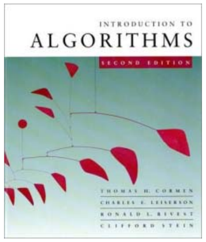
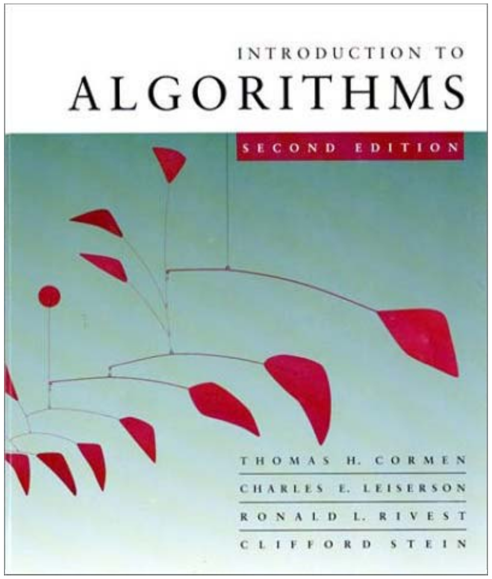


CS 3343 -- Fall 2010

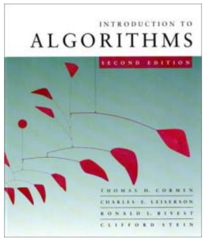


INTRODUCTION TO
ALGORITHMS
SECOND EDITION

THOMAS H. CORMEN
CHARLES E. LEISERSON
RONALD L. RIVEST
CLIFFORD STEIN

B-trees

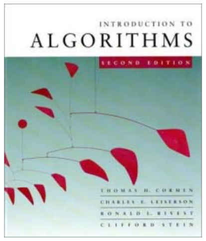
Carola Wenk



External memory dictionary

Task: Given a large amount of data that does not fit into main memory, process it into a dictionary data structure

- Need to minimize number of disk accesses
- With each disk read, read a whole block of data
- Construct a balanced search tree that uses one disk block per tree node
- Each node needs to contain more than one key

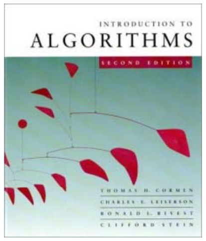


k-ary search trees

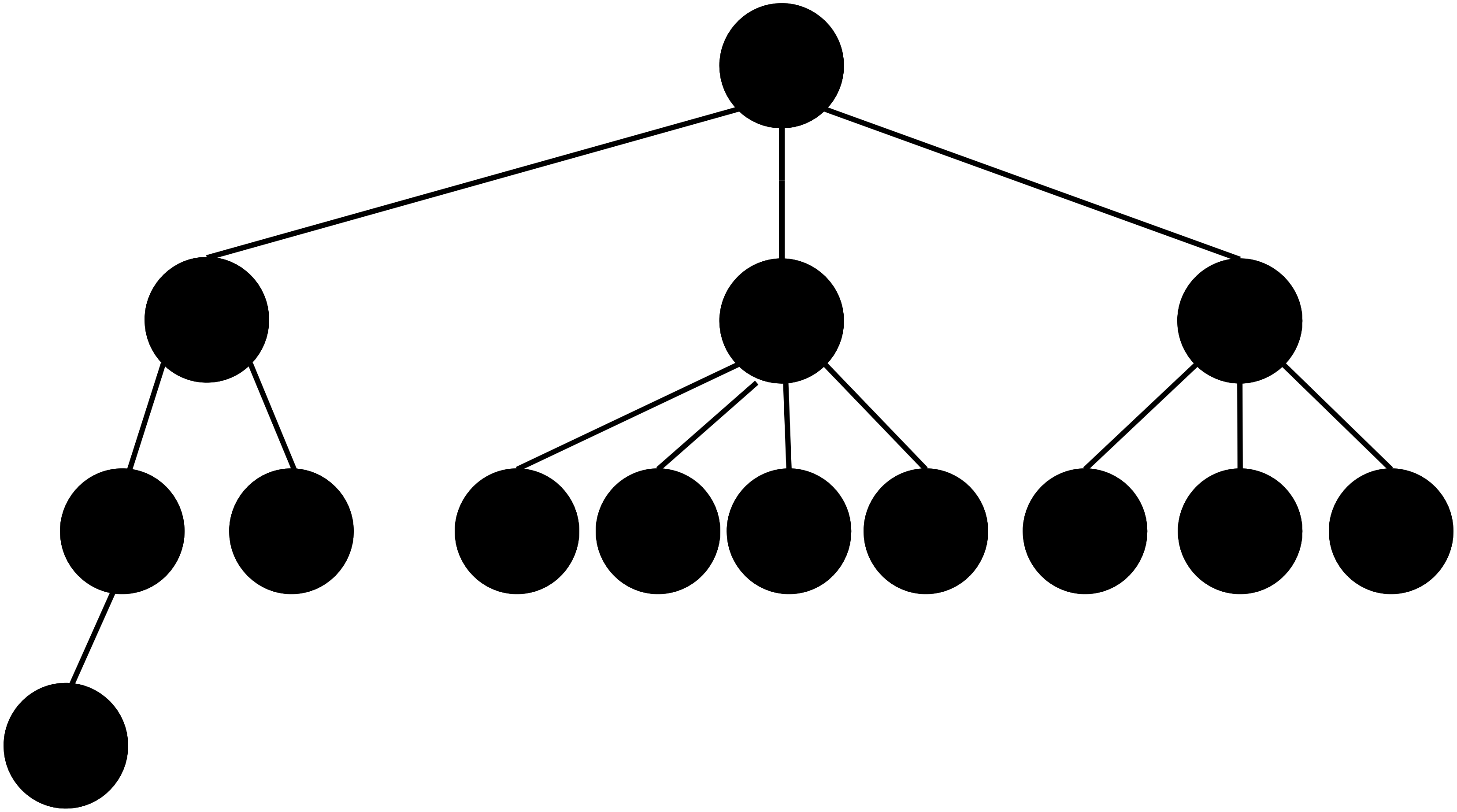
A ***k*-ary search tree** T is defined as follows:

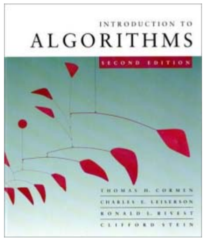
- For each node *x* of T:
  - *x* has at most *k* children (i.e., T is a *k*-ary tree)
  - *x* stores an ordered list of pointers to its children, and an ordered list of keys
  - For every internal node: #keys = #children-1
  - *x* fulfills the search tree property:

    keys in subtree rooted at *i*-th child $\leq$ *i*-th key $<$ keys in subtree rooted at $(i+1)$-st child

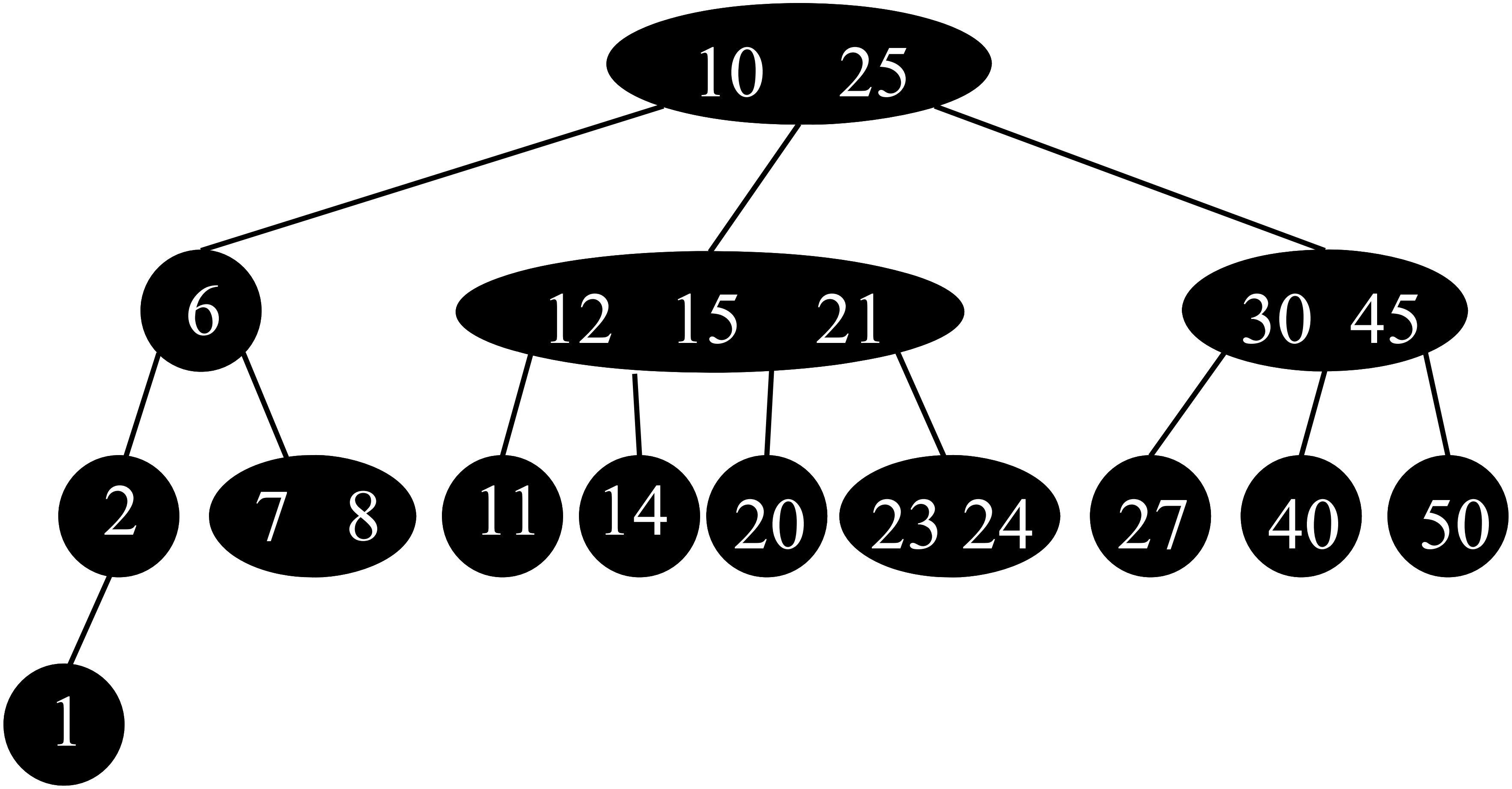


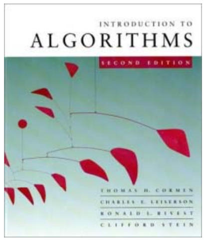
Example of a 4-ary tree





Example of a 4-ary search tree

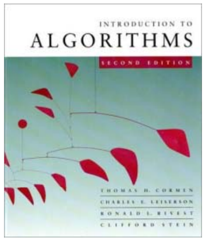




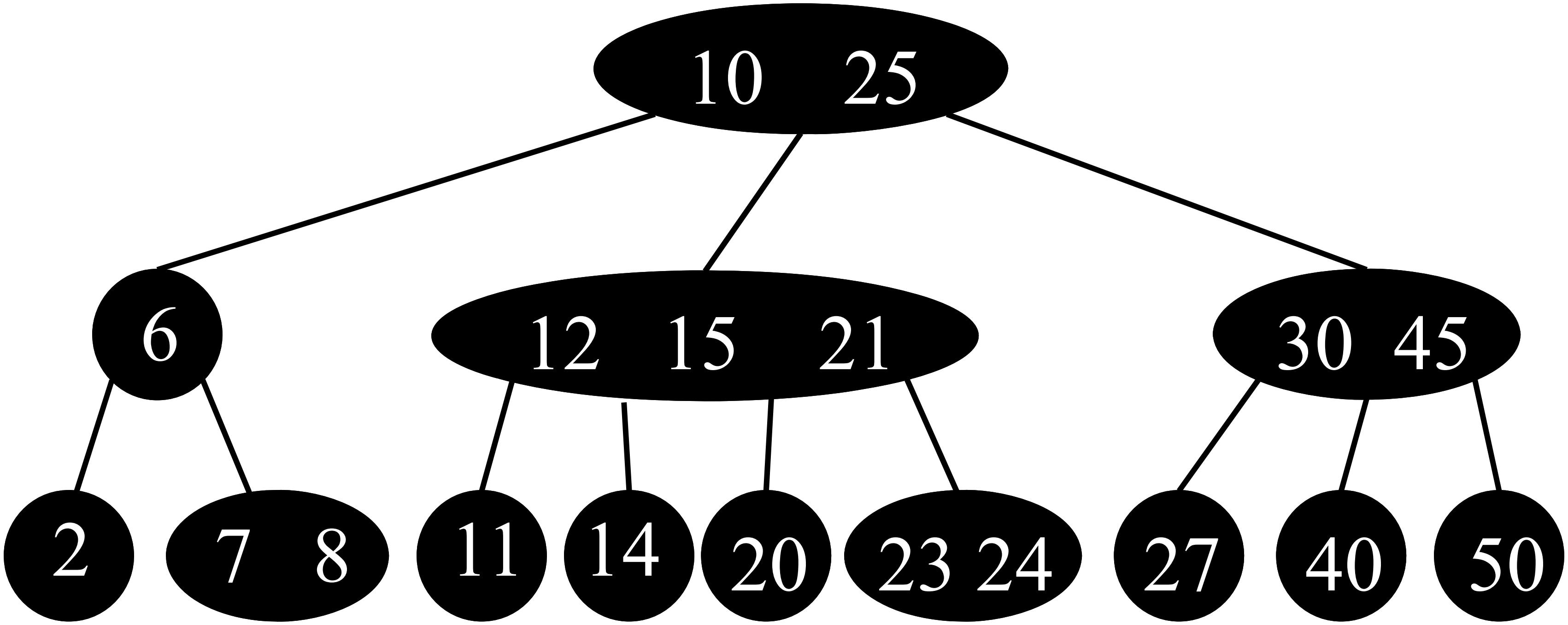
B-tree

A ***B*-tree** T with **minimum degree** $k \geq 2$ is defined as follows:

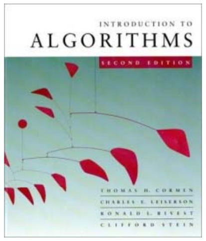
1. T is a $(2k)$-ary search tree
2. Every node, except the root, stores at least k-1 keys
   (every internal non-root node has at least k children)
3. The root must store at least one key
4. All leaves have the same depth



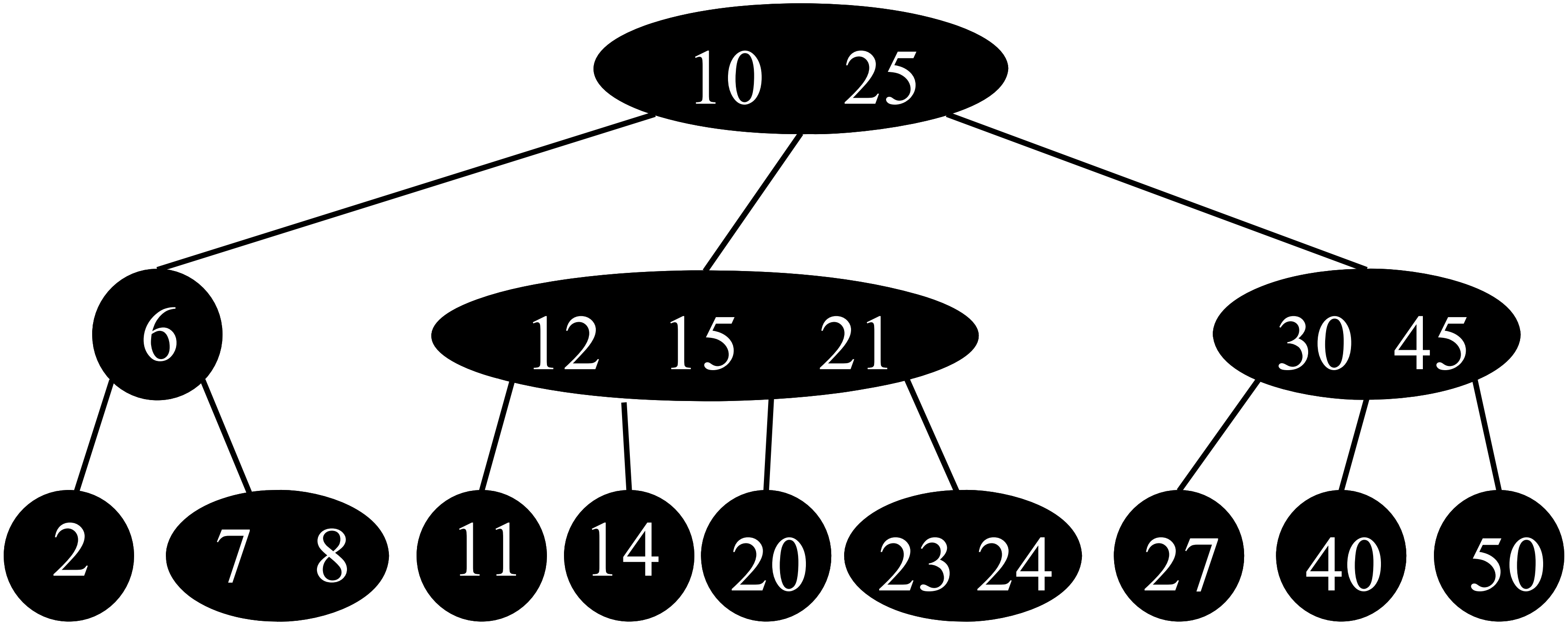
B-tree with k=2



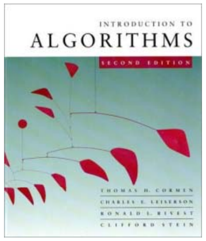
1. T is a $(2k)$-ary search tree



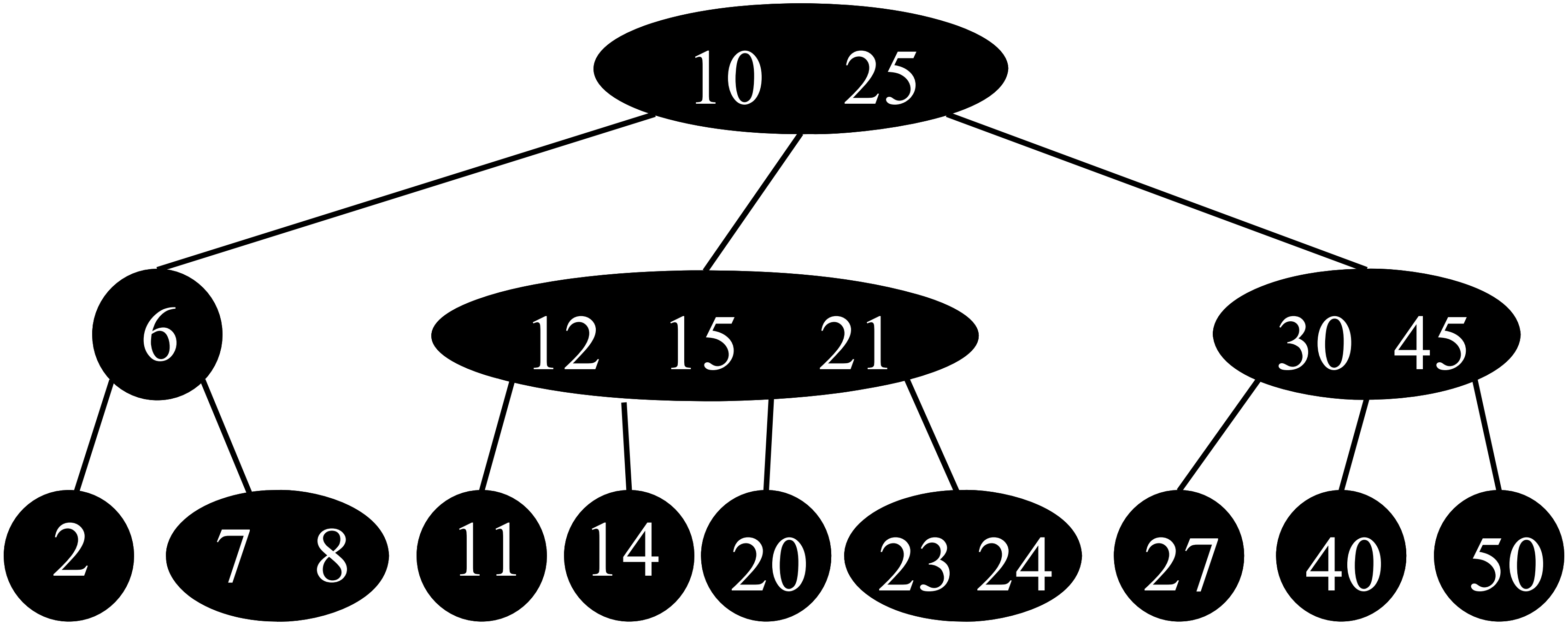
B-tree with k=2



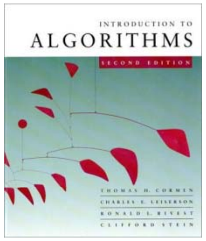
2. Every node, except the root, stores at least k-1 keys



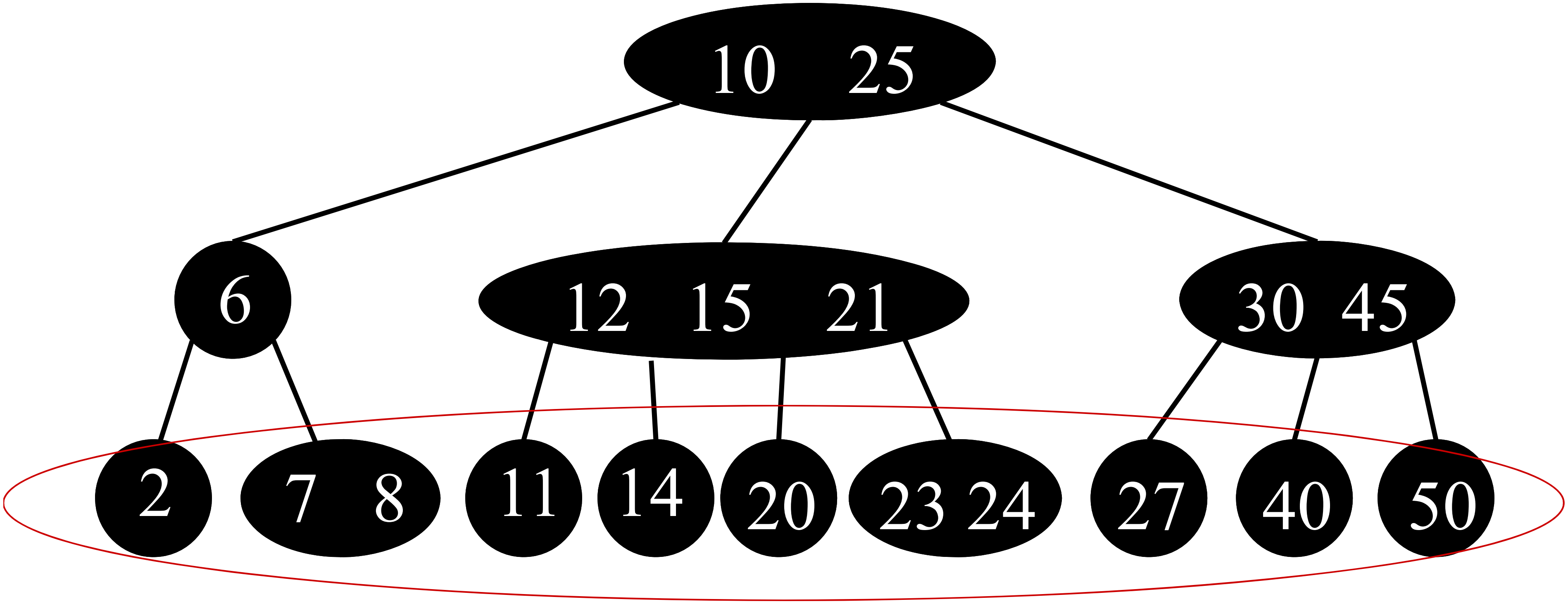
B-tree with *k*=2



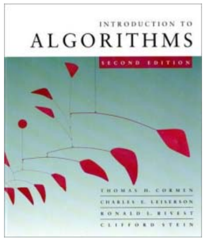
3. The root must store at least one key



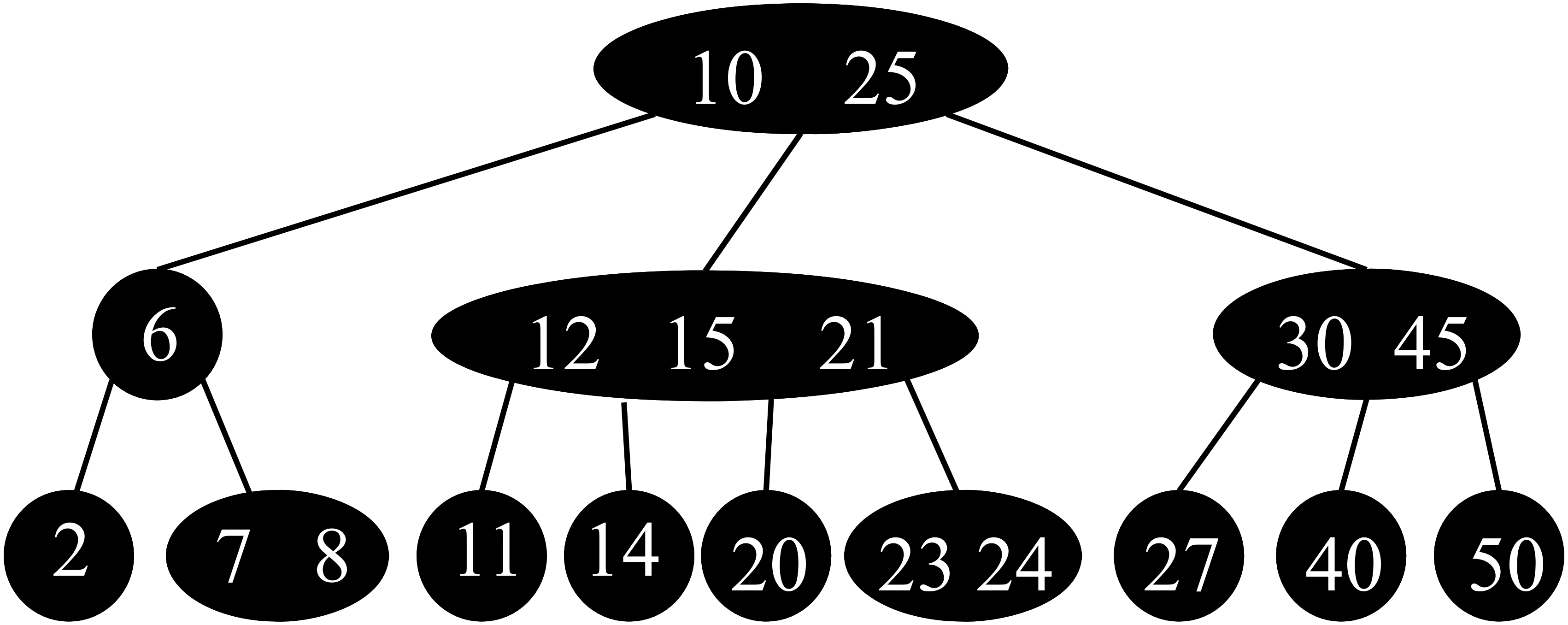
B-tree with *k*=2



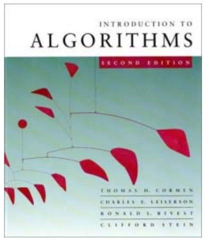
4. All leaves have the same depth



B-tree with *k*=2



Remark: This is a (2,3,4)-tree.



Height of a B-tree

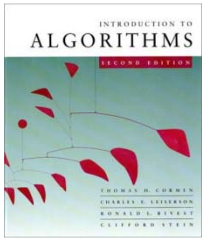
Theorem: For a B-tree with minimum degree $k \geq 2$ which stores n keys has height h holds:

$$h \leq \log_k (n+1)/2$$

Proof: #nodes $\geq \underbrace{1}_{\text{level 0}}+\underbrace{2}_{\text{level 1}}+\underbrace{2k}_{\text{level 2}}+\underbrace{2k^2}_{\text{level 3}}+\ldots+2k^{h-1}$

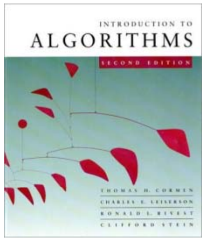
$$n = \#\text{keys} \geq 1+(k-1)\sum_{i=0}^{h-1} 2k^i = 1+2(k-1)\cdot\frac{k^h-1}{k-1} = 2k^h-1$$

$\square$



B-tree search

```
B-TREE-SEARCH(x,key)
    i ← 1
    while i≤#keys of x and key > i-th key of x
        do  i ← i+1
    if i≤#keys of x and key = i-th key of x
        then return (x,i)
    if x is a leaf
        then return NIL
    else b=DISK-READ(i-th child of x)
        return B-TREE-SEARCH(b,key)
```

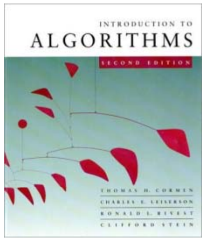


B-tree search runtime

- $O(k)$ per node
- Path has height $h = O(\log_k n)$
- CPU-time: $O(k \log_k n)$

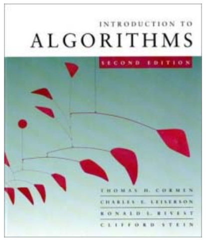
- Disk accesses: $O(\log_k n)$

  disk accesses are more expensive than CPU time



B-tree insert

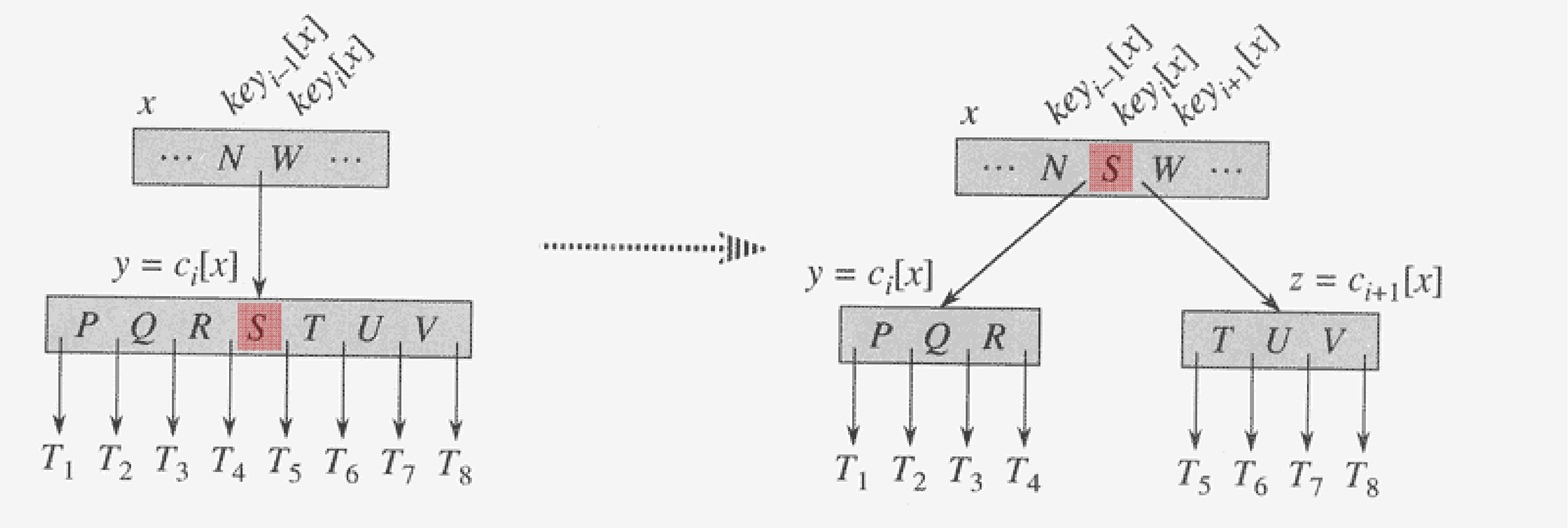
- There are different insertion strategies. We just cover one of them
- Make one pass down the tree:
  - The goal is to insert the new *key* into a leaf
  - Search where *key* should be inserted
  - **Only descend into non-full nodes:**
    - If a node is full, split it. Then continue descending.
    - **Splitting of the root node is the only way a B-tree grows in height**

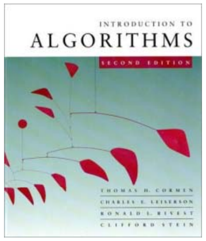


B-TREE-SPLIT-CHILD(x,i,y)

has $2k$-1 keys

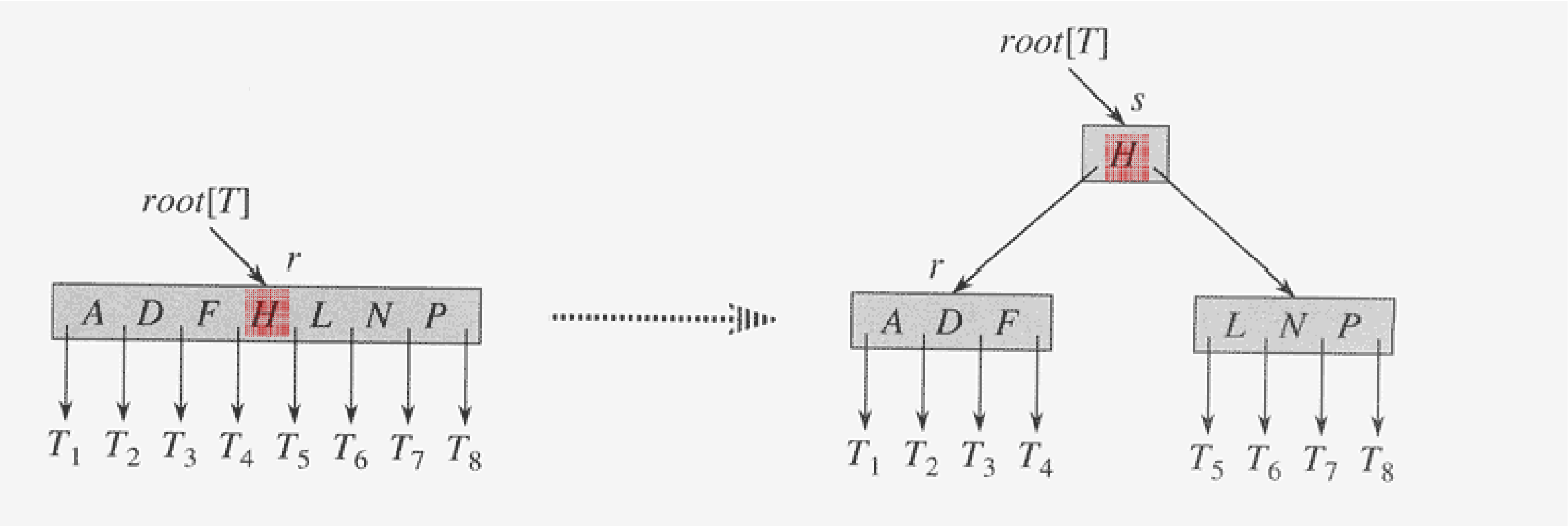
- Split full node y into two nodes y and z of k-1 keys
- Median key S of y is moved up into y's parent x
- Example below for $k = 4$

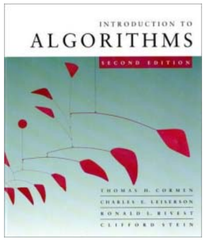




Split root: B-TREE-SPLIT-CHILD(s,1,r)

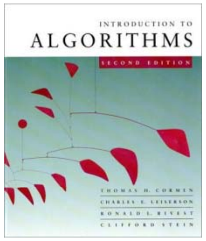
- The **full** root node r is split in two.
- A new root node s is created
- s contains the median key H of r and has the two halves of r as children
- Example below for $k = 4$





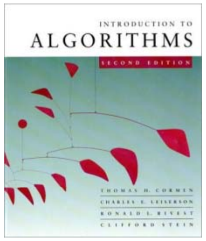
B-TREE-INSERT(*T*,*key*)

```
r = root[T]
if (# keys in r) = 2k-1 // root r is full
    //insert new root node:
    s ← ALLOCATE-NODE()
    root[T] ← s
    // split old root r to be two children of new root s
    B-TREE-SPLIT-CHILD(s,1,r)
    B-TREE-INSERT-NONFULL(s,key)
else B-TREE-INSERT-NONFULL(r,key)
```

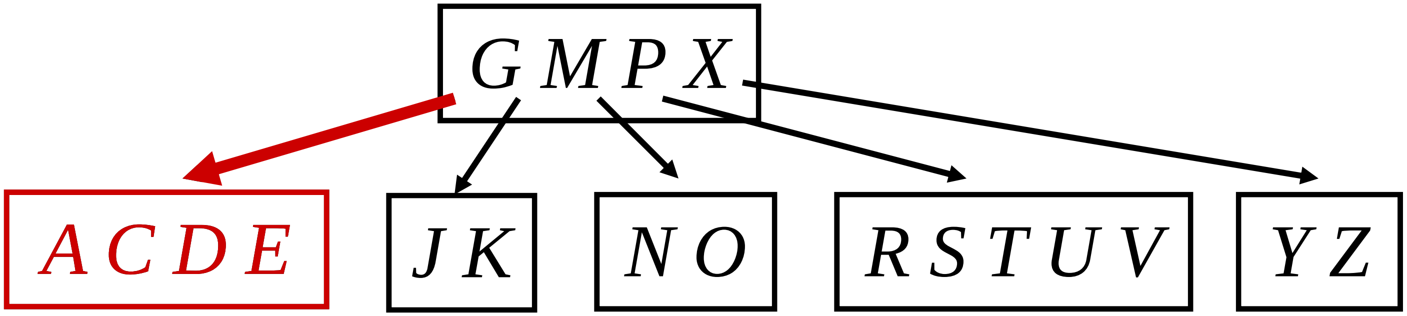


B-TREE-INSERT-NONFULL(x,key)

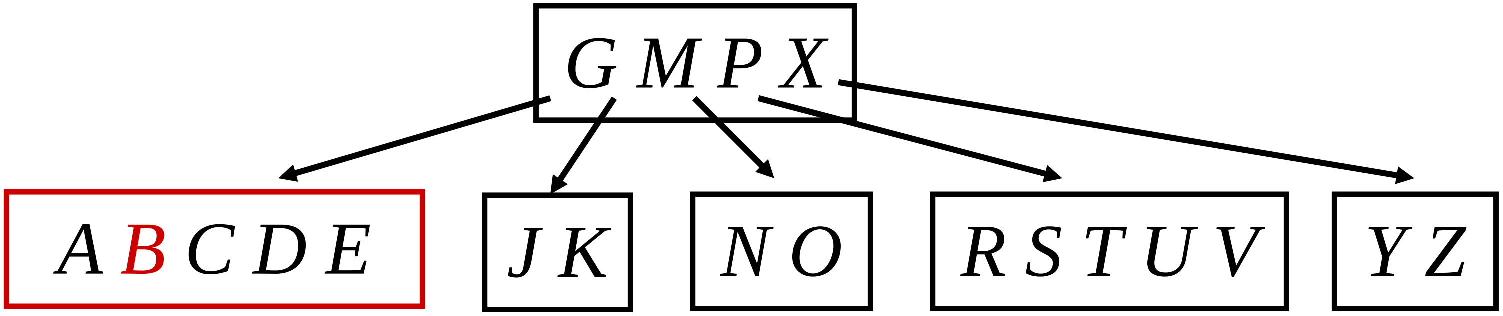
```
if x is a leaf then
    insert key at the correct (sorted) position in x
    DISK-WRITE(x)
else
    find child c of x which by the search tree property
      should contain key
    DISK-READ(c)
    if c is full then  // c contains 2k-1 keys
        B-TREE-SPLIT-CHILD(x,i,c)
        c=child of x which should contain key
    B-TREE-INSERT-NONFULL(c,key)
```

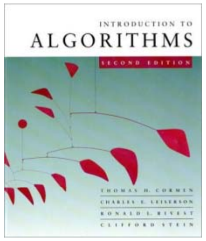


Insert example (*k*=3)

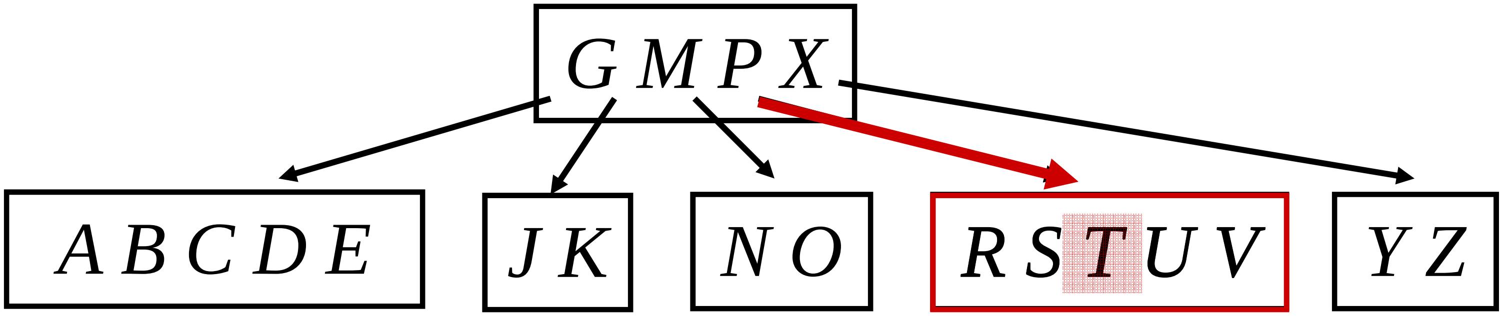


- Insert *B*:





Insert example (*k*=3) -- cont.



G M P X

A B C D E

J K

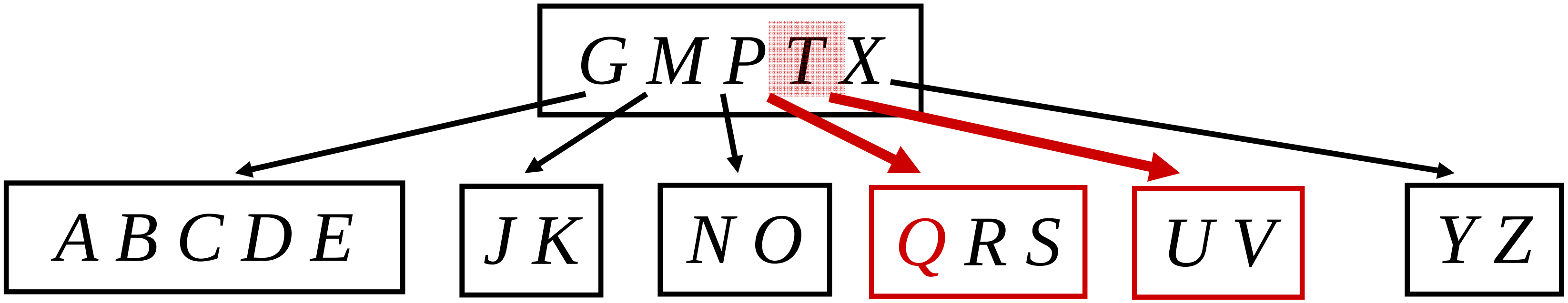
N O

R S T U V

Y Z

node is full

- Insert *Q*:



G M P T X

A B C D E

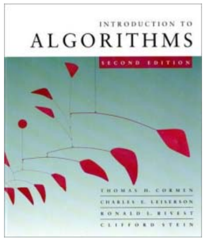
J K

N O

Q R S

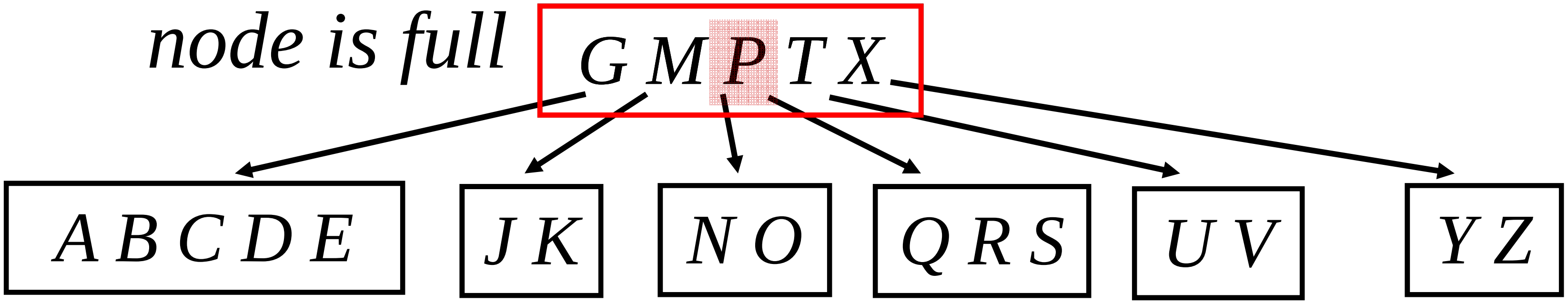
U V

Y Z



Insert example ($k=3$) -- cont.

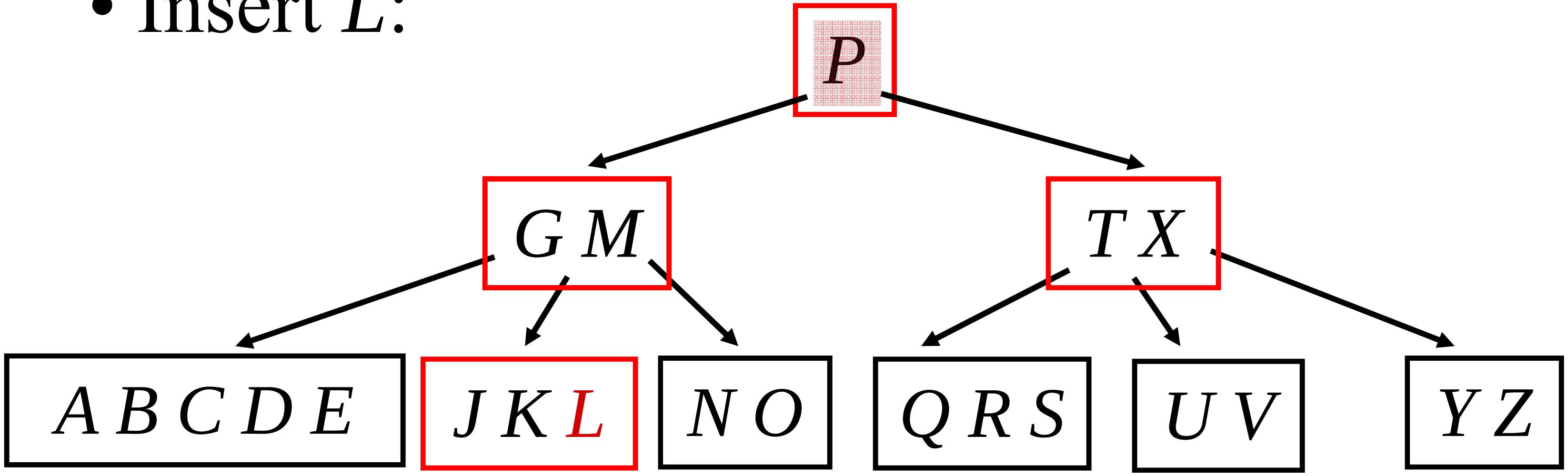
node is full



G M P T X

A B C D E *J K* *N O* *Q R S* *U V* *Y Z*

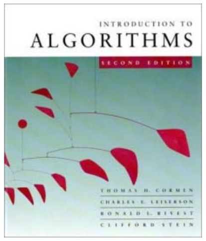
- Insert *L*:



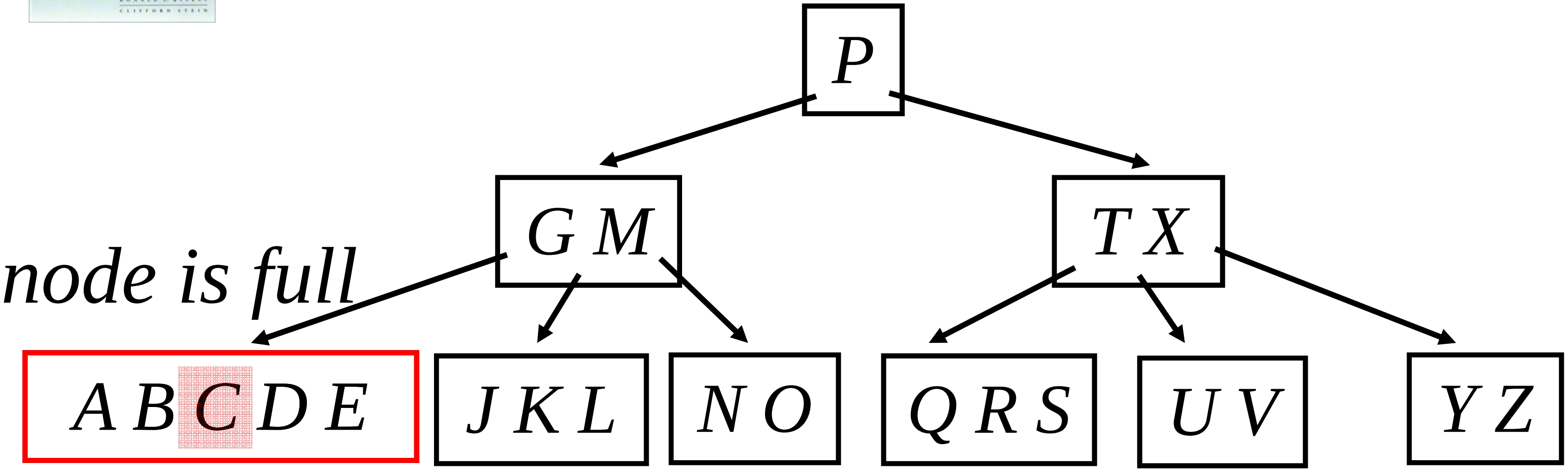
P

G M *T X*

A B C D E *J K L* *N O* *Q R S* *U V* *Y Z*



Insert example ($k=3$) -- cont.



P

G M

T X

node is full

A B C D E

J K L

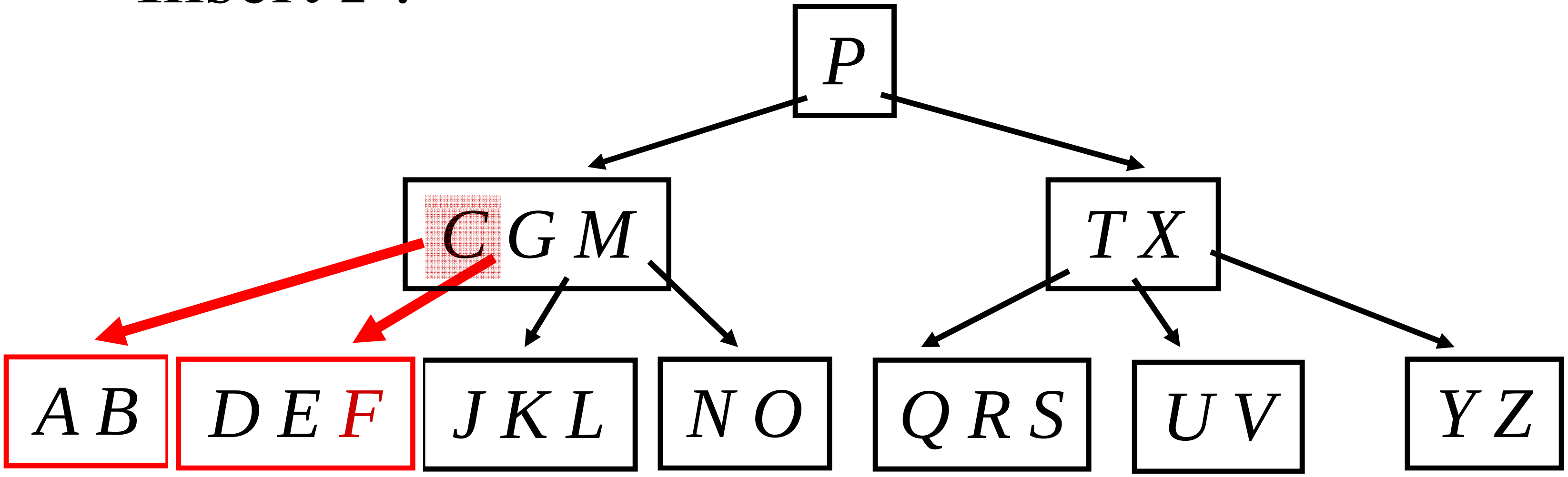
N O

Q R S

U V

Y Z

- Insert *F*:



P

C G M

T X

A B

D E F

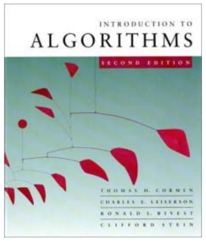
J K L

N O

Q R S

U V

Y Z

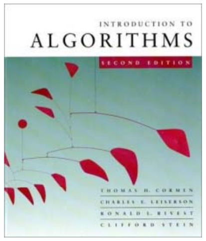


Runtime of B-TREE-INSERT

- $O(k)$ runtime per node
- Path has height $h = O(\log_k n)$
- CPU-time: $O(k \log_k n)$

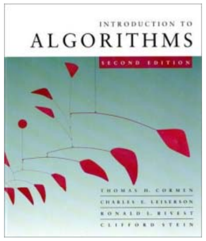
- Disk accesses: $O(\log_k n)$

  disk accesses are more expensive than CPU time



Deletion of an element

- Similar to insertion, but a bit more complicated; see book for details
- If sibling nodes get not full enough, they are **merged** into a single node
- Same complexity as insertion



B-trees -- Conclusion

- B-trees are balanced $2k$-ary search trees
- The **degree** of each node is **bounded from above and below** using the parameter k
- All leaves are at the same height
- **No rotations** are needed: During insertion (or deletion) the balance is maintained by node **splitting** (or node **merging**)
- The tree grows (shrinks) in height only by splitting (or merging) the root