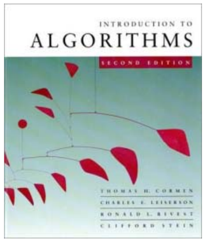
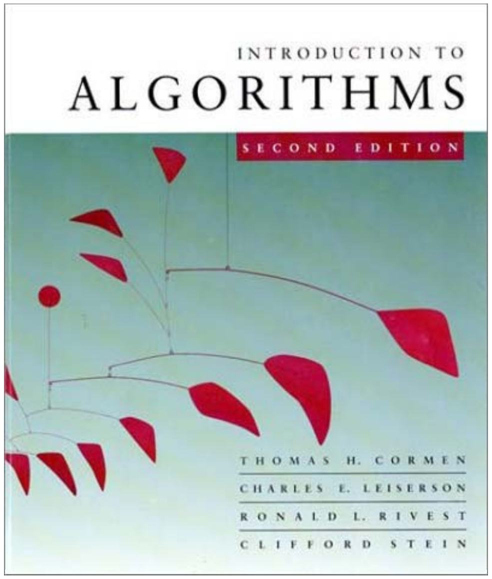


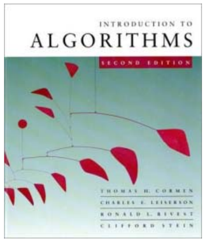

# CS 3343 – Fall 2011




# *Randomized Algorithms & Quicksort*

**Carola Wenk**

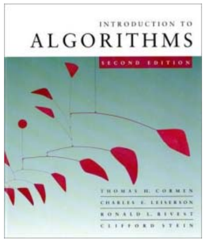
Slides courtesy of Charles Leiserson with small changes by Carola Wenk




# Deterministic Algorithms

Runtime for deterministic algorithms with input size *n*:

- Best-case runtime
  - ➔Attained by one input of size *n*
- Worst-case runtime
  - ➔ Attained by one input of size *n*
- Average runtime
  - ➔ Averaged **over all possible inputs** of size *n*




# Deterministic Algorithms: Insertion Sort

Best-case runtime: $O(n)$, input $[1,2,3,\ldots,n]$

➔ Attained by one input of size $n$

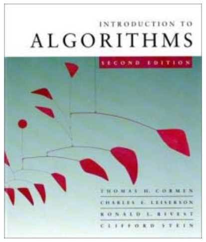
- Worst-case runtime: $O(n^2)$, input $[n, n\text{-}1, \ldots,2,1]$

➔ Attained by one input of size $n$

- Average runtime : $O(n^2)$; see book for analysis

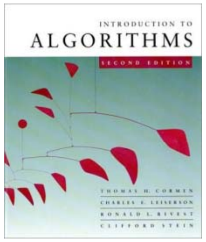
➔ Averaged **over all possible inputs** of size $n$

- What kind of inputs are there?
- How many inputs are there?




# Average Runtime

- What kind of inputs are there?
  - Do [1,2,…,$n$] and [5,6,…,$n$+5] cause different behavior of Insertion Sort?
  - No. Therefore it suffices to only consider all permutations of [1,2,…,$n$] .
- How many inputs are there?
  - There are $n$! different permutations of [1,2,…,$n$]



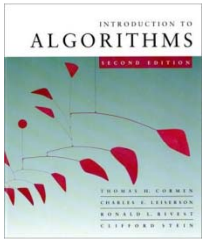

# Average Runtime Insertion Sort: *$n$=4*

```
for j=2 to n {
    key = A[j]
    // insert A[j] into sorted sequen
    i=j-1
    while(i>0 && A[i]>key){
       A[i+1]=A[i]
       i--
    }
    A[i+1]=key
}
```

- Inputs: 4!=24

| | | | |
|---|---|---|---|
| [1,2,3,4] **0** | [4,1,2,3] **3** | [4,1,3,2] **4** | [4,3,2,1] **6** |
| [2,1,3,4] **1** | [1,4,2,3] **2** | [1,4,3,2] **3** | [3,4,2,1] **5** |
| [1,3,2,4] **1** | [1,2,4,3] **1** | [1,3,4,2] **2** | [3,2,4,1] **4** |
| [3,1,2,4] **2** | [4,2,1,3] **4** | [4,3,1,2] **5** | [4,2,3,1] **5** |
| [3,2,1,4] **3** | [2,1,4,3] **2** | [3,4,1,2] **4** | [2,4,3,1] **4** |
| [2,3,1,4] **2** | [2,1,3,4] **1** | [3,1,4,2] **3** | [2,3,4,1] **3** |

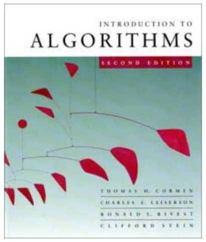
- Runtime is proportional to: 3 + **#times in while loop**
- Best: 3+**0**, Worst: 3+**6**=9, Average: 3+**70**/24 = 5.92




# Average Runtime: Insertion Sort

- The average runtime averages runtimes over all *n*! different input permutations
- Disadvantage of considering average runtime:
  - There are still worst-case inputs that will have the worst-case runtime
  - Are all inputs really equally likely? That depends on the application

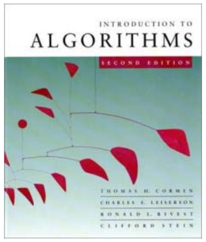
$\Rightarrow$ **Better:** Use a randomized algorithm




# Randomized Algorithm: Insertion Sort

- **Randomize the order of the input array:**
  - Either prior to calling insertion sort,
  - or during insertion sort (insert random element)
- This makes the runtime depend on a probabilistic experiment (sequence of numbers obtained from random number generator)

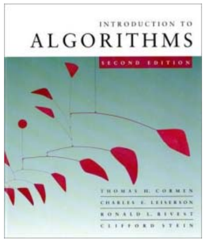
  ⇒Runtime is a random variable (maps sequence of random numbers to runtimes)
- **Expected runtime** = expected value of runtime random variable




# Randomized Algorithm: Insertion Sort

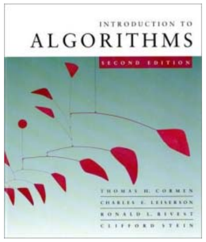
- Runtime is independent of input order ([1,2,3,4] may have good or bad runtime, depending on sequence of random numbers)
- No assumptions need to be made about input distribution
- No one specific input elicits worst-case behavior
- The worst case is determined only by the output of a random-number generator.

$\Rightarrow$ When possible use expected runtimes of randomized algorithms instead of average case analysis of deterministic algorithms




# Quicksort

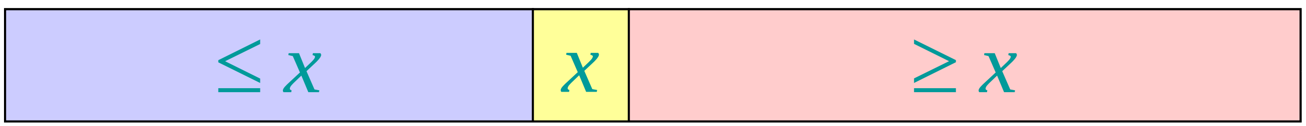
- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts “in place” (like insertion sort, but not like merge sort).
- Very practical (with tuning).
- We are going to perform an expected runtime analysis on randomized quicksort




# Quicksort: Divide and conquer

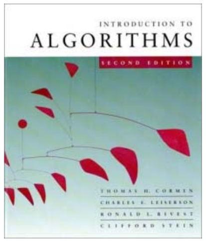
Quicksort an $n$-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** $x$ such that elements in lower subarray $\leq x \leq$ elements in upper subarray.




2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

**Key:** *Linear-time partitioning subroutine.*




# Partitioning subroutine

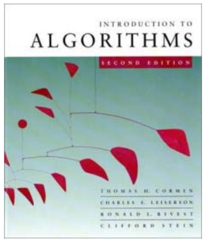
PARTITION$(A, p, q)$ ⊳ $A[p\,.\,.\,q]$
    $x \leftarrow A[p]$ ⊳ pivot $= A[p]$
    $i \leftarrow p$
    **for** $j \leftarrow p + 1$ **to** $q$
        **do if** $A[j] \leq x$
            **then** $i \leftarrow i + 1$
                exchange $A[i] \leftrightarrow A[j]$
    exchange $A[p] \leftrightarrow A[i]$
    **return** $i$

Running time $= O(n)$ for $n$ elements.

***Invariant:***

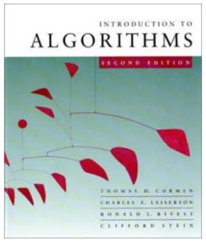
| $x$ | $\leq x$ | $\geq x$ | ? |
|---|---|---|---|

$p$ $i$ $j$ $q$




# Example of partitioning

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |
| $i$ | $j$ | | | | | | |

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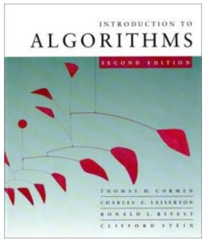
$i$ ⟶ $j$

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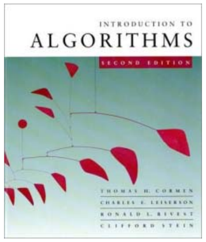


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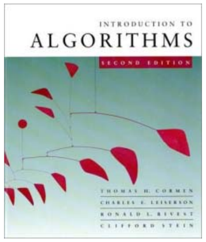
$\longrightarrow i$ $\quad\quad j$

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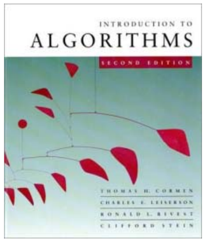


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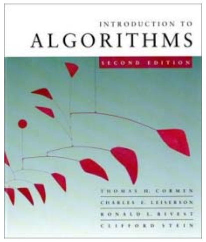


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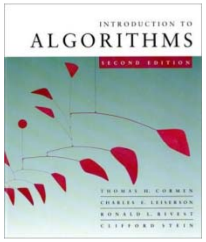
→ $i$ $j$

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|---|---|---|---|---|---|---|---|
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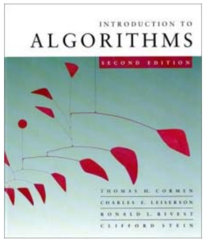
$i$ $\longrightarrow j$

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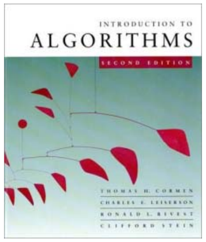
→ $i$ $j$

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| | | | | | | | |
|---|---|---|---|---|---|---|---|
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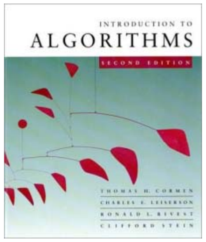
*i* → *j*

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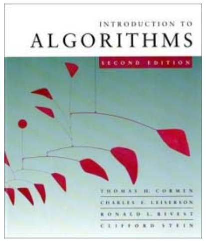
$i$ $\quad \bullet\!\!\longrightarrow j$

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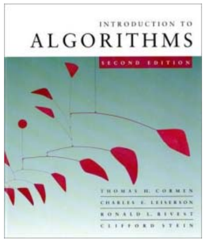
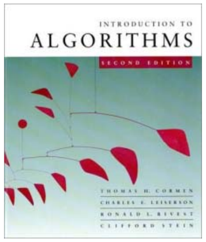
| 6 | 5 | 13 | 10 | 8 | 3 | 2 | 11 |
| 6 | 5 | 3 | 10 | 8 | 13 | 2 | 11 |
| 6 | 5 | 3 | 2 | 8 | 13 | 10 | 11 |
| 2 | 5 | 3 | 6 | 8 | 13 | 10 | 11 |
| | | | $i$ | | | | |




# Pseudocode for quicksort

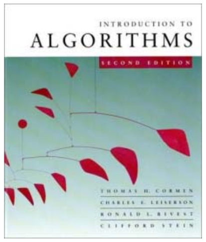
QUICKSORT$(A, p, r)$
    **if** $p < r$
        **then** $q \leftarrow$ PARTITION$(A, p, r)$
            QUICKSORT$(A, p, q–1)$
            QUICKSORT$(A, q+1, r)$

**Initial call:** QUICKSORT$(A, 1, n)$




# Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of $n$ elements.

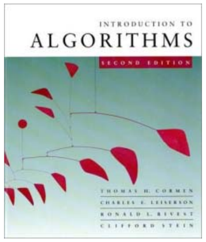



# Worst-case of quicksort

```
QUICKSORT(A, p, r)
   if p < r
      then q ← PARTITION(A, p, r)
           QUICKSORT(A, p, q–1)
           QUICKSORT(A, q+1, r)
```

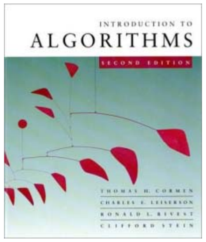
- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \qquad \textit{(arithmetic series)} \end{aligned}$$




# Worst-case recursion tree

$$T(n) = T(0) + T(n–1) + cn$$

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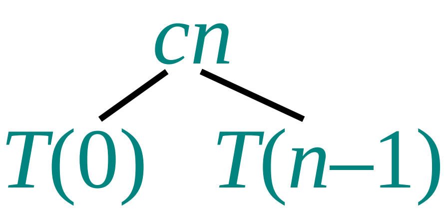
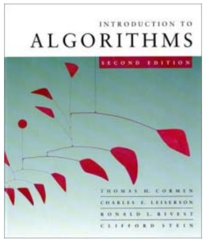
$T(n)$

# Worst-case recursion tree

$$T(n) = T(0) + T(n\text{–}1) + cn$$



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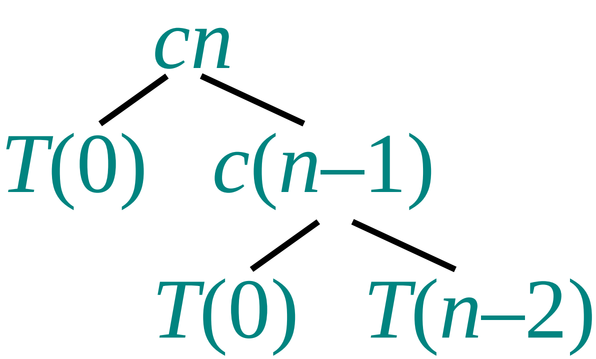

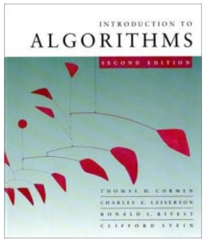


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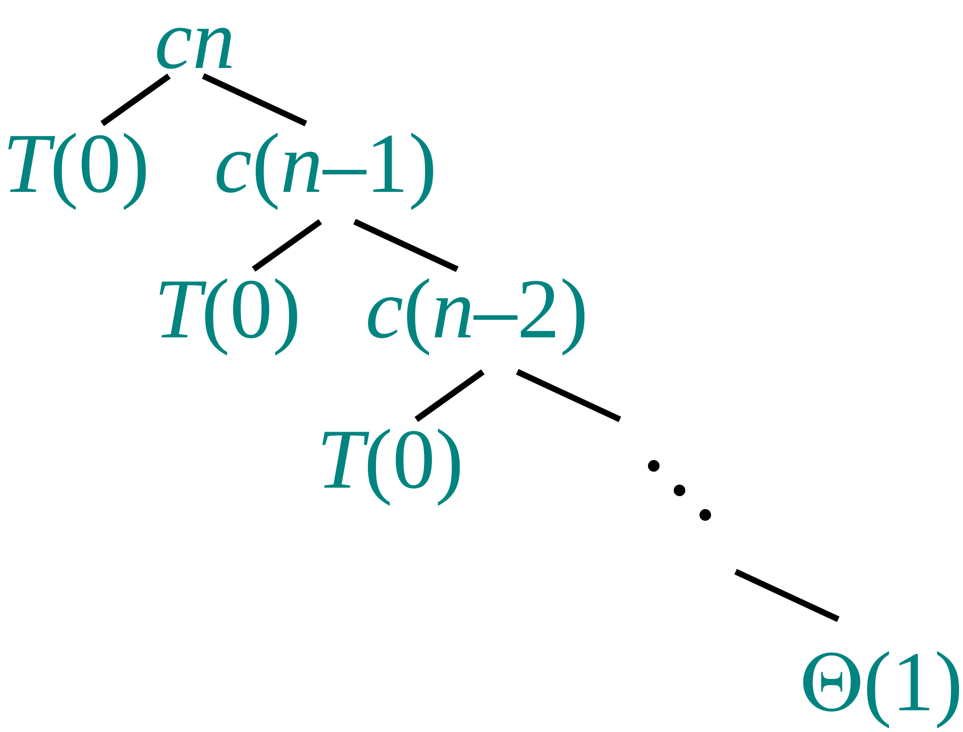

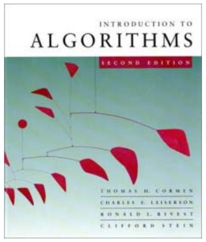


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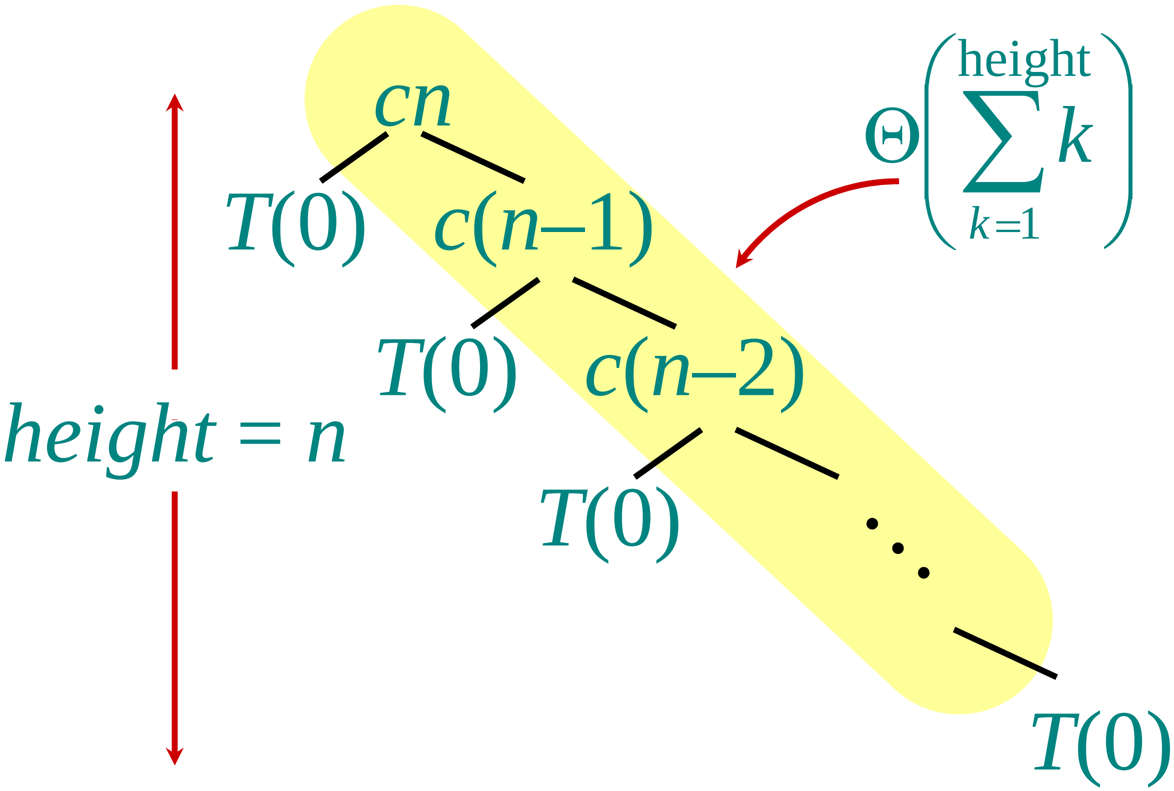

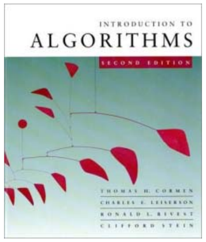


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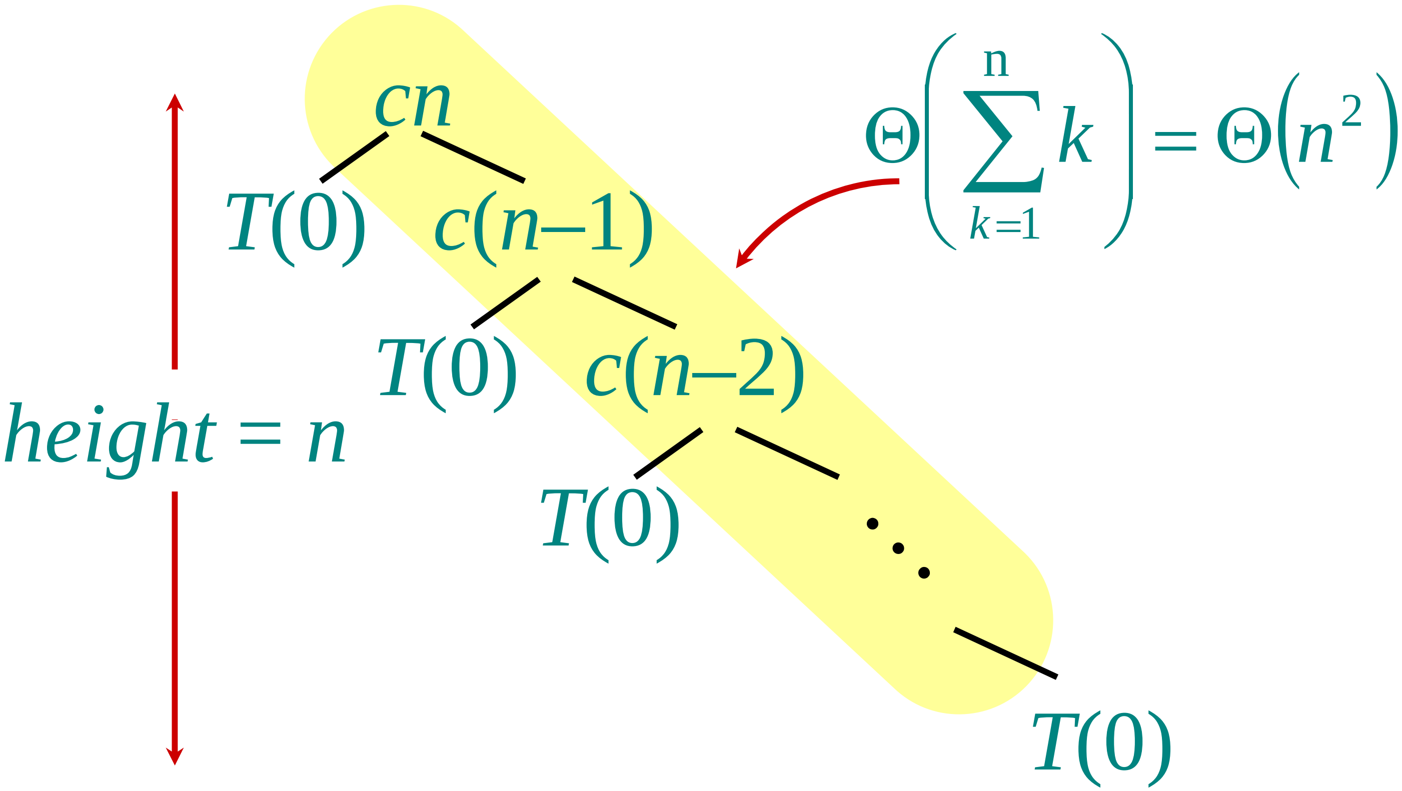
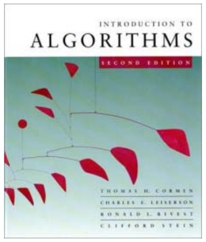
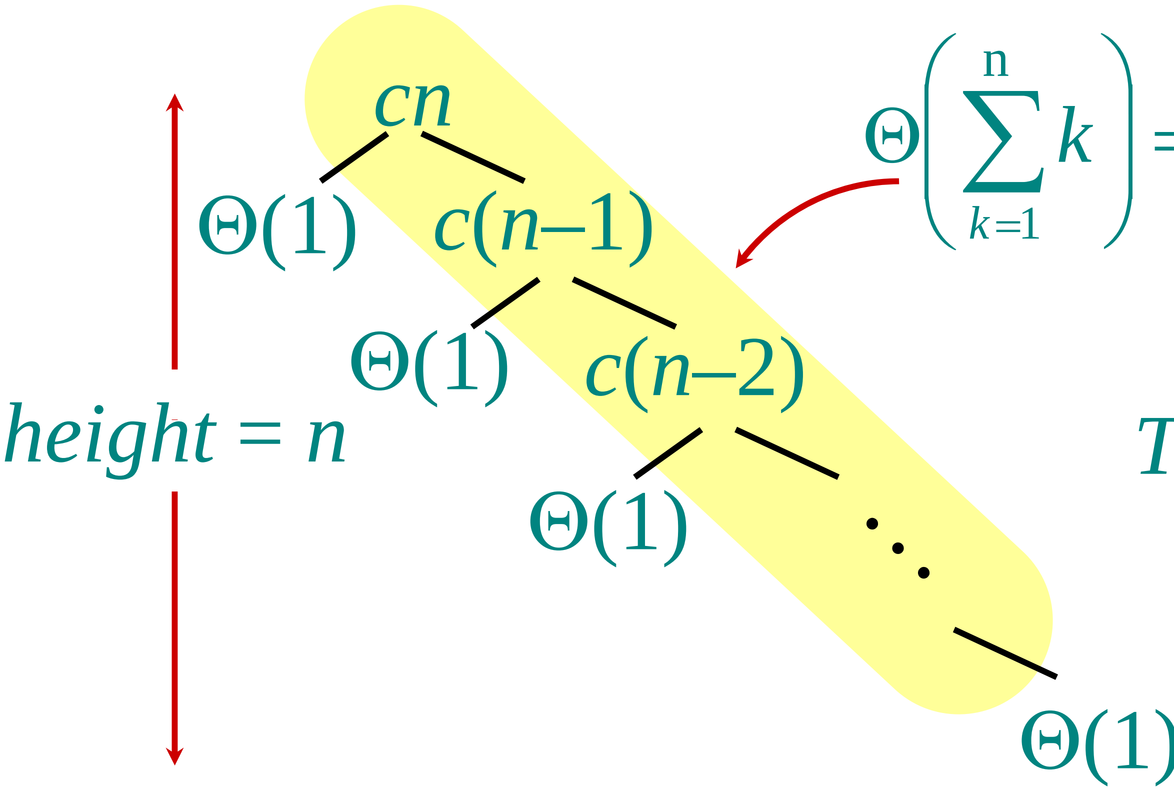




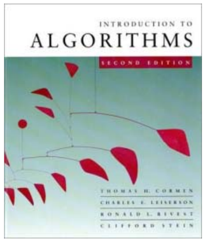
# Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$




$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$

$$T(n) = \Theta(n) + \Theta(n^2)$$
$$= \Theta(n^2)$$




# Best-case analysis

***(For intuition only!)***

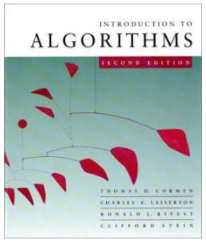
If we're lucky, PARTITION splits the array evenly:

$$T(n) = 2T(n/2) + \Theta(n)$$
$$= \Theta(n \log n)$$ (same as merge sort)

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

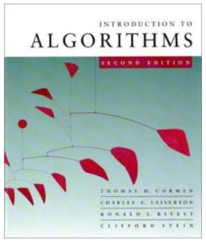
$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?




# Analysis of “almost-best” case

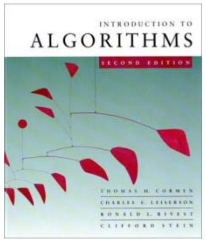
$$T(n)$$



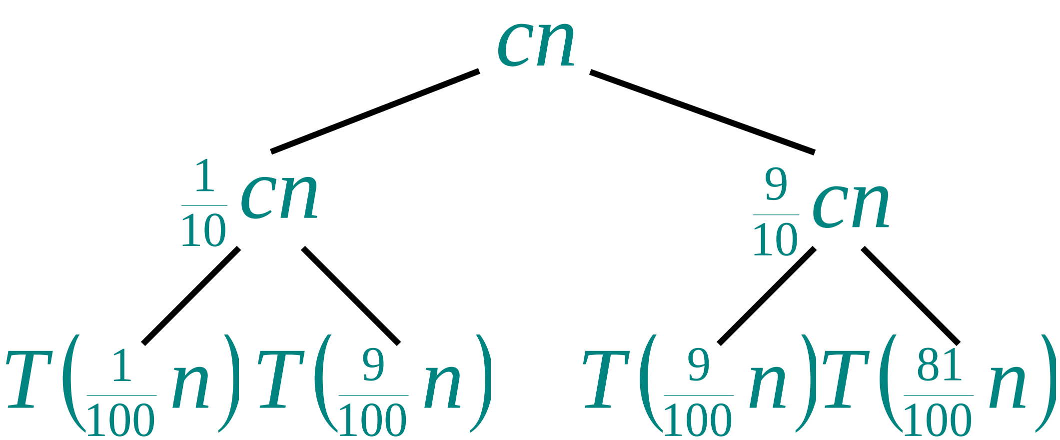

# Analysis of “almost-best” case

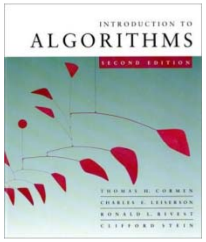
$$cn$$

$$T\left(\tfrac{1}{10}n\right) \qquad\qquad T\left(\tfrac{9}{10}n\right)$$

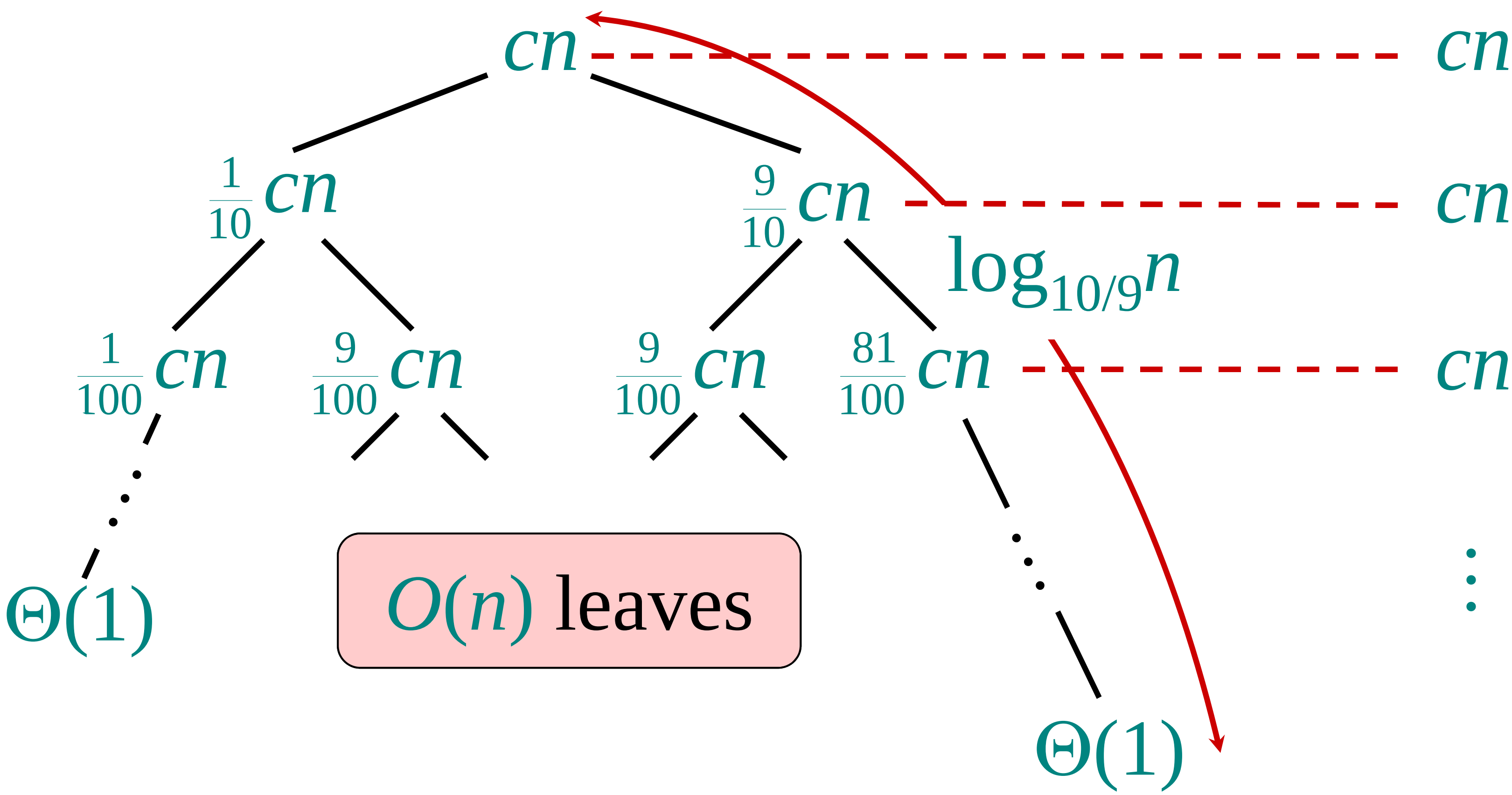


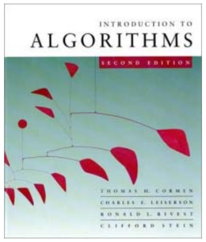

# Analysis of “almost-best” case



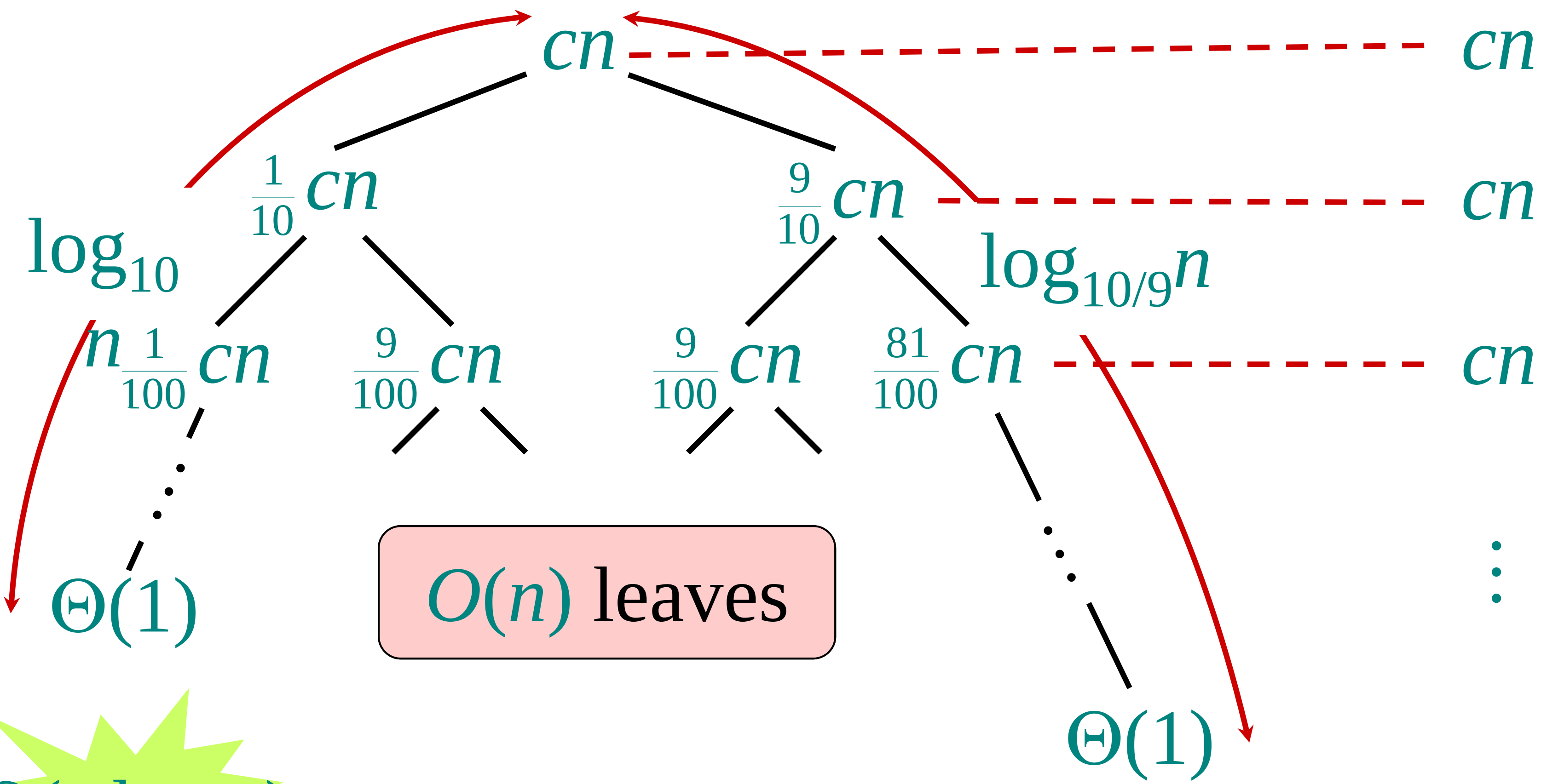



# Analysis of “almost-best” case



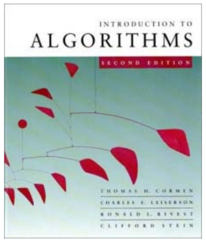



# Analysis of “almost-best” case




$\Theta(n \log n)$

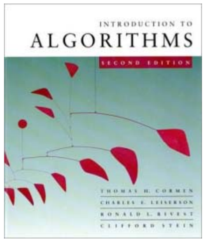
$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$




# Quicksort Runtimes

- Best case runtime $T_{best}(n) \in O(n \log n)$
- Worst case runtime $T_{worst}(n) \in O(n^2)$

- Worse than mergesort? Why is it called quicksort then?
- Its average runtime $T_{avg}(n) \in O(n \log n)$
- Better even, the expected runtime of **randomized quicksort** is $O(n \log n)$

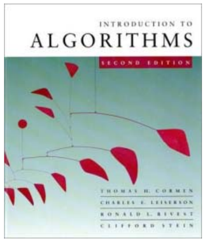



# Average Runtime

The **average runtime** $T_{avg}(n)$ for Quicksort is the average runtime over **all possible inputs** of length n.

- $T_{avg}(n)$ has to average the runtimes over all $n!$ different input permutations.
- There are still worst-case inputs that will have a $O(n^2)$ runtime

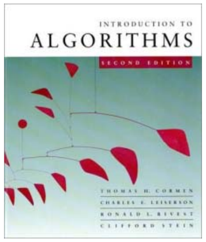
$\Rightarrow$ **Better:** Use randomized quicksort




# Randomized quicksort

**IDEA**: Partition around a ***random*** element.

- Running time is independent of the input order. It depends only on the sequence $s$ of random numbers.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the sequence $s$ of random numbers.

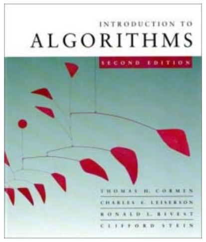



# Randomized quicksort analysis

- $T(n,s)$ = random variable for the running time of randomized quicksort on an input of size $n$, with sequence $s$ of random numbers which are assumed to be independent.

- $E(T(n))$ = expected value of $T(n,s)$, the "expected runtime" of randomized quicksort.

$$T(n,s) = \begin{cases} T(0,s) + T(n-1,s) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1,s) + T(n-2,s) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad \ldots & \\ T(n-1,s) + T(0,s) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

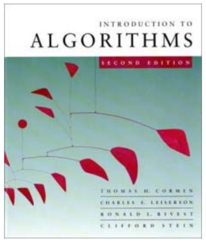



# Randomized quicksort analysis

For $k = 0, 1, \ldots, n–1$, define the ***indicator random variable***

$$X_k(s) = \begin{cases} 1 & \text{if PARTITION generates a } k : n–k–1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

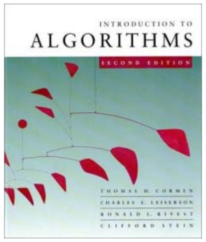
$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.




# Analysis (continued)

$$T(n,s) = \begin{cases} T(0,s) + T(n-1,s) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1,s) + T(n-2,s) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad \ldots & \\ T(n-1,s) + T(0,s) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

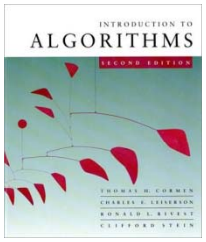
$$= \sum_{k=0}^{n-1} X_k(s)\big(T(k,s) + T(n-k-1,s) + \Theta(n)\big)$$




# Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

Take expectations of both sides.

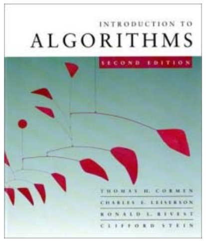



# Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

Linearity of expectation.



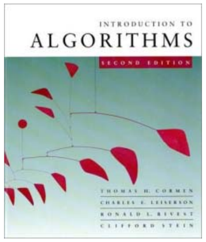

# Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

Independence of $X_k$ from other random choices.




# Calculating expectation

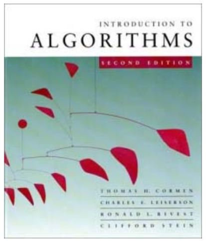
$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$ .




# Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

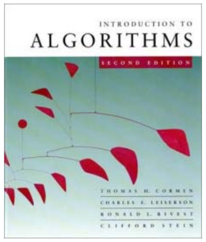
$$= \sum_{k=0}^{n-1} E\big[X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$= \frac{2}{n}\sum_{k=0}^{n-1} E[T(k)] + \Theta(n)$$

Summations have identical terms.




# Hairy recurrence

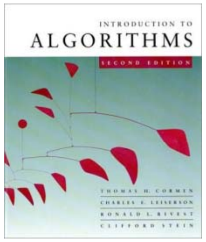
$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

**Prove:** $E[T(n)] \le a n \log n$ for constant $a > 0$.

- Choose $a$ large enough so that $a n \log n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

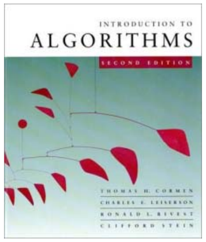
**Use fact:** $\sum_{k=2}^{n-1} k \log k \le \frac{1}{2} n^2 \log n - \frac{1}{8} n^2$ (exercise).




# Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=2}^{n-1} ak \log k + \Theta(n)$$

Substitute inductive hypothesis.

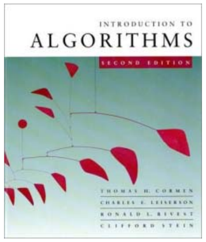



# Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \log k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2 \log n - \frac{1}{8}n^2\right) + \Theta(n)$$

Use fact.



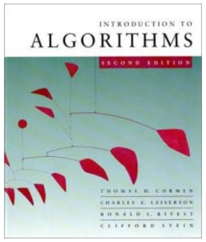

# Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\log k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\log n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an\log n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as ***desired – residual***.




# Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\log k + \Theta(n)$$

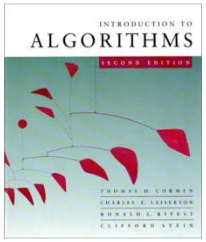
$$= \frac{2a}{n}\left(\frac{1}{2}n^2\log n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an\log n - \left(\frac{an}{4} - \Theta(n)\right)$$

$$\le an\log n$$

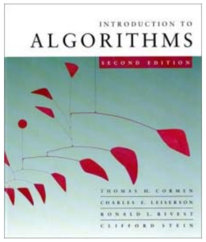
,

if $a$ is chosen large enough so that $an/4$ dominates the $\Theta(n)$.




# Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.



# Average Runtime vs. Expected Runtime

- Average runtime is averaged over all inputs of a deterministic algorithm.
- Expected runtime is the expected value of the runtime random variable of a randomized algorithm. It effectively "averages" over all sequences of random numbers.
- De facto both analyses are very similar. However in practice the randomized algorithm ensures that not one single input elicits worst case behavior.