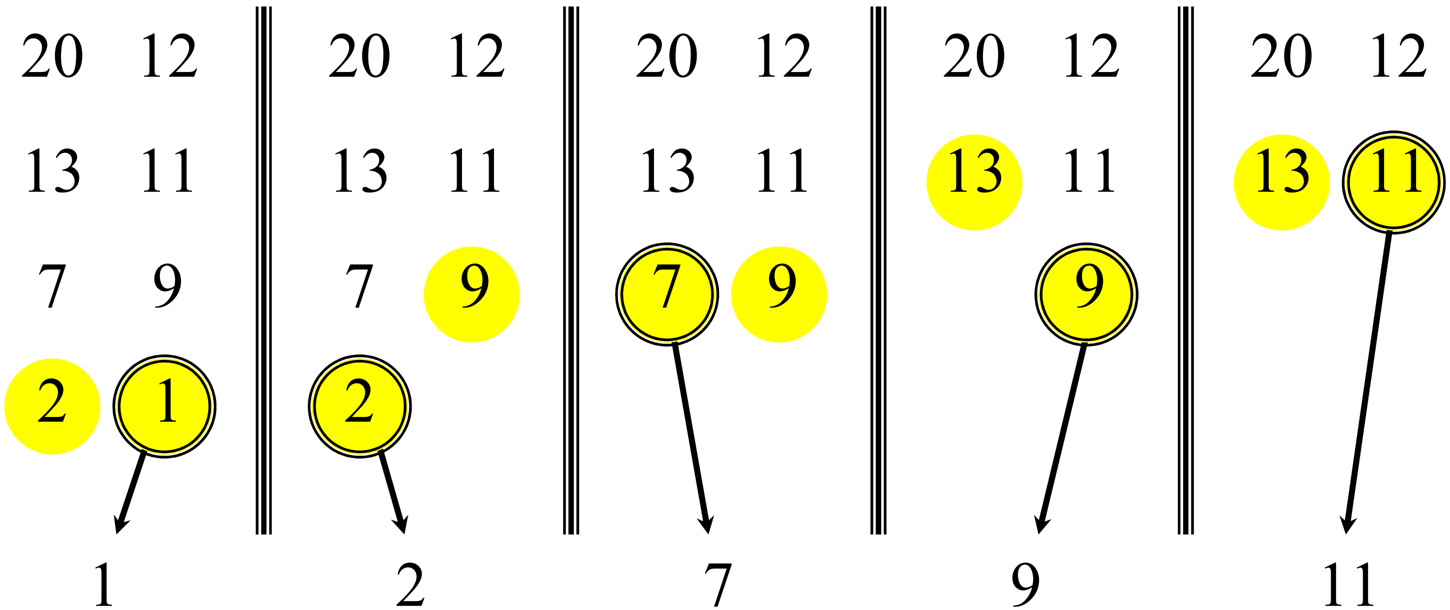


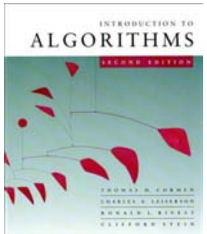
Merging two sorted arrays



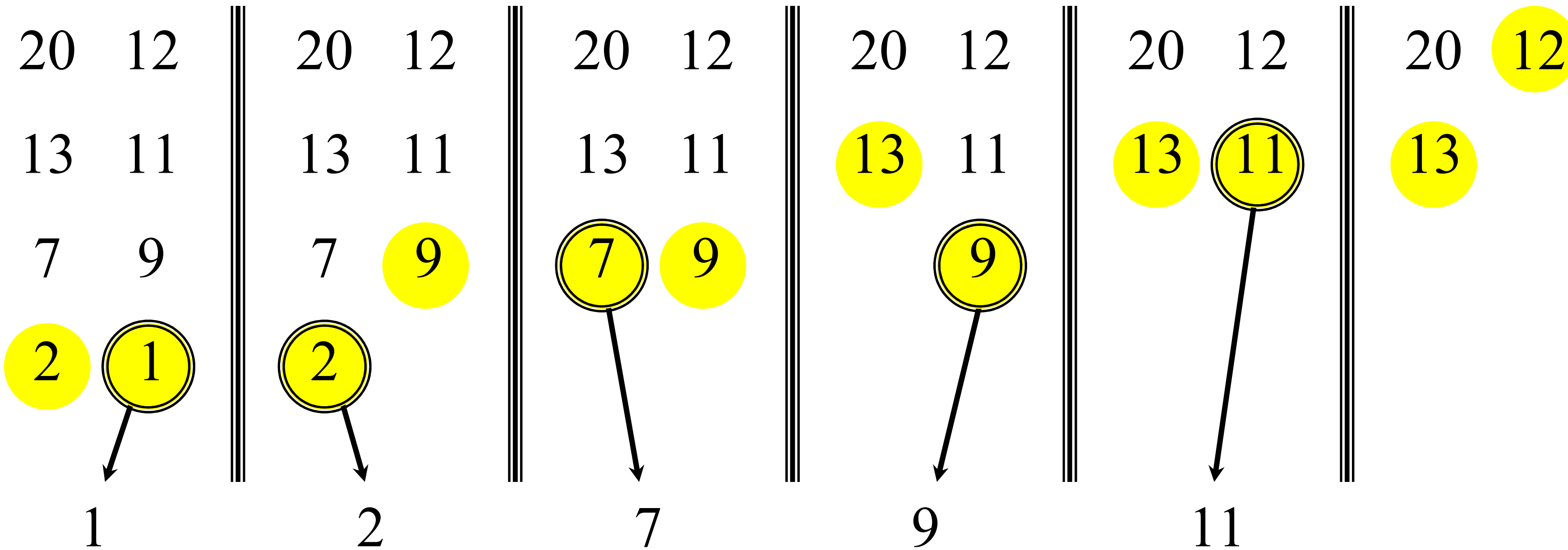


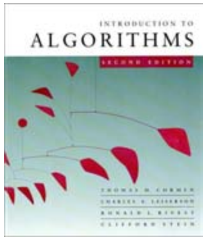
Merging two sorted arrays



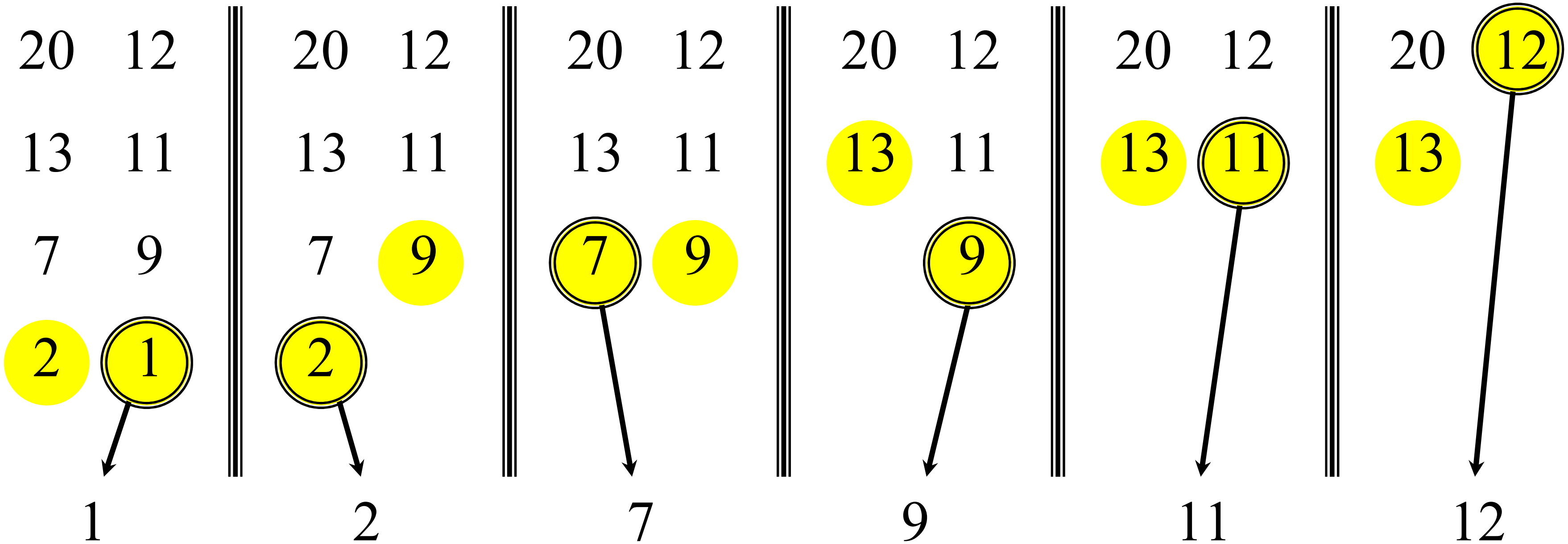


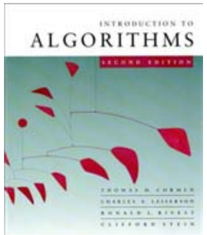
Merging two sorted arrays



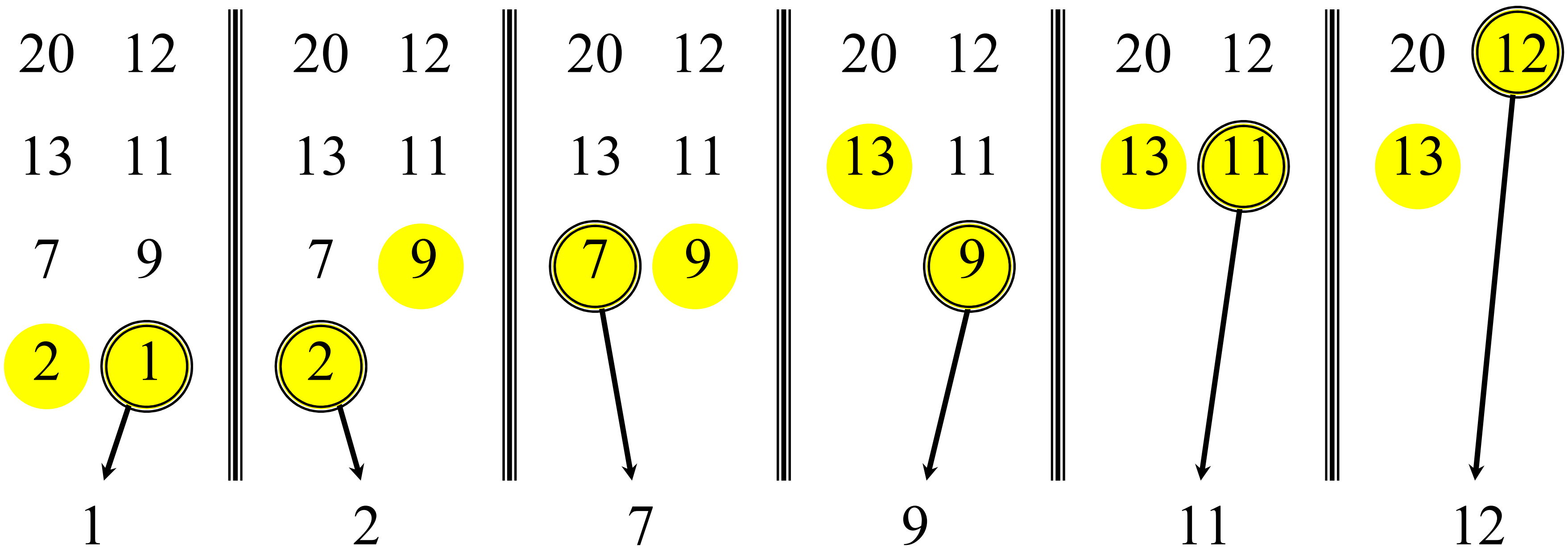


Merging two sorted arrays

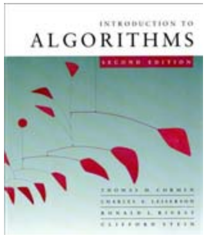




Merging two sorted arrays



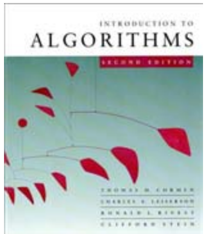
Time $dn = \Theta(n)$ to merge a total of n elements (linear time).



Analyzing merge sort

$T(n)$	**MERGE-SORT** $A[1\,.\,.\,n]$
d_0	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1\,.\,.\lceil n/2\rceil\,]$ and $A[\,\lceil n/2\rceil+1\,.\,.\,n\,]$.
dn	3. ***"Merge"*** the 2 sorted lists

Sloppiness: Should be $T(\lceil n/2\rceil) + T(\lfloor n/2\rfloor)$, but it turns out not to matter asymptotically.

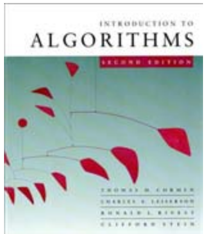


Recurrence for merge sort

$$T(n) = \begin{cases} d_0 \text{ if } n = 1; \\ 2T(n/2) + dn \text{ if } n > 1. \end{cases}$$

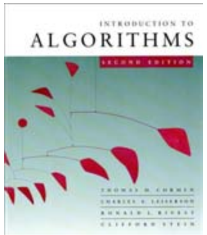
- We shall often omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.

- But what does $T(n)$ solve to? I.e., is it $O(n)$ or $O(n^2)$ or $O(n^3)$ or …?



The divide-and-conquer design paradigm

1. ***Divide*** the problem (instance) into subproblems.
2. ***Conquer*** the subproblems by solving them recursively.
3. ***Combine*** subproblem solutions.



Example: merge sort

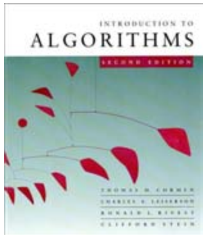
1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.

$$T(n) = 2\,T(n/2) + \Theta(n)$$

 # subproblems

subproblem size

work dividing and combining



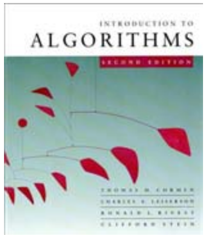
Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3	5	7	8	9	12	15



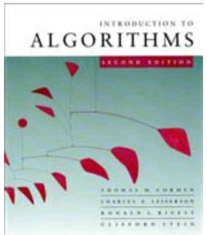
Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3 5 7 8 9 12 15



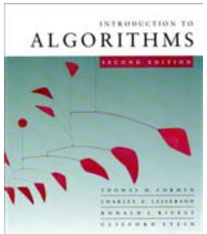
Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3 5 7 8 9 12 15



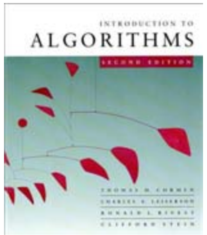
Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3 5 7 8 9 12 15



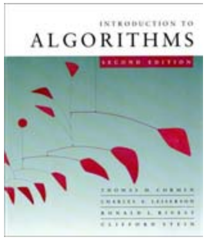
Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3 5 7 8 9 12 15



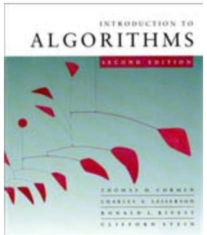
Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3 5 7 8 9 12 15



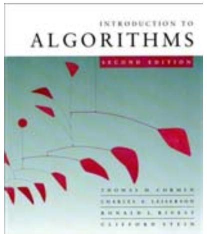
Recurrence for binary search

$$T(n) = 1\,T(n/2) + \Theta(1)$$

 # subproblems

subproblem size

work dividing and combining



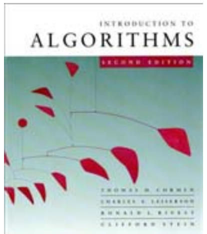
Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1; \\ 2T(n/2) + \Theta(n) & \text{if } n > 1. \end{cases}$$

- How do we solve $T(n)$? I.e., how do we found out if it is $O(n)$ or $O(n^2)$ or …?

Recursion tree

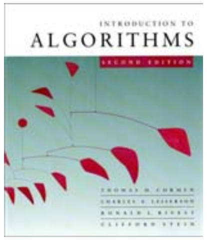
Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.



Recursion tree

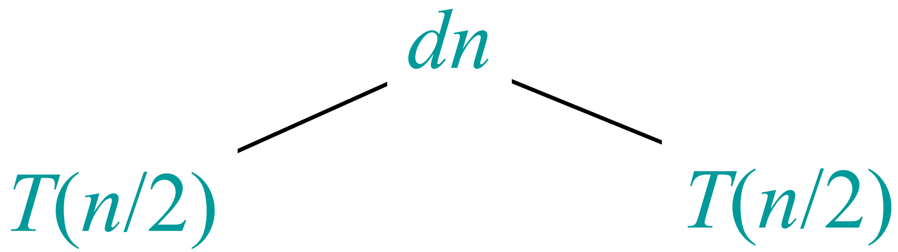
Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

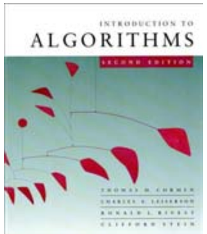
$$T(n)$$



Recursion tree

Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.





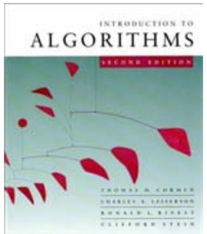
Recursion tree

Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

dn

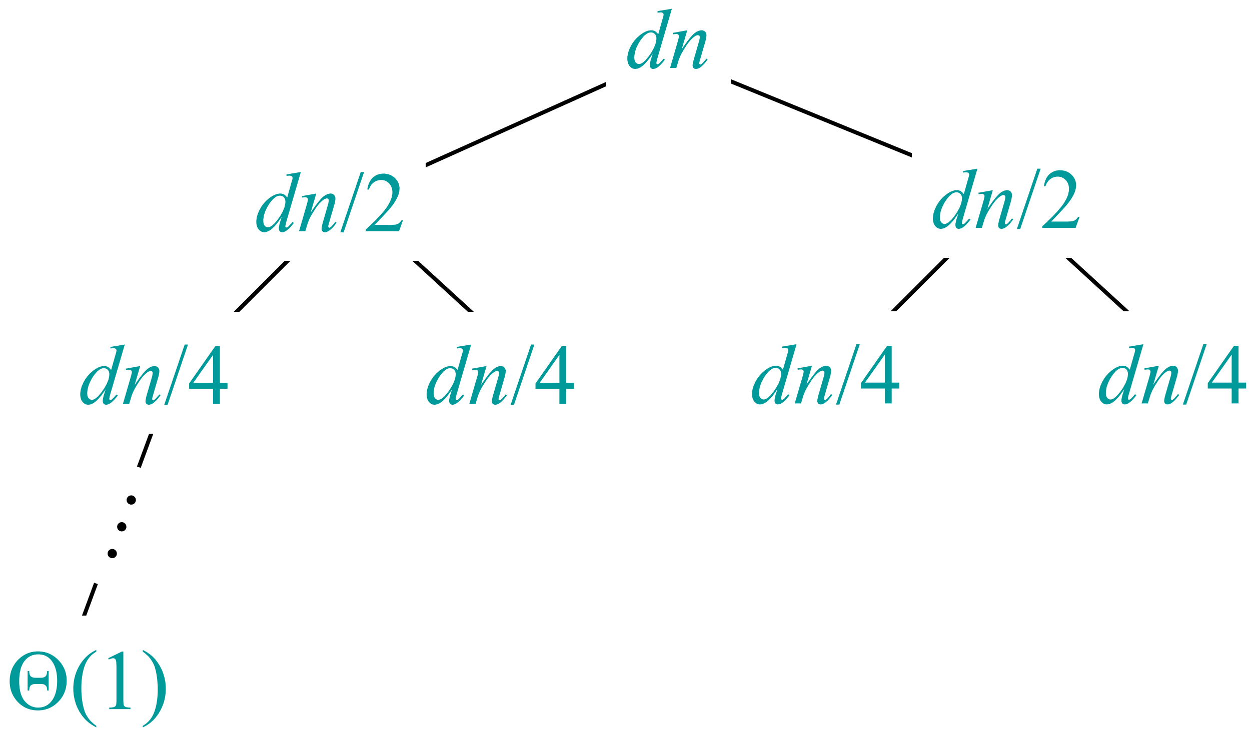
$dn/2$ $\qquad$ $dn/2$

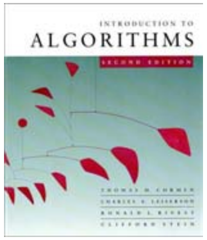
$T(n/4)$ $\quad$ $T(n/4)$ $\quad$ $T(n/4)$ $\quad$ $T(n/4)$



Recursion tree

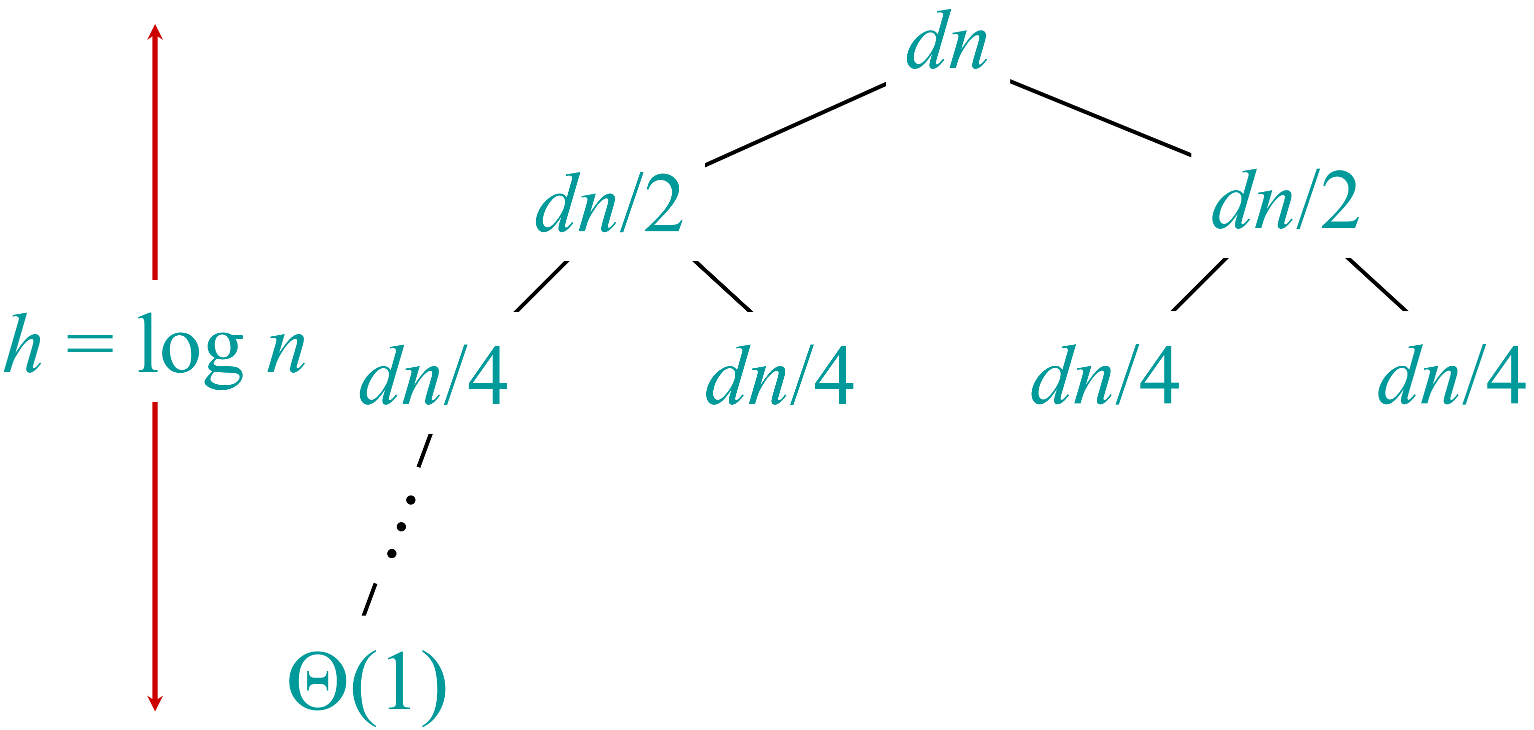
Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

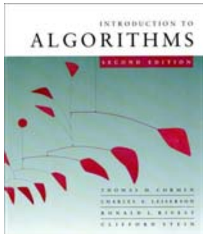




Recursion tree

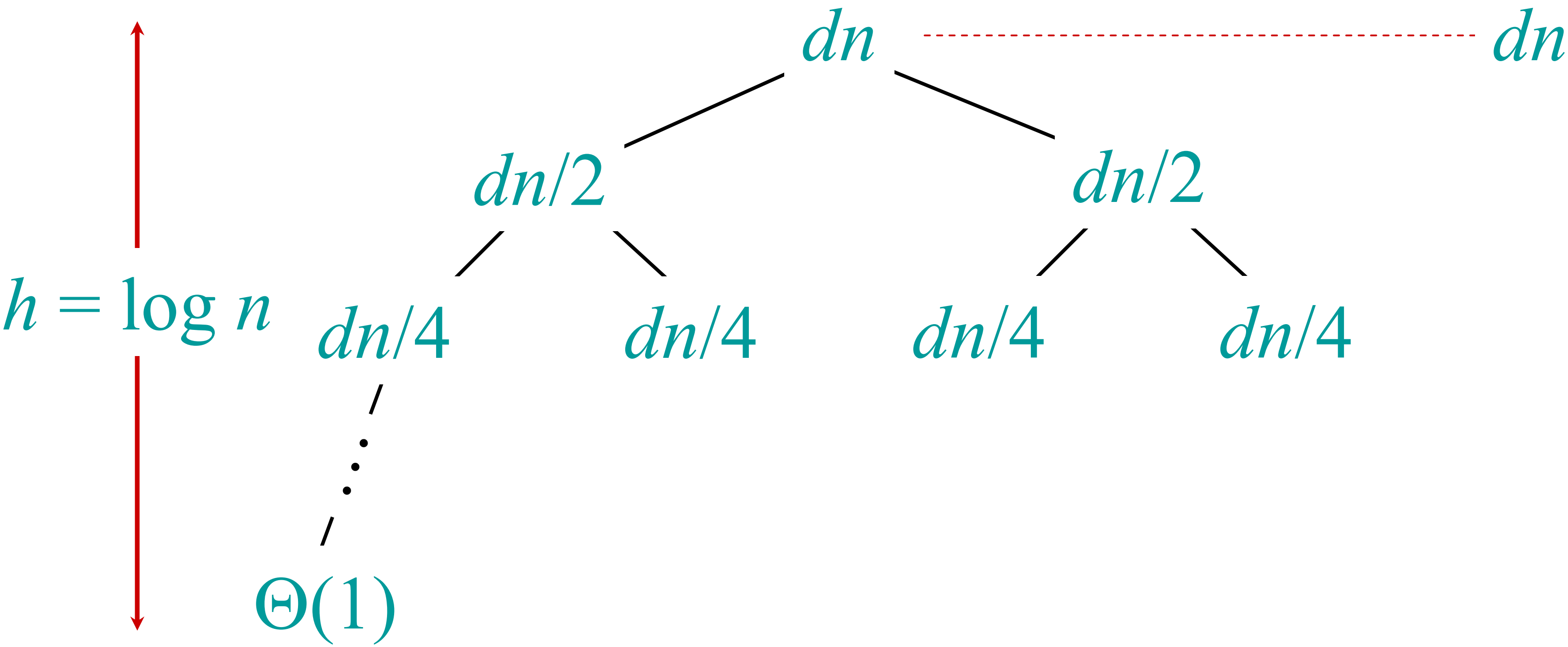
Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

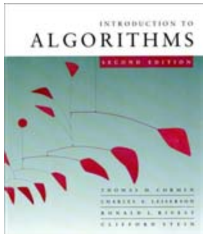




Recursion tree

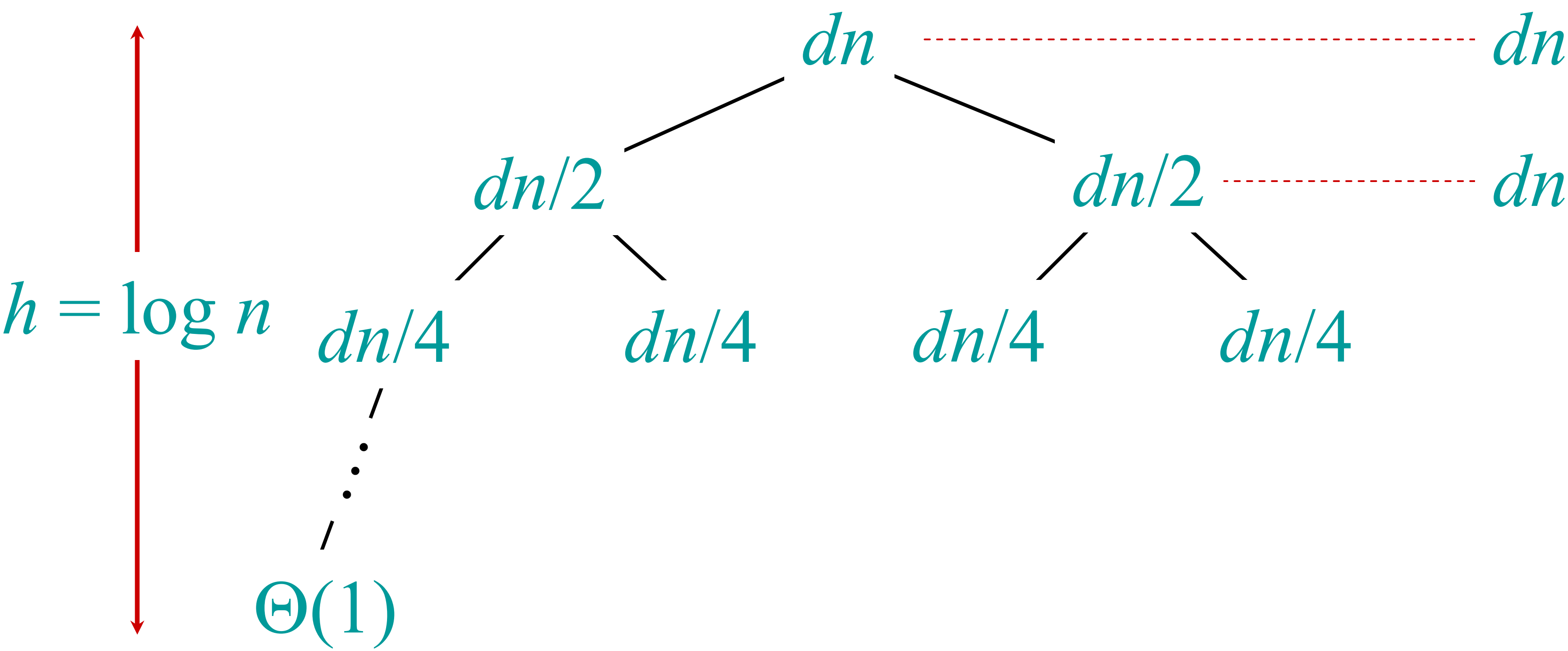
Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

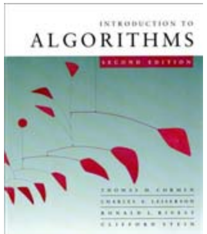




Recursion tree

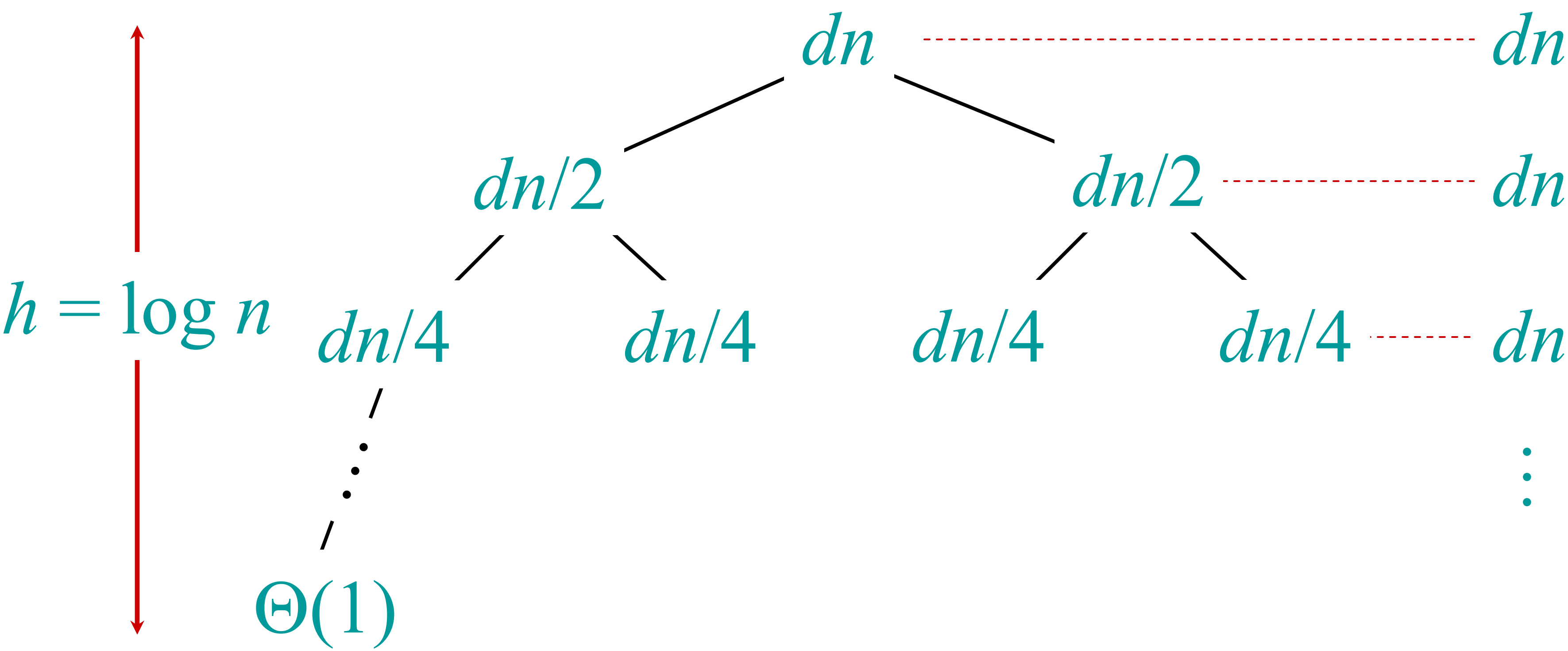
Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

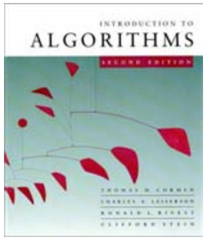




Recursion tree

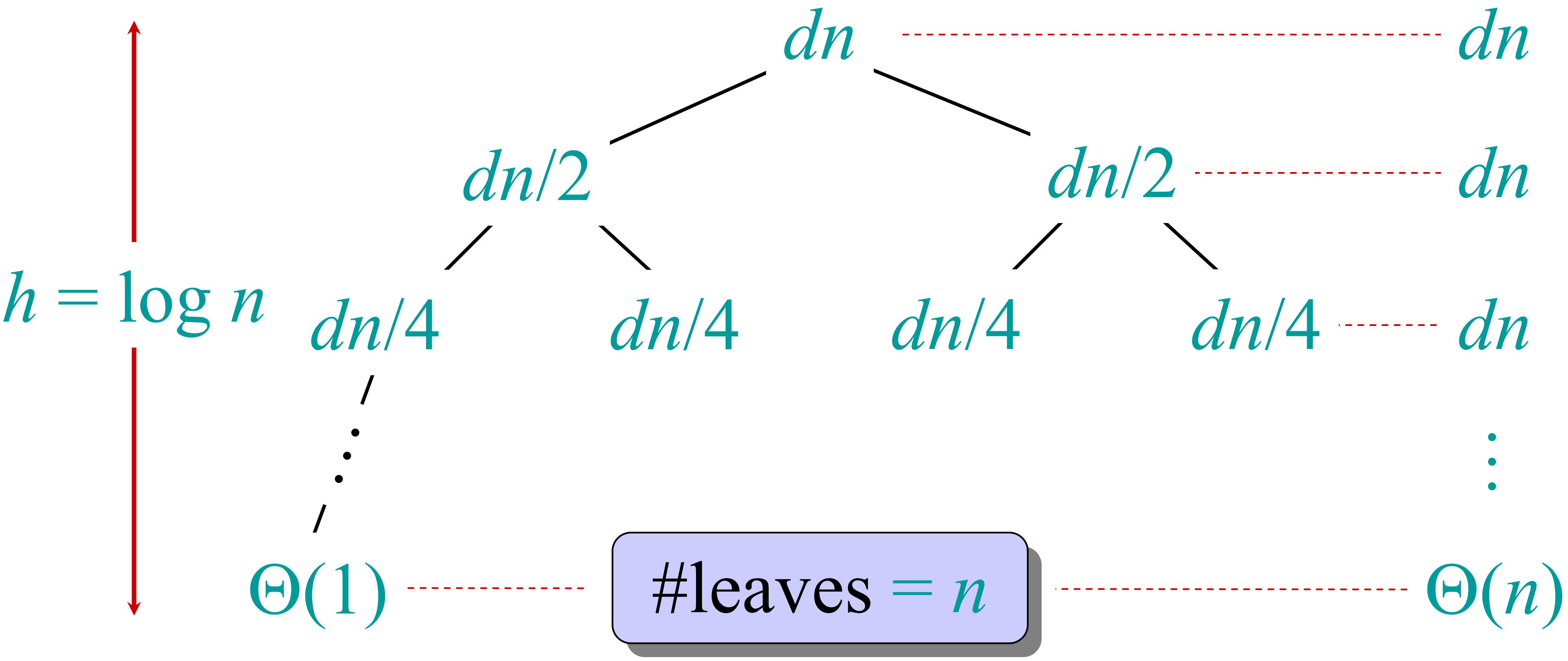
Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

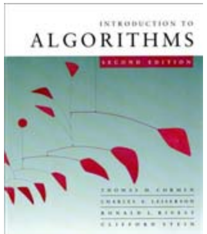




Recursion tree

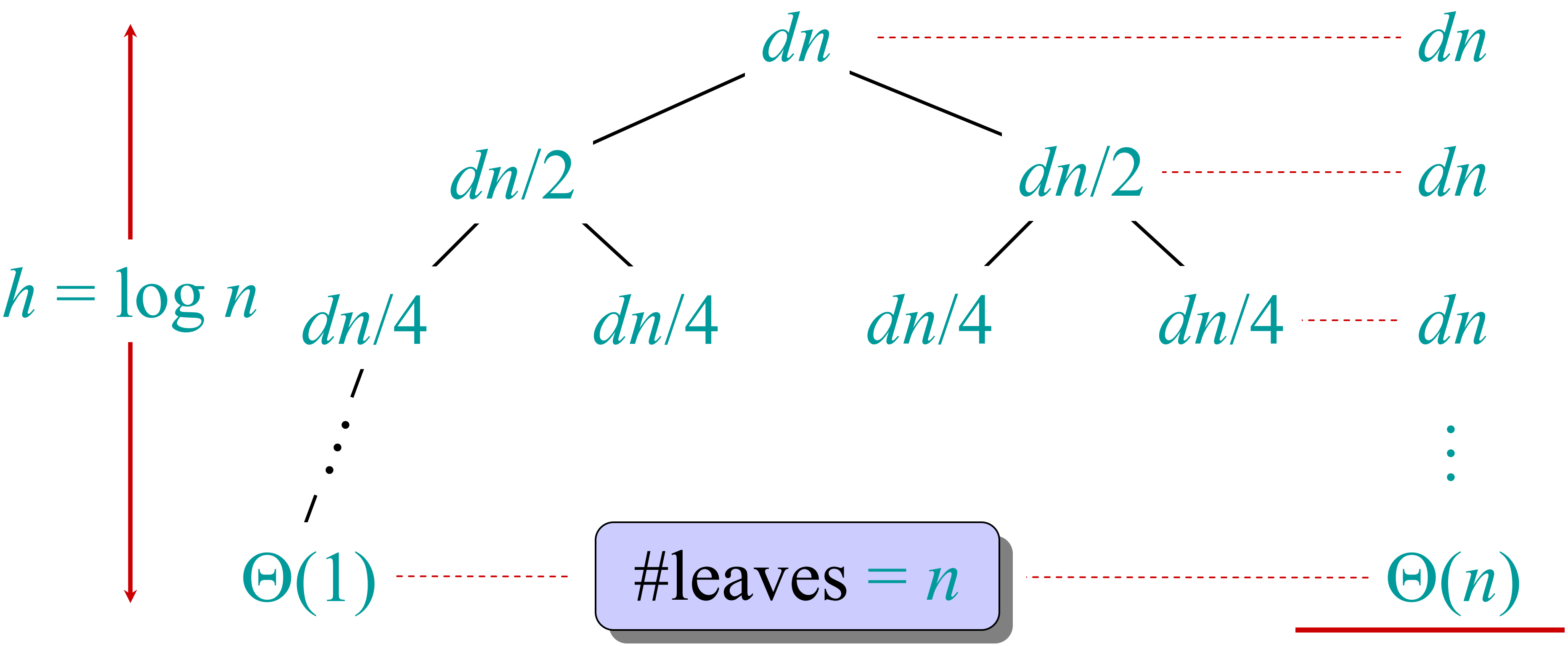
Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.



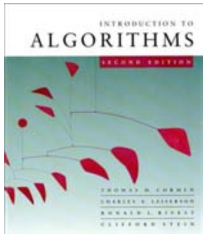


Recursion tree

Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

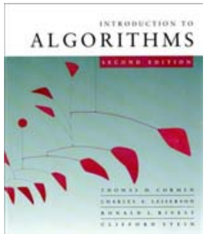


Total $\Theta(n \log n)$



Conclusions

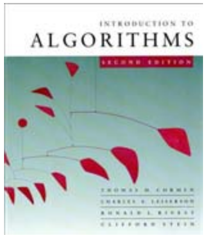
- Merge sort runs in $\Theta(n \lg n)$ time.
- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so. (Why not earlier?)



Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- It is good for generating **guesses** of what the runtime could be.

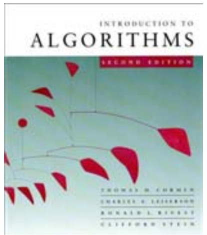
But: Need to **verify** that the guess is right.
→ Induction (substitution method)



Substitution method

The most general method to solve a recurrence (prove O and Θ separately):

1. ***Guess*** the form of the solution:
   (e.g. using recursion trees, or
   expansion)
2. ***Verify*** by induction (inductive step).
3. ***Solve*** for constants n_0 and c (base case of induction)



The divide-and-conquer design paradigm

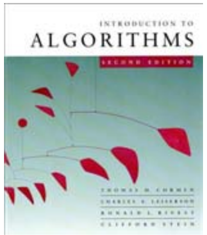
1. ***Divide*** the problem (instance) into subproblems.

   a subproblems, **each** of size n/b

2. ***Conquer*** the subproblems by solving them recursively.

3. ***Combine*** subproblem solutions.

   Runtime is $f(n)$

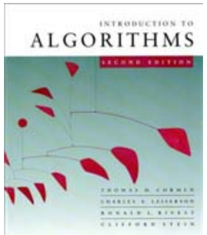


The master method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\ ,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.



Three common cases

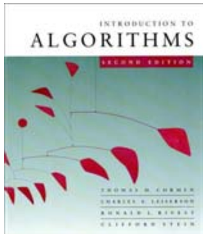
Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.

2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.



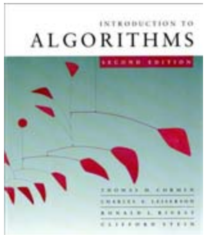
Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

   and $f(n)$ satisfies the ***regularity condition*** that $a f(n/b) \le c f(n)$ for some constant $c < 1$.

   Solution: $T(n) = \Theta(f(n))$.



Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n.$

Case 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

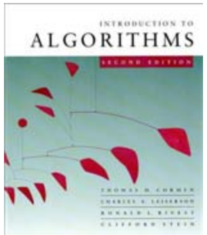
$\therefore\ T(n) = \Theta(n^2)$.

Ex. $T(n) = 4T(n/2) + n^2$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2.$

Case 2: $f(n) = \Theta(n^2 \lg^0 n)$, that is, $k = 0$.

$\therefore\ T(n) = \Theta(n^2 \lg n)$.



Examples

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^3.$

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

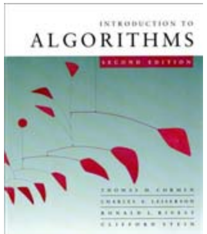
and $4(cn/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3)$.

Ex. $T(n) = 4T(n/2) + n^2/\lg n$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2/\lg n.$

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^\varepsilon = \omega(\lg n)$.



Master theorem (summary)

$$T(n) = a\,T(n/b) + f(n)$$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon})$
$\Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a} \lg^k n)$
$\Rightarrow T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$ and $a\,f(n/b) \le c\,f(n)$
$\Rightarrow T(n) = \Theta(f(n))$.

Merge sort: $a = 2,\ b = 2 \ \Rightarrow\ n^{\log_b a} = n$
$\Rightarrow$ CASE 2 $(k = 0) \ \Rightarrow\ T(n) = \Theta(n \lg n)$.