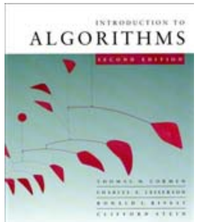
## CS 3343 -- Spring 2009




# *Graphs*

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Slides courtesy of Charles Leiserson with changes and additions by Carola Wenk



## Graphs (review)

**Definition.** A ***directed graph (digraph)*** $G = (V, E)$ is an ordered pair consisting of

- a set $V$ of ***vertices*** (singular: ***vertex***),
- a set $E \subseteq V \times V$ of ***edges***.

In an ***undirected graph*** $G = (V, E)$, the edge set $E$ consists of *unordered* pairs of vertices.

In either case, we have $|E| = O(|V|^2)$. Moreover, if $G$ is connected, then $|E| \geq |V| - 1$.

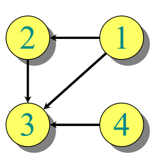
(Review CLRS, Appendix B.4 and B.5.)



## Adjacency-matrix representation

The ***adjacency matrix*** of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1 \,.\,.\, n, 1 \,.\,.\, n]$ given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E. \end{cases}$$



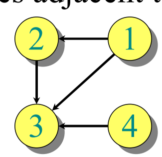

| $A$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 |
| 2 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 |

$\Theta(|V|^2)$ storage $\Rightarrow$ ***dense*** representation.



## Adjacency-list representation

An ***adjacency list*** of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to $v$.




$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3\}$

For undirected graphs, $|Adj[v]| = degree(v)$.

For digraphs, $|Adj[v]| = out\text{-}degree(v)$.

# Adjacency-list representation

**Handshaking Lemma:**

*Every edge is counted twice*

- For undirected graphs:
  $\sum_{v \in V} degree(v) = 2|E|$
- For digraphs:
  $\sum_{v \in V} in\text{-}degree(v) + \sum_{v \in V} out\text{-}degree(v) = 2\,|\,E\,|$

$\Rightarrow$ adjacency lists use $\Theta(|V| + |E|)$ storage
$\Rightarrow$ a ***sparse*** representation
$\Rightarrow$ We usually use this representation, unless stated otherwise



# Graph Traversal

Let $G=(V,E)$ be a (directed or undirected) graph, given in adjacency list representation.

$|V| = n$ , $|E| = m$

A graph traversal visits every vertex:
- Breadth-first search (BFS)
- Depth-first search (DFS)



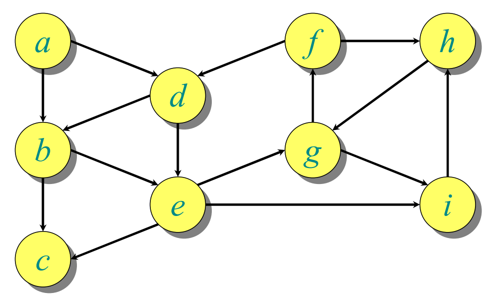
# Breadth-First Search (BFS)

```
BFS(G=(V,E))
    Mark all vertices in G as “unvisited” // time=0
    Initialize empty queue Q
    for each vertex v ∈ V do
        if v is unvisited
            visit v // time++
            Q.enqueue(v)
            BFS_iter(G)
```

```
BFS_iter(G)
    while Q is non-empty do
        v = Q.dequeue()
        for each w adjacent to v do
            if w is unvisited
                visit w   // time++
                Add edge (v,w) to T
                Q.enqueue(w)
```

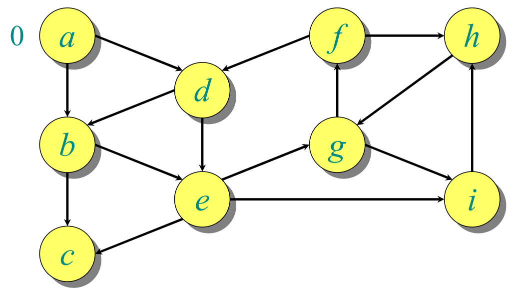


# Example of breadth-first search




$Q$:

## Example of breadth-first search

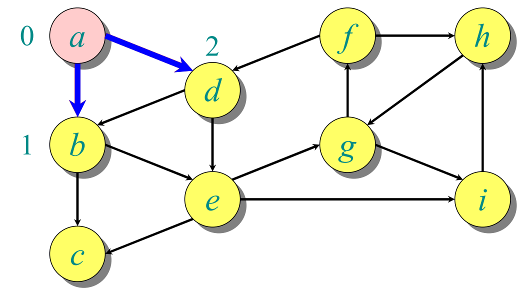



0

*Q:* *a*



## Example of breadth-first search

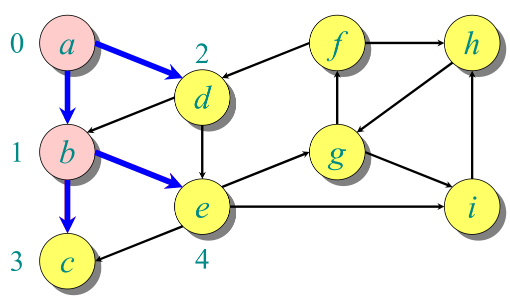



1 2

*Q:* *a* *b* *d*



## Example of breadth-first search

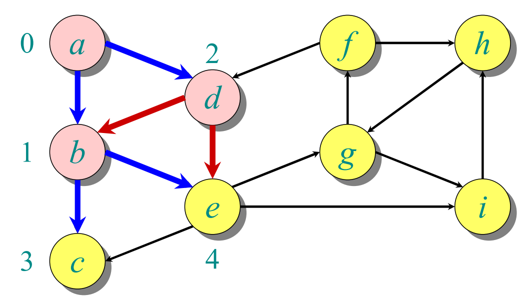



2 3 4

*Q:* *a* *b* *d* *c* *e*



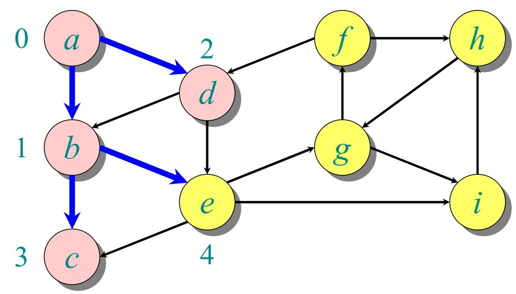
## Example of breadth-first search




3 4

*Q:* *a* *b* *d* *c* *e*

# Example of breadth-first search

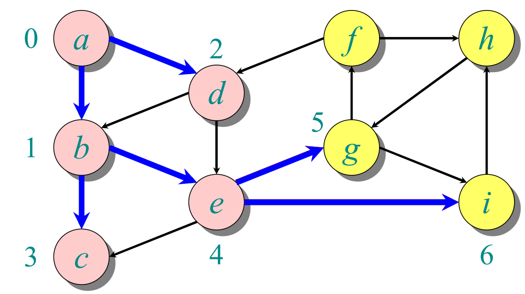



4

*Q:* *a b d c e*



# Example of breadth-first search

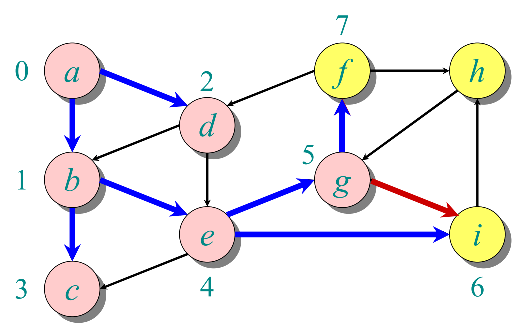



5 6

*Q:* *a b d c e g i*



# Example of breadth-first search

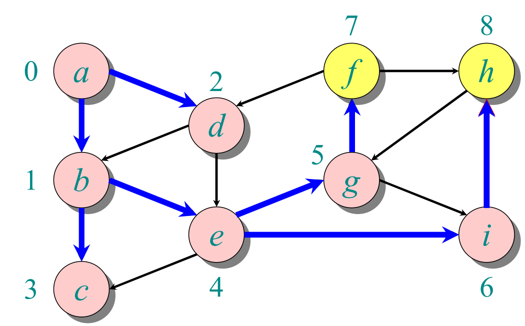



6 7

*Q:* *a b d c e g i f*



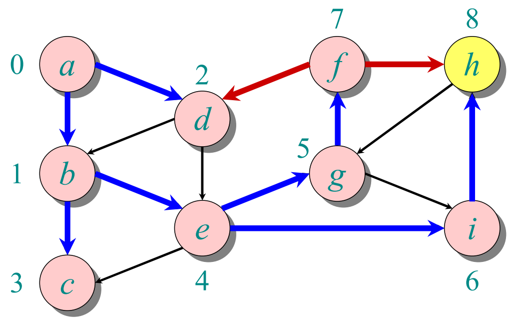
# Example of breadth-first search




7 8

*Q:* *a b d c e g i f h*

# Example of breadth-first search

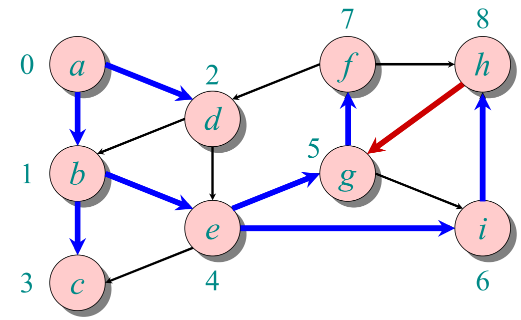



8

*Q*: *a b d c e g i f h*



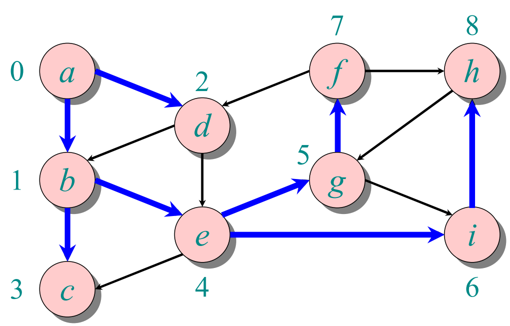
# Example of breadth-first search




*Q*: *a b d c e g i f h*



# Example of breadth-first search




*Q*: *a b d c e g i f h*



# Breadth-First Search (BFS)

BFS(*G*=(*V*,*E*))
- O(*n*) Mark all vertices in *G* as "unvisited" // time=0
- O(1) Initialize empty queue *Q*
- O(*n*) without BFS_iter: **for** each vertex *v* ∈ *V* **do**
  - **if** *v* is unvisited
    - visit *v* // time++
    - *Q*.enqueue(*v*)
    - BFS_iter(*G*)

BFS_iter(*G*) — O(*m*)
- **while** *Q* is non-empty **do**
  - *v* = *Q*.dequeue()
  - **for** each *w* adjacent to *v* **do** — O(*deg*(*v*))
    - **if** *w* is unvisited
      - visit *w* // time++
      - Add edge (*v*,*w*) to *T*
      - *Q*.enqueue(*w*)

# BFS runtime

- Each vertex is marked as unvisited in the beginning ⇒ O($n$) time
- Each vertex is marked at most once, enqueued at most once, and therefore dequeued at most once
- The time to process a vertex is proportional to the size of its adjacency list (its degree), since the graph is given in adjacency list representation
 ⇒ O($m$) time
- Total runtime is O($n$+$m$) = O(|V| + |E|)



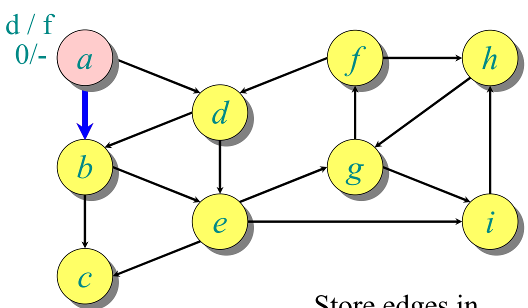
# Depth-First Search (DFS)

```
DFS(G=(V,E))
    Mark all vertices in G as "unvisited"  // time=0
    for each vertex v ∈ V do
        if v is unvisited
            DFS_rec(G,v)
```

```
DFS_rec(G, v)
    visit v  // d[v]=++time
    for each w adjacent to v do
        if w is unvisited
            Add edge (v,w) to tree T
            DFS_rec(G,w)
    // f[v]=++time
```



# Example of depth-first search

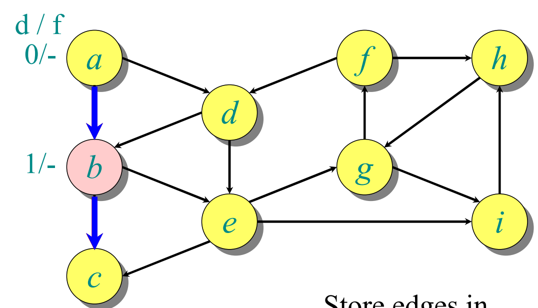



Store edges in predecessor array

π: a b c d e f g h i
- a



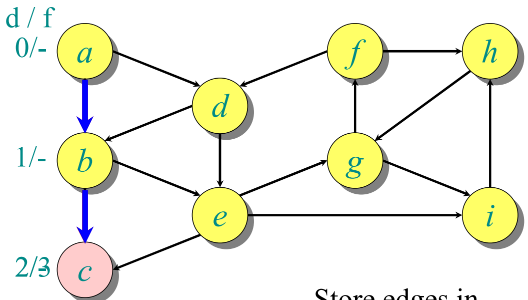
# Example of depth-first search




Store edges in predecessor array

π: a b c d e f g h i
- a b

# Example of depth-first search

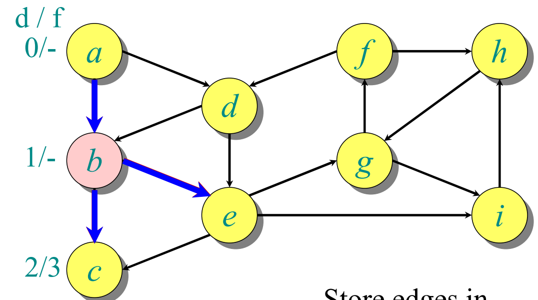



Store edges in
predecessor array

π: a b c d e f g h i
- a b



# Example of depth-first search

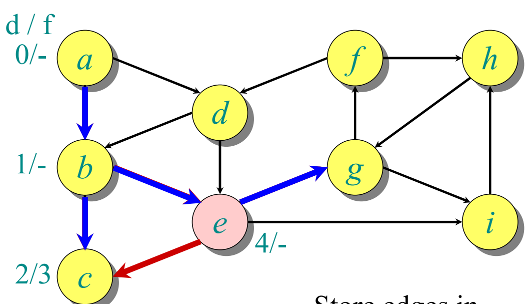



Store edges in
predecessor array

π: a b c d e f g h i
- a b b



# Example of depth-first search




Store edges in
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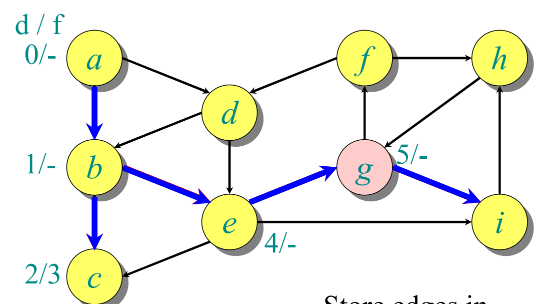
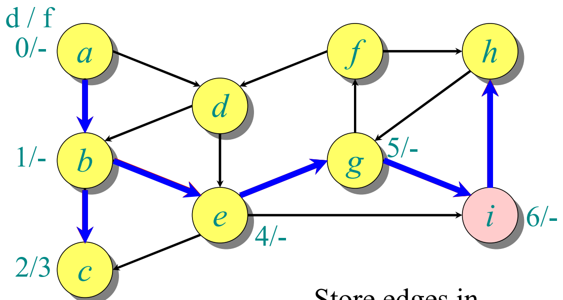



Store edges in
predecessor array

π: a b c d e f g h i
- a b b e g

## Example of depth-first search

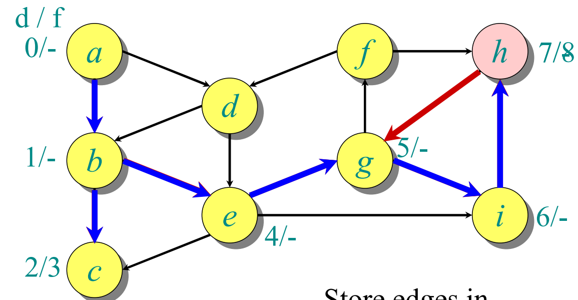



Store edges in predecessor array

| π: | a | b | c | d | e | f | g | h | i |
|---|---|---|---|---|---|---|---|---|---|
| | - | a | b | | b | | e | i | g |



## Example of depth-first search

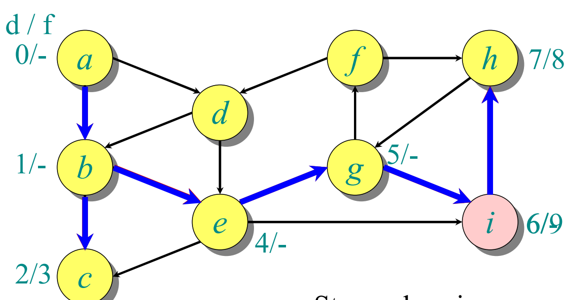



Store edges in predecessor array

| π: | a | b | c | d | e | f | g | h | i |
|---|---|---|---|---|---|---|---|---|---|
| | - | a | b | | b | | e | i | g |



## Example of depth-first search

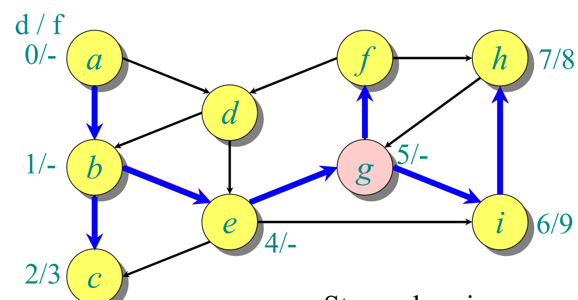



Store edges in predecessor array

| π: | a | b | c | d | e | f | g | h | i |
|---|---|---|---|---|---|---|---|---|---|
| | - | a | b | | b | | e | i | g |



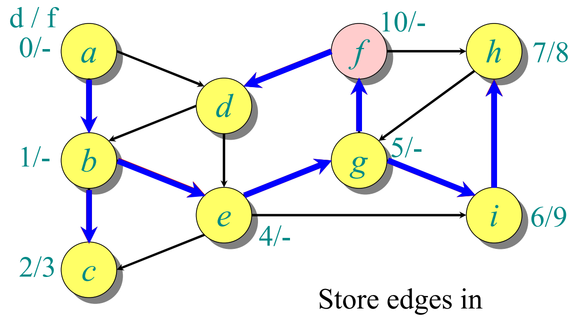
## Example of depth-first search




Store edges in predecessor array

| π: | a | b | c | d | e | f | g | h | i |
|---|---|---|---|---|---|---|---|---|---|
| | - | a | b | | b | g | e | i | g |

# Example of depth-first search

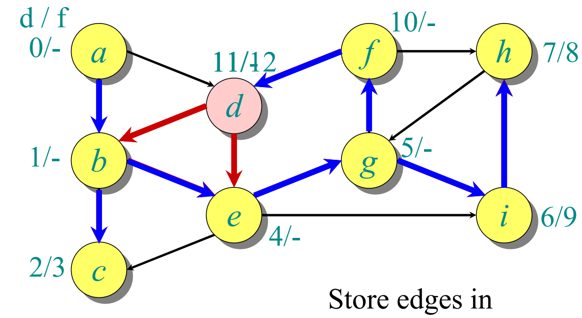



Store edges in
predecessor array

π: a b c d e f g h i
- a b f b g e i g



# Example of depth-first search

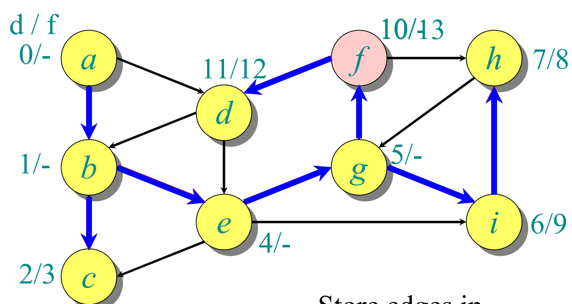



Store edges in
predecessor array

π: a b c d e f g h i
- a b f b g e i g



# Example of depth-first search

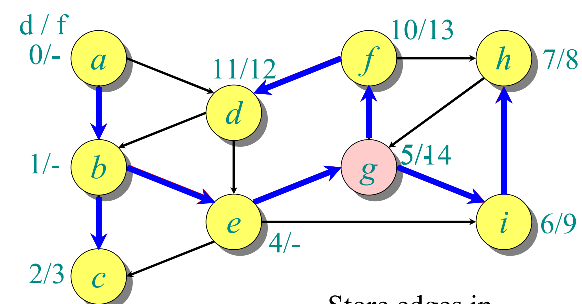



Store edges in
predecessor array

π: a b c d e f g h i
- a b f b g e i g



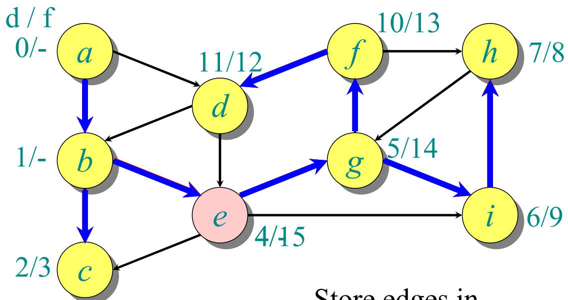
# Example of depth-first search




Store edges in
predecessor array

π: a b c d e f g h i
- a b f b g e i g

## Example of depth-first search

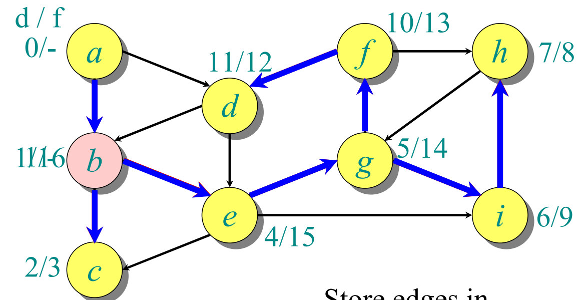



Store edges in predecessor array

| π: | a | b | c | d | e | f | g | h | i |
|---|---|---|---|---|---|---|---|---|---|
| | - | a | b | f | b | g | e | i | g |



## Example of depth-first search

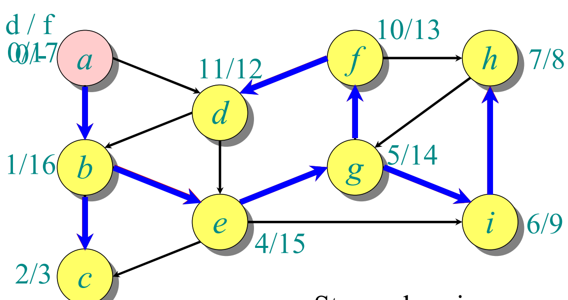



Store edges in predecessor array

| π: | a | b | c | d | e | f | g | h | i |
|---|---|---|---|---|---|---|---|---|---|
| | - | a | b | f | b | g | e | i | g |



## Example of depth-first search




Store edges in predecessor array

| π: | a | b | c | d | e | f | g | h | i |
|---|---|---|---|---|---|---|---|---|---|
| | - | a | b | f | b | g | e | i | g |



## Depth-First Search (DFS)

DFS($G=(V,E)$)
- O($n$): Mark all vertices in $G$ as "unvisited" // time=0
- O($n$) without DFS_rec: **for** each vertex $v \in V$ **do**
  - **if** $v$ is unvisited
    - DFS_rec($G,v$)

DFS_rec($G, v$)
- O(1): visit $v$ // $d[v]$=++time
- O($deg(v)$) without recursive call: **for** each $w$ adjacent to $v$ **do**
  - **if** $w$ is unvisited
    - Add edge $(v,w)$ to tree $T$
    - DFS_rec($G,w$)
- // $f[v]$=++time

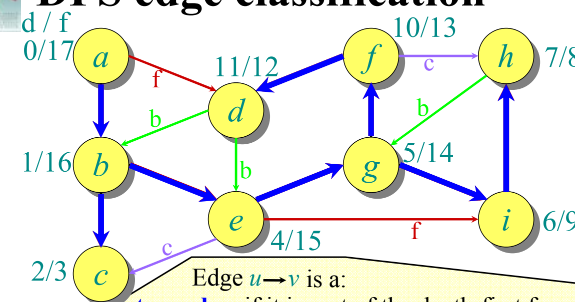
⇒ With Handshaking Lemma, all recursive calls are O($m$), for a total of O($n + m$) runtime

# DFS runtime

- Each vertex is visited at most once ⇒ $O(n)$ time
- The body of the **for** loops (except the recursive call) take constant time per graph edge
- All **for** loops take $O(m)$ time
- Total runtime is $O(n+m) = O(|V| + |E|)$



# DFS edge classification




Edge $u \rightarrow v$ is a:
- **tree edge**, if it is part of the depth-first forest.
- back edge, if $u$ connects to an ancestor $v$ in a depth-first tree. It holds $d(u)>d(v)$ and $f(u)<f(v)$.
- forward edge, if it connects $u$ to a descendant $v$ in a depth-first tree. It holds $d(u)<d(v)$.
- cross edge, if it is any other edge. It holds $d(u)>d(v)$ and $f(u)>f(v)$.



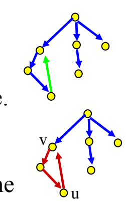
# Paths, Cycles, Connectivity

Let $G=(V,E)$ be a directed (or undirected) graph
- A **path** from $v_1$ to $v_k$ in $G$ is a sequence of vertices $v_1, v_2,\dots,v_k$ such that $(v_i,v_{\{i+1\}})\in E$ (or $\{v_i,v_{\{i+1\}}\} \in E$ if $G$ is undirected) for all $i\in\{1,\dots,k\text{-}1\}$.
- A path is **simple** if all vertices in the path are distinct.
- A path $v_1, v_2,\dots,v_k$ forms a **cycle** if $v_1=v_k$ and $k\geq 3$.
- A graph with no cycles is **acyclic**.
  - An undirected acyclic graph is called a **tree**. (Trees do not have to have a root vertex specified.)
  - A directed acyclic graph is a **DAG**. (A DAG can have undirected cycles if the direction of the edges is not considered.)
- An undirected graph is **connected** if every pair of vertices is connected by a path. A directed graph is **strongly connected** if for every pair $u,v\in V$ there is a path from $u$ to $v$ and there is a path from $v$ to $u$.
- The **(strongly) connected components** of a graph are the equivalence classes of vertices under this reachability relation.



# DAG Theorem

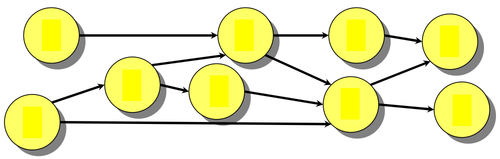
**Theorem:** A directed graph G is acyclic
⇔ a depth-first search of G yields no back edges.
**Proof:**
"⇒": Suppose there is a back edge (u,v). Then by definition of a back edge there would be a cycle.
"⇐": Suppose G contains a cycle c. Let v be the first vertex to be discovered in c, and let u be the preceding vertex in c. v is an ancestor of u in the depth-first forest, hence (u,v) is a back edge.

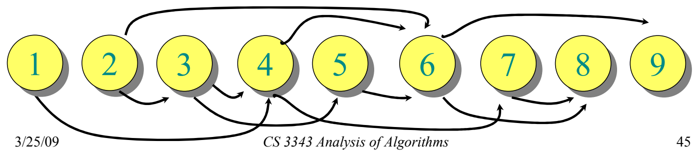


# Topological Sort

***Topologically sort*** the vertices of a ***directed acyclic graph*** (***DAG***):

- Determine $f : V \rightarrow \{1, 2, \ldots, |V|\}$ such that $(u, v) \in E \Rightarrow f(u) < f(v)$.

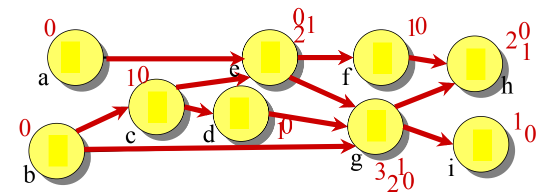







# Topological Sort Algorithm

- Store vertices with in-degree 0 in a queue Q.
- While Q is not empty
  - Dequeue vertex v, and give it the next number
  - Decrease in-degree of all adjacent vertices by 1
  - Enqueue all vertices with in-degree 0




Q: a , b , c, e, d, f, g, i, h



# Topological Sort Runtime

**Runtime:**

- O(|V|+|E|) because every edge is touched once, and every vertex is enqueued and dequeued exactly once



# Depth-First Search Revisited

```
DFS(G=(V,E))
    Mark all vertices in G as "unvisited"  // time=0
    for each vertex v ∈ V do
        if v is unvisited
            DFS_rec(G,v)
```

```
DFS_rec(G, v)
    visit v  // d[v]=++time
    for each w adjacent to v do
        if w is unvisited
            Add edge (v,w) to tree T
            DFS_rec(G,w)
    // f[v]=++time
```

## DFS-Based Topological Sort Algorithm

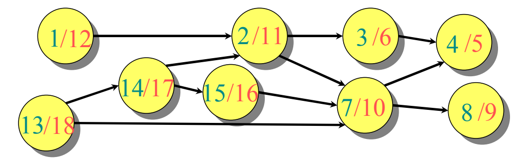
- Call DFS on the directed acyclic graph $G=(V,E)$
  $\Rightarrow$ Finish time for every vertex
- Reverse the finish times (highest finish time becomes the lowest finish time,…)
  $\Rightarrow$ Valid function $f': V \rightarrow \{1, 2, \ldots, |V|\}$ such that
  $(u, v) \in E \Rightarrow f'(u) < f'(v)$

Runtime: $O(|V|+|E|)$

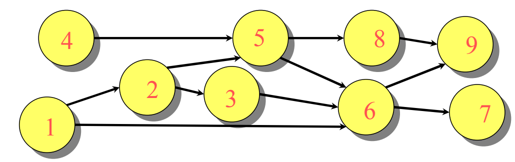


## DFS-Based Topological Sort

- Run DFS:

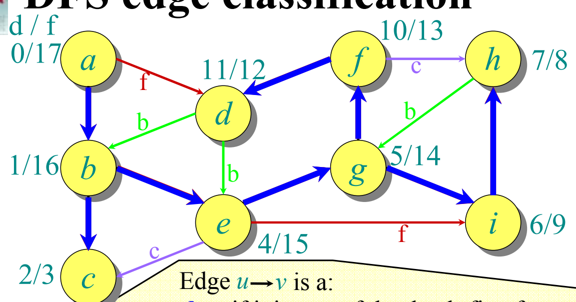



- Reverse finish times:






## DFS edge classification




Edge $u \rightarrow v$ is a:
- **tree edge**, if it is part of the depth-first forest.
- back edge, if $u$ connects to an ancestor $v$ in a depth-first tree. It holds $d(u)>d(v)$ and $f(u)<f(v)$.
- forward edge, if it connects $u$ to a descendant $v$ in a depth-first tree. It holds $d(u)<d(v)$.
- cross edge, if it is any other edge. It holds $d(u)>d(v)$ and $f(u)>f(v)$.



## DFS-Based Top. Sort Correctness

- Need to show that for any $(u, v) \in E$ holds $f(v) < f(u)$.
  (since we consider reversed finish times)
- Consider exploring edge $(u, v)$ in DFS:
  - $v$ cannot be visited and unfinished (and hence an ancestor in the depth first tree), since then $(u,v)$ would be a back edge (which by the DAG lemma cannot happen).
  - If $v$ has not been visited yet, it becomes a descendant of $u$, and hence $f(v)<f(u)$ . (tree edge)
  - If $v$ has been finished, $f(v)$ has been set, and $u$ is still being explored, hence $f(u)>f(v)$ (forward edge, cross edge) .

# Topological Sort Runtime

**Runtime:**

- O(|V|+|E|) because every edge is touched once, and every vertex is enqueued and dequeued exactly once
- DFS-based algorithm: O(|V| + |E|)