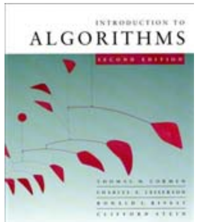
CS 3343 -- Spring 2009



Quicksort

Carola Wenk

Slides courtesy of Charles Leiserson with small changes by Carola Wenk

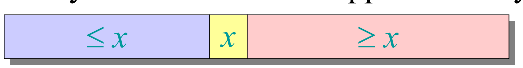
Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts "in place" (like insertion sort, but not like merge sort).
- Very practical (with tuning).

Divide and conquer

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.



2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

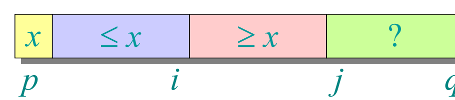
Key: *Linear-time partitioning subroutine.*

Partitioning subroutine

```
PARTITION(A, p, q)  ▷ A[p . . q]
   x ← A[p]          ▷ pivot = A[p]
   i ← p
   for j ← p + 1 to q
      do if A[j] ≤ x
            then i ← i + 1
                 exchange A[i] ↔ A[j]
   exchange A[p] ↔ A[i]
   return i
```

Running time $= O(n)$ for n elements.

Invariant:



Example of partitioning

6	10	13	5	8	3	2	11

i *j*

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6	10	13	5	8	3	2	11

i → *j*

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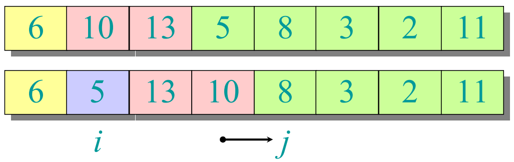
→ *i* *j*

Example of partitioning



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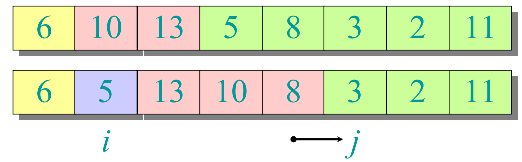

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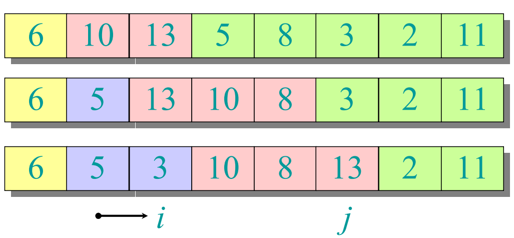
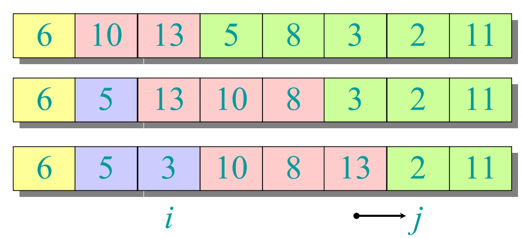


Example of partitioning



Example of partitioning

6	10	13	5	8	3	2	11
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→ i j

Example of partitioning

6	10	13	5	8	3	2	11
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i → j

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i → j

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11
2	5	3	6	8	13	10	11

i

Pseudocode for quicksort

```
QUICKSORT(A, p, r)
  if p < r
    then q ← PARTITION(A, p, r)
         QUICKSORT(A, p, q–1)
         QUICKSORT(A, q+1, r)
```

Initial call: QUICKSORT(A, 1, n)

Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.

Worst-case of quicksort

```
QUICKSORT(A, p, r)
  if p < r
    then q ← PARTITION(A, p, r)
         QUICKSORT(A, p, q–1)
         QUICKSORT(A, q+1, r)
```

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \Theta(n^2) \qquad \textit{(arithmetic series)}
\end{aligned}
$$

Worst-case recursion tree

$$T(n) = T(0) + T(n–1) + cn$$

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$T(n)$

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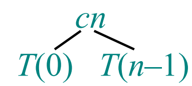

Worst-case recursion tree

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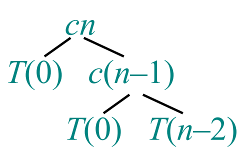

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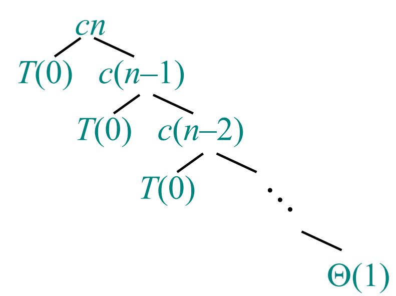

Worst-case recursion tree

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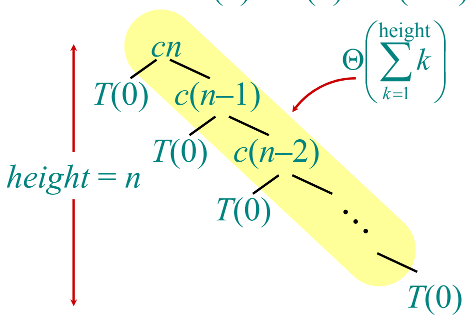

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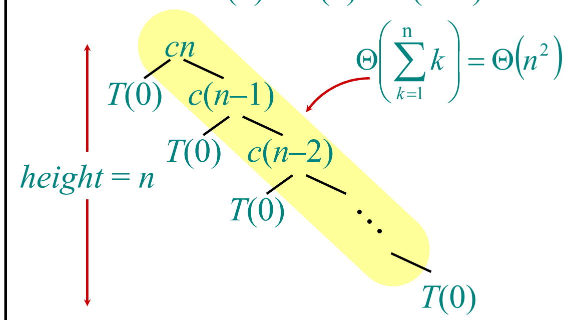
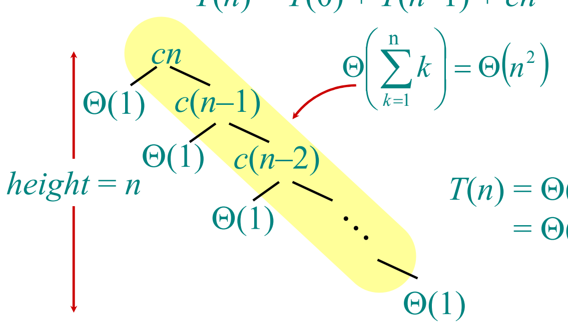


Worst-case recursion tree

$T(n) = T(0) + T(n–1) + cn$



$$T(n) = \Theta(n) + \Theta(n^2) \\ = \Theta(n^2)$$

Best-case analysis

(For intuition only!)

If we're lucky, PARTITION splits the array evenly:

$$T(n) = 2T(n/2) + \Theta(n) \\ = \Theta(n \log n)$$ (same as merge sort)

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

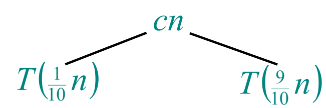
$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

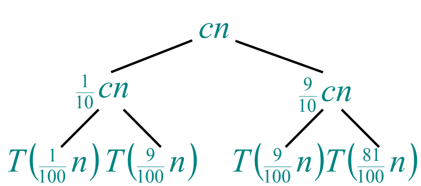
Analysis of "almost-best" case

$T(n)$

Analysis of "almost-best" case



Analysis of "almost-best" case



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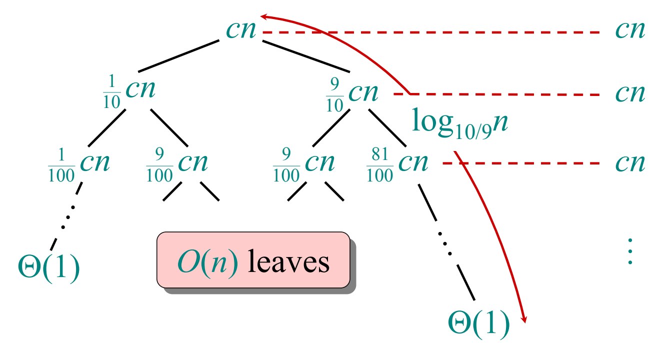
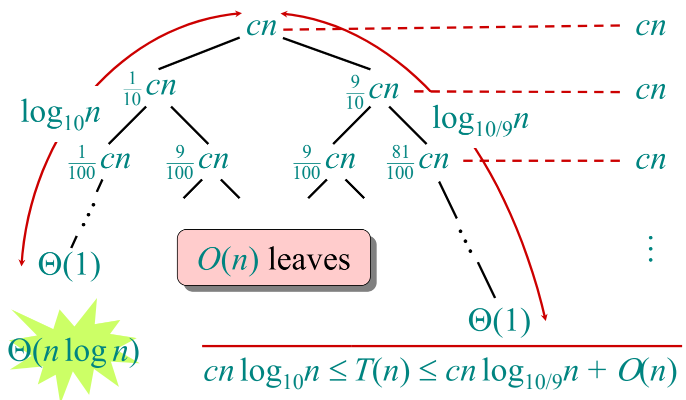


Analysis of "almost-best" case



$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$

Quicksort Runtimes

- Best case runtime $T_{best}(n) \in O(n \log n)$
- Worst case runtime $T_{worst}(n) \in O(n^2)$

- Worse than mergesort? Why is it called quicksort then?
- Its average runtime $T_{avg}(n) \in O(n \log n)$
- Better even, the expected runtime of **randomized quicksort** is $O(n \log n)$

Average Runtime

The **average runtime** $T_{avg}(n)$ for Quicksort is the average runtime over **all possible inputs** of length n.

- What kind of inputs are there?
- How many inputs are there?

Average Runtime

- What kind of inputs are there?
  - Do $[1,2,\ldots,n]$ and $[5,6,\ldots,n+5]$ cause different runtimes of Quicksort?
  - No. Therefore only consider all permutations of $[1,2,\ldots,n]$.
- How many inputs are there?
  - There are $n!$ different permutations of $[1,2,\ldots,n]$

Average Runtime

- Therefore, $T_{avg}(n)$ has to average the runtimes over all $n!$ different input permutations
- Disadvantage of considering average runtime:
  - There are still worst-case inputs that will have a $O(n^2)$ runtime
  - Are all inputs really equally likely? That depends on the application

$\Rightarrow$ **Better:** Use randomized quicksort

Randomized quicksort

IDEA: Partition around a ***random*** element.

- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.

Randomized quicksort analysis

- $T(n)$ = random variable for the running time of randomized quicksort on an input of size n, assuming random numbers are independent.
- $E(T(n))$ = expected value of $T(n)$, the "expected runtime" of randomized quicksort.

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad \dots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

Randomized quicksort analysis

For $k = 0, 1, \dots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.

Analysis (continued)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad\dots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n) \right).$$

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n) \right) \right]$$

Take expectations of both sides.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n) \right) \right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n) \right) \right]$$

Linearity of expectation.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n) \right) \right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n) \right) \right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\left[T(k) + T(n-k-1) + \Theta(n) \right]$$

Independence of X_k from other random choices.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k)+T(n-k-1)+\Theta(n)\big)\right]$$
$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k)+T(n-k-1)+\Theta(n)\big)\big]$$
$$= \sum_{k=0}^{n-1} E[X_k]\cdot E[T(k)+T(n-k-1)+\Theta(n)]$$
$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1}\Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k)+T(n-k-1)+\Theta(n)\big)\right]$$
$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k)+T(n-k-1)+\Theta(n)\big)\big]$$
$$= \sum_{k=0}^{n-1} E[X_k]\cdot E[T(k)+T(n-k-1)+\Theta(n)]$$
$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1}\Theta(n)$$
$$= \frac{2}{n}\sum_{k=0}^{n-1} E[T(k)] + \Theta(n)$$

Summations have identical terms.

Hairy recurrence

$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \le a\,n\log n$ for constant $a > 0$.

- Choose a large enough so that $a\,n\log n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

Use fact: $\sum_{k=2}^{n-1} k\log k \le \frac{1}{2}n^2\log n - \frac{1}{8}n^2$ (exercise).

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\log k + \Theta(n)$$

Substitute inductive hypothesis.

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\log k + \Theta(n)$$
$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\log n - \frac{1}{8}n^2\right) + \Theta(n)$$

Use fact.

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\log k + \Theta(n)$$
$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\log n - \frac{1}{8}n^2\right) + \Theta(n)$$
$$= an\log n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as ***desired* – *residual***.

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\log k + \Theta(n)$$
$$= \frac{2a}{n}\left(\frac{1}{2}n^2\log n - \frac{1}{8}n^2\right) + \Theta(n)$$
$$= an\log n - \left(\frac{an}{4} - \Theta(n)\right)$$
$$\le an\log n\ ,$$

if a is chosen large enough so that $an/4$ dominates the $\Theta(n)$.

Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.