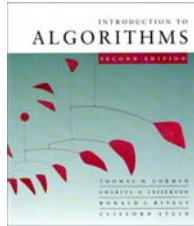




CS 3343 -- Spring 2009



Dynamic Programming

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Slides courtesy of Charles Leiserson with small changes by Carola Wenk



Dynamic programming

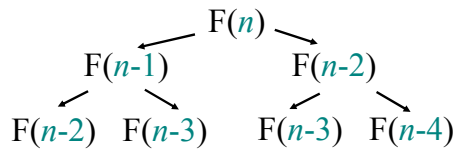
- Algorithm design technique
- A technique for solving problems that have
 - overlapping subproblems
 - and, when used for optimization, have an optimal substructure property
- **Idea:** Do not repeatedly solve the same subproblems, but solve them only once and store the solutions in a **dynamic programming table**



Example: Fibonacci numbers

• $F(0)=0$; $F(1)=1$; $F(n)=F(n-1)+F(n-2)$ for $n \geq 2$

• Implement this recursion naively:



Solve same subproblems many times !

Runtime is exponential in n .

• Store 1D DP-table and fill bottom-up in $O(n)$ time:

F:

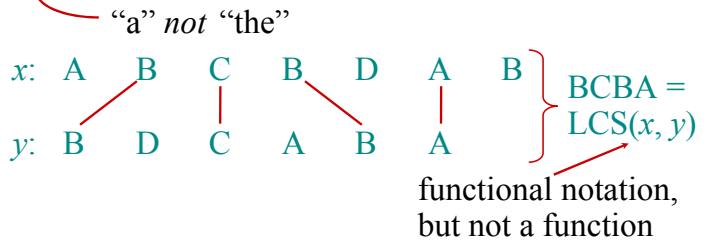
0	1	1	2	3	5	8				
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Longest Common Subsequence

Example: Longest Common Subsequence (LCS)

• Given two sequences $x[1 \dots m]$ and $y[1 \dots n]$, find a longest subsequence common to them both.





Brute-force LCS algorithm

Check every subsequence of $x[1 \dots m]$ to see if it is also a subsequence of $y[1 \dots n]$.

Analysis

- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).
- Hence, the runtime would be exponential !



Towards a better algorithm

Two-Step Approach:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|s|$.

Strategy: Consider *prefixes* of x and y .

- Define $c[i, j] = |\text{LCS}(x[1 \dots i], y[1 \dots j])|$.
- Then, $c[m, n] = |\text{LCS}(x, y)|$.

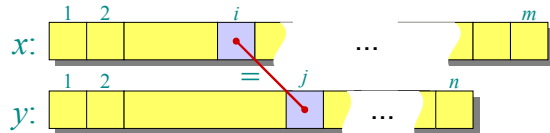


Recursive formulation

Theorem.

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

Proof. Case $x[i] = y[j]$:



Let $z[1 \dots k] = \text{LCS}(x[1 \dots i], y[1 \dots j])$, where $c[i, j] = k$. Then, $z[k] = x[i]$, or else z could be extended. Thus, $z[1 \dots k-1]$ is CS of $x[1 \dots i-1]$ and $y[1 \dots j-1]$.



Proof (continued)

Claim: $z[1 \dots k-1] = \text{LCS}(x[1 \dots i-1], y[1 \dots j-1])$.

Suppose w is a longer CS of $x[1 \dots i-1]$ and $y[1 \dots j-1]$, that is, $|w| > k-1$. Then, **cut and paste:** $w \parallel z[k]$ (w concatenated with $z[k]$) is a common subsequence of $x[1 \dots i]$ and $y[1 \dots j]$ with $|w \parallel z[k]| > k$. Contradiction, proving the claim.

Thus, $c[i-1, j-1] = k-1$, which implies that $c[i, j] = c[i-1, j-1] + 1$.

Other cases are similar. ◻



Dynamic-programming hallmark #1

Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

→ *Recurrence*

If $z = \text{LCS}(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y .

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Recursive algorithm for LCS

```

LCS(x, y, i, j)
  if x[i] = y[j]
    then c[i, j] ← LCS(x, y, i-1, j-1) + 1
    else c[i, j] ← max { LCS(x, y, i-1, j),
                        LCS(x, y, i, j-1) }

```

Worst-case: $x[i] \neq y[j]$, in which case the algorithm evaluates two subproblems, each with only one parameter decremented.

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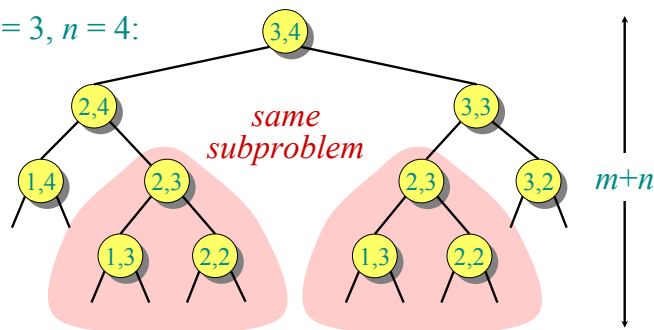
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Recursion tree

$m = 3, n = 4$:



Height = $m + n \Rightarrow$ work potentially exponential, but we're solving subproblems already solved!

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Dynamic-programming hallmark #2

Overlapping subproblems

A recursive solution contains a "small" number of distinct subproblems repeated many times.

The number of distinct LCS subproblems for two strings of lengths m and n is only mn .

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Dynamic-programming

There are two variants of dynamic programming:

1. Memoization
2. Bottom-up dynamic programming (often referred to as “dynamic programming”)



Memoization algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

```

for all i, j: c[i,0]=0 and c[0,j]=0
LCS(x, y, i, j)
  if c[i, j] = NIL
    then if x[i] = y[j]
      then c[i, j] ← LCS(x, y, i-1, j-1) + 1
      else c[i, j] ← max { LCS(x, y, i-1, j),
                          LCS(x, y, i, j-1) }
  } same as before

```

Time = $\Theta(mn)$ = constant work per table entry.
Space = $\Theta(mn)$.



Memoization

LCS(x,y,7,6)

		1	2	3	4	5	6	7
x:	A	B	C	B	D	A	B	
y:	0	0	0	0	0	0	0	0
1 B	0	0	1	nil	nil	nil	nil	nil
2 D	0	0	1	nil	nil	nil	nil	nil
3 C	0	0	2	nil	nil	nil	nil	nil
4 A	0	1	nil	nil	nil	nil	nil	nil
5 B	0	nil	2	nil	nil	nil	nil	nil
6 A	0	nil	nil	nil	nil	nil	nil	nil

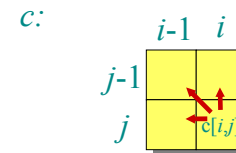
Recursion tree for LCS(x,y,7,6):

- (6,6) → (5,5) → (4,5)
- (7,5) → (6,4) → (5,4) → (5,3)



Recursive formulation

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max \{ c[i-1, j], c[i, j-1] \} & \text{otherwise.} \end{cases}$$





Bottom-up dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

	A	B	C	B	D	A	B
0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1
D	0	0	1	1	1	2	2
C	0	0	1	2	2	2	2
A	0	1	1	2	2	2	3
B	0	1	2	2	3	3	4
A	0	1	2	2	3	3	4



Bottom-up dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct
LCS by back-
tracing.

Space = $\Theta(mn)$.

Exercise:
 $O(\min\{m, n\})$.

	A	B	C	B	D	A	B
0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1
D	0	0	1	1	1	2	2
C	0	0	1	2	2	2	2
A	0	1	1	2	2	2	3
B	0	1	2	2	3	3	4
A	0	1	2	2	3	3	4