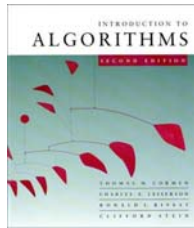




CS 3343 -- Spring 2009



Order Statistics

Carola Wenk

Slides courtesy of Charles Leiserson with small changes by Carola Wenk



Order statistics

Select the i th smallest of n elements (the element with **rank** i).

- $i = 1$: **minimum**;
- $i = n$: **maximum**;
- $i = \lfloor (n+1)/2 \rfloor$ or $\lceil (n+1)/2 \rceil$: **median**.

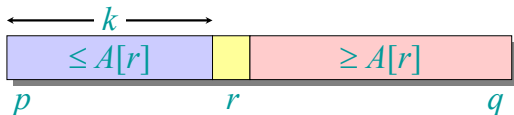
Naive algorithm: Sort and index i th element.

Worst-case running time = $\Theta(n \log n) + \Theta(1)$
= $\Theta(n \log n)$,
using merge sort or heapsort (*not* quicksort).



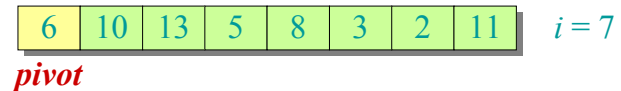
Randomized divide-and-conquer algorithm

$\Theta(n)$ $T(n)$
RAND-SELECT(A, p, q, i) \triangleright i -th smallest of $A[p..q]$
 if $p = q$ then return $A[p]$
 $r \leftarrow$ **RAND-PARTITION**(A, p, q)
 $k \leftarrow r - p + 1$ $\triangleright k = \text{rank}(A[r])$
 if $i = k$ then return $A[r]$
 if $i < k$
 then return **RAND-SELECT**($A, p, r - 1, i$)
 else return **RAND-SELECT**($A, r + 1, q, i - k$)

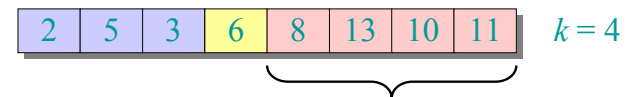


Example

Select the $i = 7$ th smallest:



Partition:



Select the $7 - 4 = 3$ rd smallest recursively.



Intuition for analysis

(All our analyses today assume that all elements are distinct.)

Lucky:

$$T(n) = T(9n/10) + \Theta(n) \quad n^{\log_{10}/9} = n^0 = 1$$

$$= \Theta(n) \quad \text{CASE 3}$$

Unlucky:

$$T(n) = T(n-1) + \Theta(n) \quad \text{arithmetic series}$$

$$= \Theta(n^2)$$

Worse than sorting!



Analysis of expected time

The analysis follows that of randomized quicksort, but it's a little different.

Let $T(n)$ = the random variable for the running time of RAND-SELECT on an input of size n , assuming random numbers are independent.

For $k = 0, 1, \dots, n-1$, define the **indicator random variable**

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$



Analysis (continued)

To obtain an upper bound, assume that the i th element always falls in the larger side of the partition:

$$T(n) = \begin{cases} T(\max\{0, n-1\}) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(\max\{1, n-2\}) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \vdots \\ T(\max\{n-1, 0\}) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k (T(\max\{k, n-k-1\}) + \Theta(n))$$

$$\leq 2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} X_k (T(k) + \Theta(n))$$



Calculating expectation

$$E[T(n)] = E \left[2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} X_k (T(k) + \Theta(n)) \right]$$

Take expectations of both sides.



Calculating expectation

$$\begin{aligned}
 E[T(n)] &= E\left[2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} X_k (T(k) + \Theta(n))\right] \\
 &= 2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[X_k (T(k) + \Theta(n))]
 \end{aligned}$$

Linearity of expectation.



Calculating expectation

$$\begin{aligned}
 E[T(n)] &= E\left[2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} X_k (T(k) + \Theta(n))\right] \\
 &= 2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[X_k (T(k) + \Theta(n))] \\
 &= 2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[X_k] \cdot E[T(k) + \Theta(n)]
 \end{aligned}$$

Independence of X_k from other random choices.



Calculating expectation

$$\begin{aligned}
 E[T(n)] &= E\left[2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} X_k (T(k) + \Theta(n))\right] \\
 &= 2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[X_k (T(k) + \Theta(n))] \\
 &= 2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[X_k] \cdot E[T(k) + \Theta(n)] \\
 &= \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} \Theta(n)
 \end{aligned}$$

Linearity of expectation; $E[X_k] = 1/n$.



Calculating expectation

$$\begin{aligned}
 E[T(n)] &= E\left[2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} X_k (T(k) + \Theta(n))\right] \\
 &= 2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[X_k (T(k) + \Theta(n))] \\
 &= 2 \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[X_k] \cdot E[T(k) + \Theta(n)] \\
 &= \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} \Theta(n) \\
 &= \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \Theta(n)
 \end{aligned}$$



Hairy recurrence

(But not quite as hairy as the quicksort one.)

$$E[T(n)] = \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \Theta(n)$$

Prove: $E[T(n)] \leq cn$ for constant $c > 0$.

- The constant c can be chosen large enough so that $E[T(n)] \leq cn$ for the base cases.

Use fact: $\sum_{k=\lfloor n/2 \rfloor}^{n-1} k \leq \frac{3}{8}n^2$ (exercise).



Substitution method

$$E[T(n)] \leq \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

Substitute inductive hypothesis.



Substitution method

$$\begin{aligned}
 E[T(n)] &\leq \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n) \\
 &\leq \frac{2c}{n} \left(\frac{3}{8}n^2 \right) + \Theta(n)
 \end{aligned}$$

Use fact.



Substitution method

$$\begin{aligned}
 E[T(n)] &\leq \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n) \\
 &\leq \frac{2c}{n} \left(\frac{3}{8}n^2 \right) + \Theta(n) \\
 &= cn - \left(\frac{cn}{4} - \Theta(n) \right)
 \end{aligned}$$

Express as **desired – residual**.



Substitution method

$$\begin{aligned}
E[T(n)] &\leq \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n) \\
&\leq \frac{2c}{n} \left(\frac{3}{8} n^2 \right) + \Theta(n) \\
&= cn - \left(\frac{cn}{4} - \Theta(n) \right) \\
&\leq cn,
\end{aligned}$$

if c is chosen large enough so that $cn/4$ dominates the $\Theta(n)$.



Summary of randomized order-statistic selection

- Works fast: linear expected time.
- Excellent algorithm in practice.
- But, the worst case is **very** bad: $\Theta(n^2)$.

Q. Is there an algorithm that runs in linear time in the worst case?

A. Yes, due to Blum, Floyd, Pratt, Rivest, and Tarjan [1973].

IDEA: Generate a good pivot recursively.



Worst-case linear-time order statistics

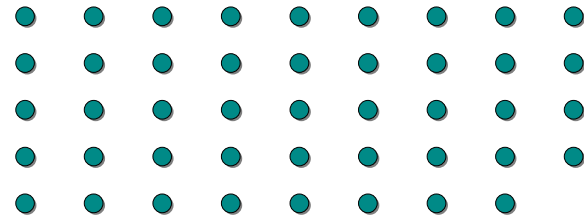
SELECT(i, n)

1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.
3. Partition around the pivot x . Let $k = \text{rank}(x)$.
4. **if** $i = k$ **then return** x
elseif $i < k$
then recursively SELECT the i th smallest element in the lower part
else recursively SELECT the $(i-k)$ th smallest element in the upper part

} Same as
RAND-
SELECT

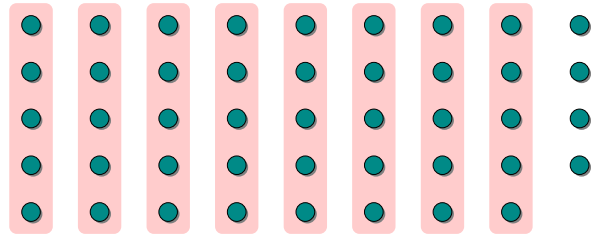


Choosing the pivot





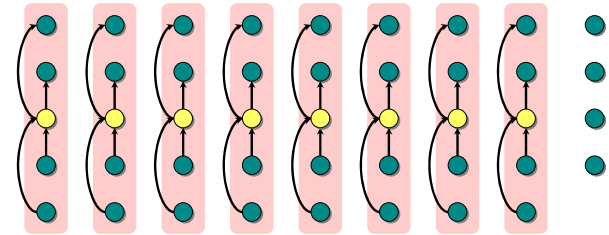
Choosing the pivot



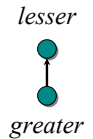
1. Divide the n elements into groups of 5.



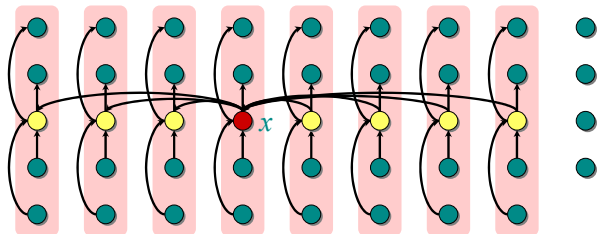
Choosing the pivot



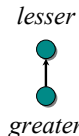
1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.



Choosing the pivot



1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.

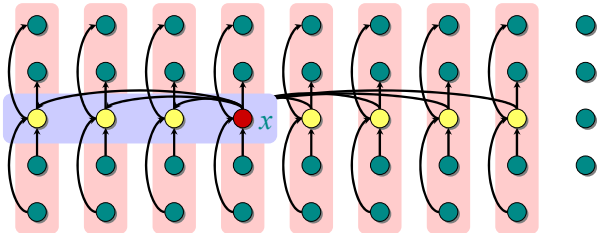


Developing the recurrence

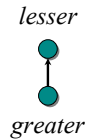
- $T(n)$ SELECT(i, n)
- $\Theta(n)$ {
1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
 2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.
 3. Partition around the pivot x . Let $k = \text{rank}(x)$.
 4. **if** $i = k$ **then return** x
elseif $i < k$
then recursively SELECT the i th smallest element in the lower part
else recursively SELECT the $(i-k)$ th smallest element in the upper part
- $T(?)$ }



Analysis (Assume all elements are distinct.)



At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$ group medians.



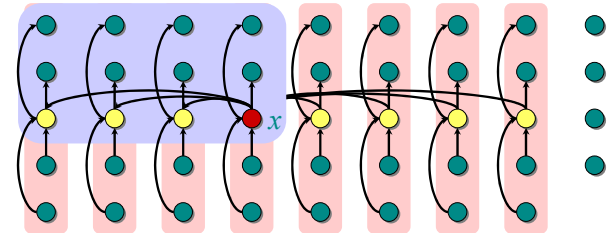
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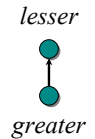


Analysis (Assume all elements are distinct.)



At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$ group medians.

- Therefore, at least $3\lfloor n/10 \rfloor$ elements are $\leq x$.



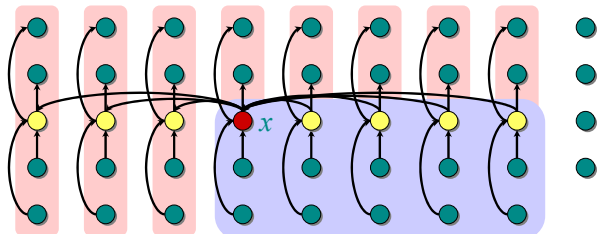
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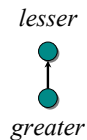


Analysis (Assume all elements are distinct.)



At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$ group medians.

- Therefore, at least $3\lfloor n/10 \rfloor$ elements are $\leq x$.
- Similarly, at least $3\lfloor n/10 \rfloor$ elements are $\geq x$.



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Analysis (Assume all elements are distinct.)

Need “at most” for worst-case runtime

- At least $3\lfloor n/10 \rfloor$ elements are $\leq x$
 \Rightarrow at most $n - 3\lfloor n/10 \rfloor$ elements are $\geq x$
- At least $3\lfloor n/10 \rfloor$ elements are $\geq x$
 \Rightarrow at most $n - 3\lfloor n/10 \rfloor$ elements are $\leq x$
- The recursive call to SELECT in Step 4 is executed recursively on $n - 3\lfloor n/10 \rfloor$ elements.

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Analysis (Assume all elements are distinct.)

- Use fact that $\lfloor a/b \rfloor \geq ((a-(b-1))/b)$ (page 51)
- $n-3\lfloor n/10 \rfloor \leq n-3\cdot(n-9)/10 = (10n-3n+27)/10 \leq 7n/10 + 3$
- The recursive call to SELECT in Step 4 is executed recursively on at most $7n/10+3$ elements.



Developing the recurrence

$T(n)$	SELECT(i, n)
$\Theta(n)$	<ol style="list-style-type: none"> 1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote. 2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.
$T(n/5)$	
$\Theta(n)$	<ol style="list-style-type: none"> 3. Partition around the pivot x. Let $k = \text{rank}(x)$. 4. if $i = k$ then return x elseif $i < k$ then recursively SELECT the ith smallest element in the lower part else recursively SELECT the $(i-k)$th smallest element in the upper part
$T(7n/10 + 3)$	



Solving the recurrence

$$T(n) = T\left(\frac{1}{5}n\right) + T\left(\frac{7}{10}n + 3\right) + dn \quad \text{for } \Theta(n)$$

Substitution: $T(n) \leq c\left(\frac{1}{5}n - 4\right) + c\left(\frac{7}{10}n + 3 - 4\right) + dn$

$$T(n) \leq c(n - 4)$$

$$\leq \frac{9}{10}cn - 4c + dn$$

$$= c(n - 4) - \frac{1}{10}cn + dn$$

$$\leq c(n - 4),$$

if c is chosen large enough, e.g., $c = 10d$

Technical trick. This shows that $T(n) \in O(n)$



Conclusions

- Since the work at each level of recursion is basically a constant fraction ($9/10$) smaller, the work per level is a geometric series dominated by the linear work at the root.
- In practice, this algorithm runs slowly, because the constant in front of n is large.
- The randomized algorithm is far more practical.

Exercise: Try to divide into groups of 3 or 7.