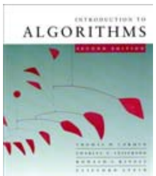
CS 3343 -- Fall 2007



Matrix Multiplication

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Slides courtesy of Charles Leiserson with small changes by Carola Wenk

Powering a number

Problem: Compute a^n, where $n \in \mathbf{N}$.

Naive algorithm: $\Theta(n)$.

Divide-and-conquer algorithm: **(recursive squaring)**

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

$$T(n) = T(n/2) + \Theta(1) \;\Rightarrow\; T(n) = \Theta(\log n)\,.$$

Matrix multiplication

Input: $A = [a_{ij}], B = [b_{ij}].$
Output: $C = [c_{ij}] = A \cdot B.$ $\Big\}\; i, j = 1, 2, \ldots, n.$

$$\begin{bmatrix} c_{11} & c_{12} & \Lambda & c_{1n} \\ c_{21} & c_{22} & \Lambda & c_{2n} \\ M & M & O & M \\ c_{n1} & c_{n2} & \Lambda & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \Lambda & a_{1n} \\ a_{21} & a_{22} & \Lambda & a_{2n} \\ M & M & O & M \\ a_{n1} & a_{n2} & \Lambda & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \Lambda & b_{1n} \\ b_{21} & b_{22} & \Lambda & b_{2n} \\ M & M & O & M \\ b_{n1} & b_{n2} & \Lambda & b_{nn} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Standard algorithm

for $i \leftarrow 1$ **to** n
  do for $j \leftarrow 1$ **to** n
    do $c_{ij} \leftarrow 0$
      for $k \leftarrow 1$ **to** n
        do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$

Running time = $\Theta(n^3)$

Divide-and-conquer algorithm

IDEA:
$n \times n$ matrix = 2×2 matrix of $(n/2) \times (n/2)$ submatrices:

$$\left[\begin{array}{c|c} r & s \\ \hline t & u \end{array}\right] = \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \cdot \left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right]$$

$$C \quad = \quad A \quad \cdot \quad B$$

$$\left.\begin{aligned} r &= a \cdot e + b \cdot g \\ s &= a \cdot f + b \cdot h \\ t &= c \cdot e + d \cdot h \\ u &= c \cdot f + d \cdot g \end{aligned}\right\}$$

8 recursive mults of $(n/2) \times (n/2)$ submatrices
4 adds of $(n/2) \times (n/2)$ submatrices

Analysis of D&C algorithm

$$T(n) = \boxed{8\, T(n/2)} + \boxed{\Theta(n^2)}$$

 # submatrices

submatrix size

work adding submatrices

$$n^{\log_b a} = n^{\log_2 8} = n^3 \;\Rightarrow\; \text{CASE 1} \;\Rightarrow\; T(n) = \Theta(n^3).$$

No better than the ordinary algorithm.

Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$
\begin{aligned}
P_1 &= a \cdot (f - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (g - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}
\qquad
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
s &= P_1 + P_2 \\
t &= P_3 + P_4 \\
u &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

7 mults, 18 adds/subs.
Note: No reliance on commutativity of mult!

Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$
\begin{aligned}
P_1 &= a \cdot (f - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (g - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}
\qquad
\begin{aligned}
s &= P_1 + P_2 \\
&= a \cdot (f - h) + (a + b) \cdot h \\
&= af - ah + ah + bh \\
&= af + bh
\end{aligned}
$$

Strassen's algorithm

1. ***Divide:*** Partition A and B into $(n/2)\times(n/2)$ submatrices. Form P-terms to be multiplied using $+$ and $-$.
2. ***Conquer:*** Perform 7 multiplications of $(n/2)\times(n/2)$ submatrices recursively.
3. ***Combine:*** Form C using $+$ and $-$ on $(n/2)\times(n/2)$ submatrices.

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

Analysis of Strassen

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

$$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \;\Rightarrow\; \text{CASE 1} \;\Rightarrow\; T(n) = \Theta(n^{\log 7}).$$

The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 30$ or so.

Best to date (of theoretical interest only): $\Theta(n^{2.376\ldots})$.