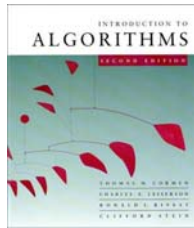




CS 3343 -- Spring 2009



More on Shortest Paths

Carola Wenk

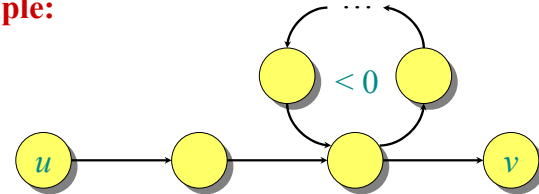
Slides courtesy of Charles Leiserson with small changes by Carola Wenk



Negative-weight cycles

Recall: If a graph $G = (V, E)$ contains a negative-weight cycle, then some shortest paths may not exist.

Example:



Bellman-Ford algorithm: Finds all shortest-path weights from a **source** $s \in V$ to all $v \in V$ or determines that a negative-weight cycle exists.



Bellman-Ford algorithm

$d[s] \leftarrow 0$
for each $v \in V - \{s\}$ do $d[v] \leftarrow \infty$ } initialization

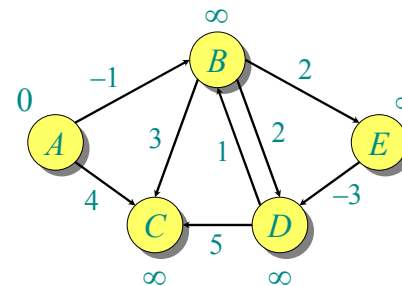
for $i \leftarrow 1$ to $|V| - 1$ do
for each edge $(u, v) \in E$ do
if $d[v] > d[u] + w(u, v)$ then
 $d[v] \leftarrow d[u] + w(u, v)$
 $\pi[v] \leftarrow u$ } relaxation step

for each edge $(u, v) \in E$
do if $d[v] > d[u] + w(u, v)$
then report that a negative-weight cycle exists
At the end, $d[v] = \delta(s, v)$. Time = $O(|V||E|)$.



Example of Bellman-Ford

Order of edges: $(B, E), (D, B), (B, D), (A, B), (A, C), (D, C), (B, C), (E, D)$

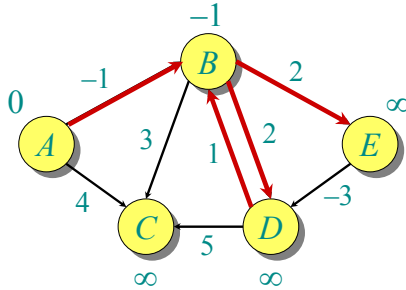


| A | B | C | D | E |
|---|----------|----------|----------|----------|
| 0 | ∞ | ∞ | ∞ | ∞ |



Example of Bellman-Ford

Order of edges: (B,E) , (D,B) , (B,D) , (A,B) , (A,C) , (D,C) , (B,C) , (E,D)



| A | B | C | D | E |
|---|----------|----------|----------|----------|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | -1 | ∞ | ∞ | ∞ |

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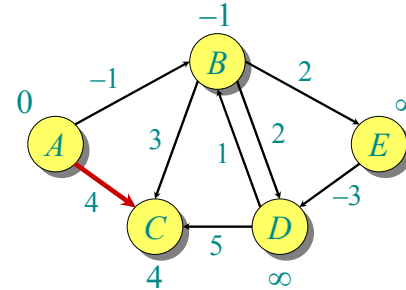
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Example of Bellman-Ford

Order of edges: (B,E) , (D,B) , (B,D) , (A,B) , (A,C) , (D,C) , (B,C) , (E,D)



| A | B | C | D | E |
|---|----------|----------|----------|----------|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | -1 | ∞ | ∞ | ∞ |
| 0 | -1 | 4 | ∞ | ∞ |

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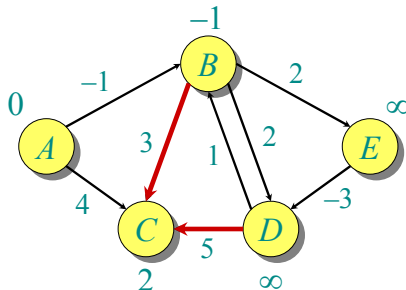
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Example of Bellman-Ford

Order of edges: (B,E) , (D,B) , (B,D) , (A,B) , (A,C) , (D,C) , (B,C) , (E,D)



| A | B | C | D | E |
|---|----------|----------|----------|----------|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | -1 | ∞ | ∞ | ∞ |
| 0 | -1 | 4 | ∞ | ∞ |
| 0 | -1 | 2 | ∞ | ∞ |

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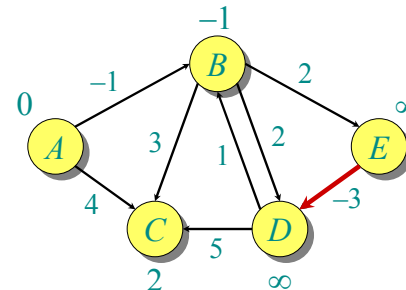
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Example of Bellman-Ford

Order of edges: (B,E) , (D,B) , (B,D) , (A,B) , (A,C) , (D,C) , (B,C) , (E,D)



| A | B | C | D | E |
|---|----------|----------|----------|----------|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | -1 | ∞ | ∞ | ∞ |
| 0 | -1 | 4 | ∞ | ∞ |
| 0 | -1 | 2 | ∞ | ∞ |

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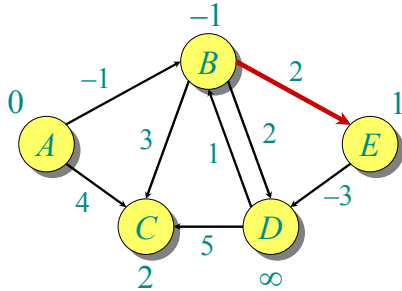
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Example of Bellman-Ford

Order of edges: (B,E) , (D,B) , (B,D) , (A,B) , (A,C) , (D,C) , (B,C) , (E,D)



| A | B | C | D | E |
|-----|----------|----------|----------|----------|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | -1 | ∞ | ∞ | ∞ |
| 0 | -1 | 4 | ∞ | ∞ |
| 0 | -1 | 2 | ∞ | ∞ |
| 0 | -1 | 2 | ∞ | 1 |

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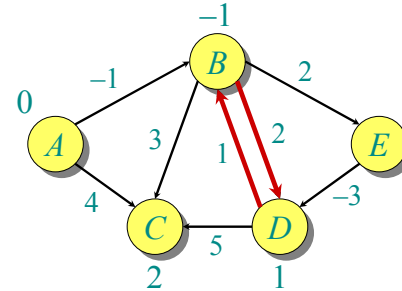
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Example of Bellman-Ford

Order of edges: (B,E) , (D,B) , (B,D) , (A,B) , (A,C) , (D,C) , (B,C) , (E,D)



| A | B | C | D | E |
|-----|----------|----------|----------|----------|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | -1 | ∞ | ∞ | ∞ |
| 0 | -1 | 4 | ∞ | ∞ |
| 0 | -1 | 2 | ∞ | ∞ |
| 0 | -1 | 2 | ∞ | 1 |
| 0 | -1 | 2 | 1 | 1 |

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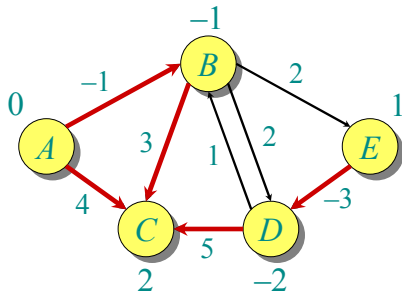
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Example of Bellman-Ford

Order of edges: (B,E) , (D,B) , (B,D) , (A,B) , (A,C) , (D,C) , (B,C) , (E,D)



| A | B | C | D | E |
|-----|----------|----------|----------|----------|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | -1 | ∞ | ∞ | ∞ |
| 0 | -1 | 4 | ∞ | ∞ |
| 0 | -1 | 2 | ∞ | ∞ |
| 0 | -1 | 2 | ∞ | 1 |
| 0 | -1 | 2 | 1 | 1 |
| 0 | -1 | 2 | -2 | 1 |

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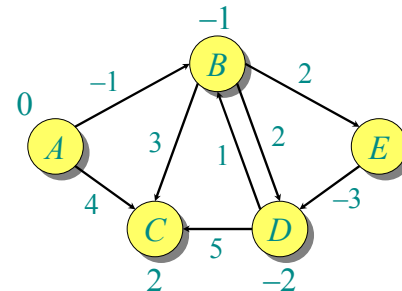
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Example of Bellman-Ford

Order of edges: (B,E) , (D,B) , (B,D) , (A,B) , (A,C) , (D,C) , (B,C) , (E,D)



| A | B | C | D | E |
|-----|----------|----------|----------|----------|
| 0 | ∞ | ∞ | ∞ | ∞ |
| 0 | -1 | ∞ | ∞ | ∞ |
| 0 | -1 | 4 | ∞ | ∞ |
| 0 | -1 | 2 | ∞ | ∞ |
| 0 | -1 | 2 | ∞ | 1 |
| 0 | -1 | 2 | 1 | 1 |
| 0 | -1 | 2 | -2 | 1 |

Note: Values decrease monotonically.

... and 2 more iterations

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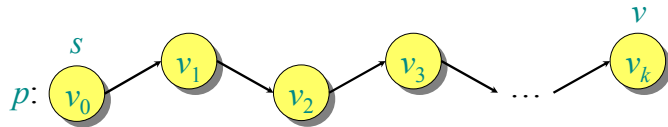
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Correctness

Theorem. If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.

Proof. Let $v \in V$ be any vertex, and consider a shortest path p from s to v with the minimum number of edges.

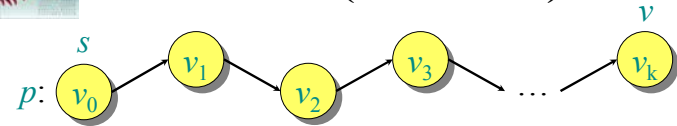


Since p is a shortest path, we have

$$\delta(s, v_i) = \delta(s, v_{i-1}) + w(v_{i-1}, v_i).$$



Correctness (continued)



Initially, $d[v_0] = 0 = \delta(s, v_0)$, and $d[s]$ is unchanged by subsequent relaxations.

- After 1 pass through E , we have $d[v_1] = \delta(s, v_1)$.
- After 2 passes through E , we have $d[v_2] = \delta(s, v_2)$.

...

- After k passes through E , we have $d[v_k] = \delta(s, v_k)$.

Since G contains no negative-weight cycles, p is simple. Longest simple path has $\leq |V| - 1$ edges. \square



Detection of negative-weight cycles

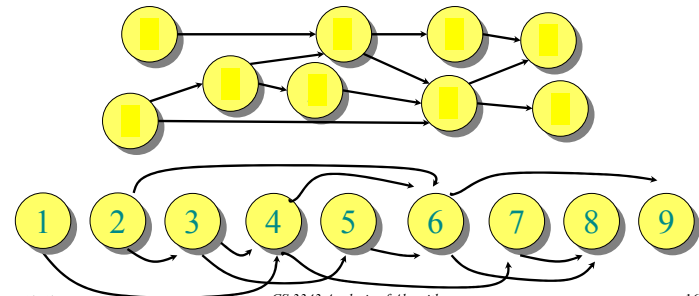
Corollary. If a value $d[v]$ fails to converge after $|V| - 1$ passes, there exists a negative-weight cycle in G reachable from s . \square



DAG shortest paths

If the graph is a **directed acyclic graph (DAG)**, we first **topologically sort** the vertices.

- Determine $f: V \rightarrow \{1, 2, \dots, |V|\}$ such that $(u, v) \in E \Rightarrow f(u) < f(v)$.

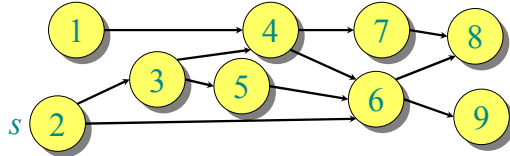




DAG shortest paths

If the graph is a **directed acyclic graph (DAG)**, we first **topologically sort** the vertices.

- Determine $f: V \rightarrow \{1, 2, \dots, |V|\}$ such that $(u, v) \in E \Rightarrow f(u) < f(v)$.
- $O(|V| + |E|)$ time



- Walk through the vertices $u \in V$ in this order, relaxing the edges in $Adj[u]$, thereby obtaining the shortest paths from s in a total of $O(|V| + |E|)$ time.



Shortest paths

Single-source shortest paths

- Nonnegative edge weights
 - Dijkstra's algorithm: $O(|E| \log |V|)$
- General: Bellman-Ford: $O(|V||E|)$
- DAG: One pass of Bellman-Ford: $O(|V| + |E|)$

All-pairs shortest paths

- Nonnegative edge weights
 - Dijkstra's algorithm $|V|$ times: $O(|V||E| \log |V|)$
- General
 - Bellman-Ford $|V|$ times: $O(|V|^2|E|)$
 - Floyd-Warshall: $O(|V|^3)$



All-pairs shortest paths

Input: Digraph $G = (V, E)$, where $|V| = n$, with edge-weight function $w: E \rightarrow \mathbb{R}$.

Output: $n \times n$ matrix of shortest-path lengths $\delta(i, j)$ for all $i, j \in V$.

Algorithm #1:

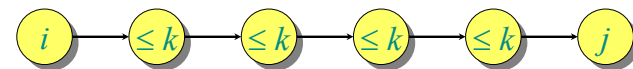
- Run Bellman-Ford once from each vertex.
- Time = $O(|V|^2|E|)$.
- But: Dense graph $\Rightarrow O(|V|^4)$ time.



Floyd-Warshall algorithm

- Dynamic programming algorithm.
- Assume $V = \{1, 2, \dots, n\}$, and assume G is given in an adjacency matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ where a_{ij} is the weight of the edge from i to j .

Define $c_{ij}^{(k)}$ = weight of a shortest path from i to j with intermediate vertices belonging to the set $\{1, 2, \dots, k\}$.



Thus, $\delta(i, j) = c_{ij}^{(n)}$. Also, $c_{ij}^{(0)} = a_{ij}$.

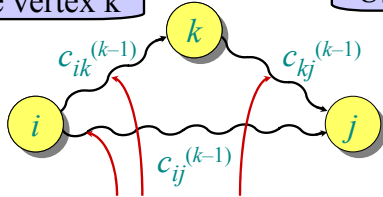


Floyd-Warshall recurrence

$$c_{ij}^{(k)} = \min \{c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)}\}$$

Do not use vertex k

Use vertex k



intermediate vertices in $\{1, 2, \dots, k-1\}$



Pseudocode for Floyd-Warshall

```

for k ← 1 to n do
  for i ← 1 to n do
    for j ← 1 to n do
      if  $c_{ij}^{(k-1)} > c_{ik}^{(k-1)} + c_{kj}^{(k-1)}$  then } relaxation
         $c_{ij}^{(k)} \leftarrow c_{ik}^{(k-1)} + c_{kj}^{(k-1)}$ 
      else
         $c_{ij}^{(k)} \leftarrow c_{ij}^{(k-1)}$ 

```

- Runs in $\Theta(n^3)$ time and space
- Simple to code.
- Efficient in practice.



Shortest paths

Single-source shortest paths

- Nonnegative edge weights
 - Dijkstra's algorithm: $O(|E| \log |V|)$
 - General: Bellman-Ford: $O(|V||E|)$
 - DAG: One pass of Bellman-Ford: $O(|V| + |E|)$
- } adj. list

All-pairs shortest paths

- Nonnegative edge weights
 - Dijkstra's algorithm $|V|$ times: $O(|V||E| \log |V|)$
 - General
 - Bellman-Ford $|V|$ times: $O(|V|^2|E|)$
 - Floyd-Warshall: $O(|V|^3)$
- adj. list
adj. matrix



Johnson's algorithm

1. Compute a weight function \hat{w} from w such that $\hat{w}(u, v) \geq 0$ for all $(u, v) \in E$. (Or determine that a negative-weight cycle exists, and stop.)
 - Can be done in $O(|V||E|)$ time (details skipped)
2. Run Dijkstra's algorithm from each vertex using \hat{w} .
 - Time = $O(|V||E| \log |V|)$.
3. Reweight each shortest-path length $\hat{w}(p)$ to produce the shortest-path lengths $w(p)$ of the original graph.
 - Time = $O(|V|^2)$ (details skipped)

Total time = $O(|V||E| \log |V|)$.



Shortest paths

Single-source shortest paths

- Nonnegative edge weights
 - Dijkstra's algorithm: $O(|E| \log |V|)$
 - General: Bellman-Ford: $O(|V||E|)$
 - DAG: One pass of Bellman-Ford: $O(|V| + |E|)$
- } adj. list

All-pairs shortest paths

- Nonnegative edge weights
 - Dijkstra's algorithm $|V|$ times: $O(|V||E| \log |V|)$ adj. list
- General
 - Bellman-Ford $|V|$ times: $O(|V|^2|E|)$ adj. list
 - Floyd-Warshall: $O(|V|^3)$ adj. matrix
 - Johnson's algorithm: $O(|V| |E| \log |V|)$ adj. list