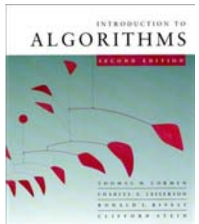
CS 3343 – Fall 2007



P and NP

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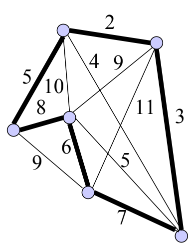
Slides courtesy of Piotr Indyk with small changes by Carola Wenk

Have seen so far

- Algorithms for various problems
  - Running times $O(nm^2)$, $O(n^2)$, $O(n \log n)$, $O(n)$, etc.
  - I.e., polynomial in the input size
- Can we solve all (or most of) interesting problems in polynomial time ?
- Not really…

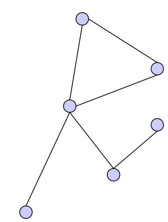
Example difficult problem

- Traveling Salesperson Problem (TSP)
  - Input: undirected graph with lengths on edges
  - Output: shortest tour that visits each vertex exactly once
- Best known algorithm: $O(n\,2^n)$ time.



Another difficult problem

- Clique:
  - Input: undirected graph $G=(V,E)$
  - Output: largest subset C of V such that every pair of vertices in C has an edge between them
- Best known algorithm: $O(n\,2^n)$ time



What can we do ?

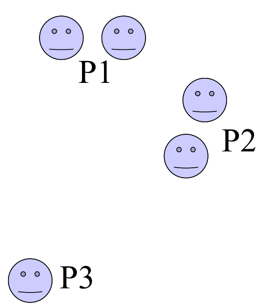
- Spend more time designing algorithms for those problems
  - People tried for a few decades, no luck
- Prove there is no polynomial time algorithm for those problems
  - Would be great
  - Seems *really* difficult
  - Best lower bounds for “natural” problems:
    - $\Omega(n^2)$ for restricted computational models
    - $4.5n$ for unrestricted computational models

What else can we do ?

- Show that those hard problems are essentially equivalent. I.e., if we can solve one of them in polynomial time, then all others can be solved in polynomial time as well.
- Works for at least 10 000 hard problems

The benefits of equivalence

- Combines research efforts
- If one problem has polynomial time solution, then all of them do
- More realistically: Once an exponential lower bound is shown for one problem, it holds for all of them



Summing up

- If we show that a problem $\prod$ is equivalent to ten thousand other well studied problems without efficient algorithms, then we get a very strong evidence that $\prod$ is hard.
- We need to:
  - Identify the class of problems of interest
  - Define the notion of equivalence
  - Prove the equivalence(s)

Class of problems: NP

- Decision problems: answer YES or NO. E.g.,"is there a tour of length $\leq K$" ?
- Solvable in *non-deterministic polynomial* time:
  - Intuitively: the solution can be verified in polynomial time
  - E.g., if someone gives us a tour T, we can verify in *polynomial* time if T is a tour of length $\leq K$.
- Therefore, TSP is in NP.

Formal definitions of P and NP

- A decision problem $\prod$ is solvable in polynomial time (or $\prod \in P$), if there is a polynomial time algorithm $A(.)$ such that for any input x:

  $\prod(x)$=YES iff $A(x)$=YES

- A decision problem $\prod$ is solvable in non-deterministic polynomial time (or $\prod \in NP$), if there is a polynomial time algorithm $A(. , .)$ such that for any input x:

  $\prod(x)$=YES iff there exists a certificate y of size poly($|x|$) such that $A(x,y)$=YES

Examples of problems in NP

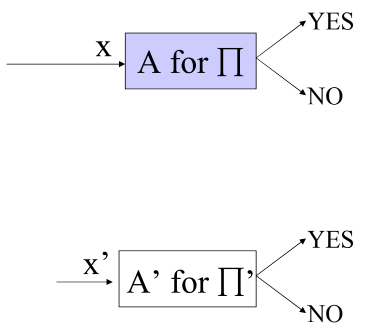
- Is "Does there exist a clique in G of size $\geq K$" in NP ?

  Yes: $A(x,y)$ interprets x as a graph G, y as a set C, and checks if all vertices in C are adjacent and if $|C| \geq K$
- Is Sorting in NP ?

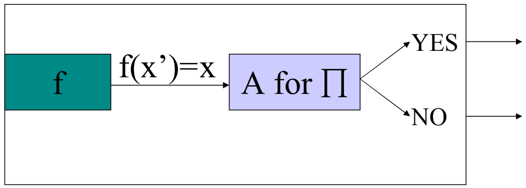
  No, not a decision problem.
- Is "Sortedness" in NP ?

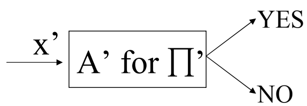
  Yes: ignore y, and check if the input x is sorted.

Reductions: ∏' to ∏

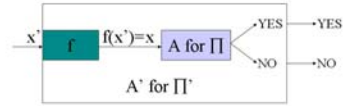


Reductions: ∏’ to ∏





Reductions



- ∏’ is *polynomial time reducible* to ∏ (∏’ ≤ ∏) iff there is a polynomial time function f that maps inputs x’ for ∏’ into inputs x for ∏, such that for any x’

  ∏’(x’)=∏(f(x’))

- Fact 1: if ∏∈P and ∏’ ≤ ∏ then ∏’∈P
- Fact 2: if ∏∈NP and ∏’ ≤ ∏ then ∏’∈NP
- Fact 3 (transitivity):

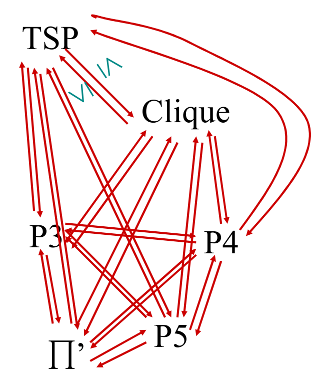
  if ∏’’ ≤ ∏’ and ∏’ ≤ ∏ then ∏” ≤ ∏

Recap

- We defined a large class of interesting problems, namely NP
- We have a way of saying that one problem is not harder than another (∏’ ≤ ∏)
- Our goal: show equivalence between hard problems

Showing equivalence between difficult problems

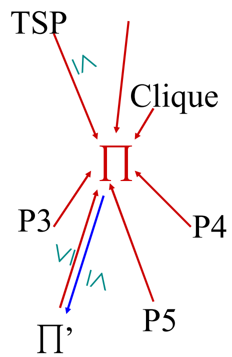
- Options:
  - Show reductions between all pairs of problems
  - Reduce the number of reductions using transitivity of “≤”



Showing equivalence between difficult problems

- Options:
  - Show reductions between all pairs of problems
  - Reduce the number of reductions using transitivity of "$\le$"
  - Show that *all* problems in NP are reducible to a *fixed* $\prod$.

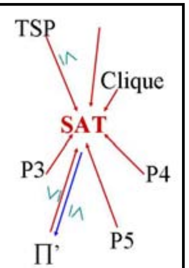
  To show that some problem $\prod' \in NP$ is equivalent to all difficult problems, we only show $\prod \le \prod'$.



The first problem $\prod$

- Satisfiability problem (SAT):
  - Given: a formula φ with m clauses over n variables, e.g., $x_1 \vee x_2 \vee x_5$, $x_3 \vee \neg x_5$
  - Check if there exists TRUE/FALSE assignments to the variables that makes the formula satisfiable

SAT is NP-complete



- Fact: SAT $\in$NP
- Theorem [Cook'71]: For any $\prod' \in NP$ we have $\prod' \le SAT$.
- Definition: A problem $\prod$ such that for any $\prod' \in NP$ we have $\prod' \le \prod$, is called *NP-hard*
- Definition: An NP-hard problem that belongs to NP is called *NP-complete*
- Corollary: SAT is NP-complete.

Plan of attack:

SAT
↓
Clique
↓
Independent set
↓
Vertex cover

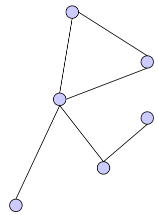
(thanks, Steve ☺)

Follow from Cook's Theorem

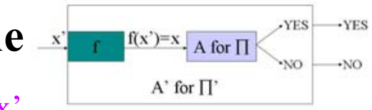
Conclusion: all of the above problems are NP-complete

Clique again

- Clique (decision variant):
  - Input: undirected graph $G=(V,E)$, K
  - Output: is there a subset C of V, $|C|\geq K$, such that every pair of vertices in C has an edge between them



SAT ≤ Clique



- Given a $\overbrace{\text{SAT formula } \varphi=C_1,\ldots,C_m \text{ over}}^{x'}$ $x_1,\ldots,x_n$, we need to produce $\underbrace{G=(V,E) \text{ and } K,}_{f(x')=x}$

  such that φ satisfiable iff G has a clique of size $\geq K$.
- Notation: a literal is either x_i or $\neg x_i$

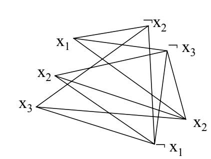
SAT ≤ Clique reduction

- For each literal t occurring in φ, create a vertex v_t
- Create an edge $v_t - v_{t'}$ iff:
  - t and t' are not in the same clause, and
  - t is not the negation of t'

SAT ≤ Clique example

Edge $v_t - v_{t'}$ ⇔	• t and t' are not in the same clause, and • t is not the negation of t'

- Formula: $x_1 \vee x_2 \vee x_3 ,\ \neg x_2 \vee \neg x_3,\ \neg x_1 \vee x_2$
- Graph:



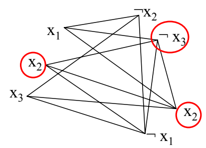
- Claim: φ satisfiable iff G has a clique of size $\geq m$

Proof

Edge $v_t - v_{t'}$ $\Leftrightarrow$
- t and t' are not in the same clause, and
- t is not the negation of t'

- "→" part:
  - Take any assignment that satisfies φ.

    E.g., x_1=F, x_2=T, x_3=F
  - Let the set C contain one satisfied literal per clause
  - C is a clique

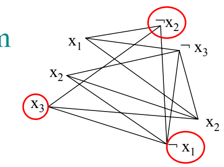


Proof

Edge $v_t - v_{t'}$ $\Leftrightarrow$
- t and t' are not in the same clause, and
- t is not the negation of t'

- "←" part:
  - Take any clique C of size $\geq$ m (i.e., = m)
  - Create a set of equations that satisfies selected literals.

    E.g., x_3=T, x_2=F, x_1=F
  - The set of equations is consistent and the solution satisfies φ

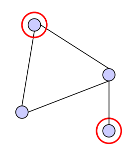


Altogether

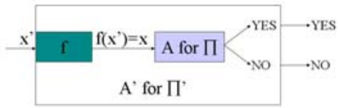
- We constructed a reduction that maps:
  - YES inputs to SAT to YES inputs to Clique
  - NO inputs to SAT to NO inputs to Clique
- The reduction works in polynomial time
- Therefore, SAT $\leq$ Clique →Clique NP-hard
- Clique is in NP → Clique is NP-complete

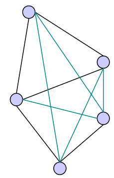
Independent set (IS)

- Input: undirected graph G=(V,E)
- Output: is there a subset S of V, |S|≥K such that no pair of vertices in S has an edge between them



Clique ≤ IS

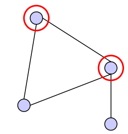




- Given an input $\overbrace{G=(V,E), K}^{x'}$ to Clique, need to construct an input $\underbrace{G'=(V',E'), K'}_{f(x')=x}$ to IS,

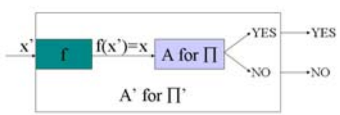
  such that G has clique of size ≥K iff G' has IS of size ≥K'.
- Construction: $K'=K, V'=V, E'=\overline{E}$
- Reason: C is a clique in G iff it is an IS in G's complement.

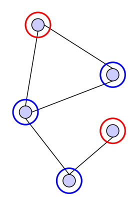
Vertex cover (VC)



- Input: undirected graph G=(V,E)
- Output: is there a subset C of V, $|C| \leq K$, such that each edge in E is incident to at least one vertex in C.

IS ≤ VC





- Given an input $\overbrace{G=(V,E), K}^{x'}$ to IS, need to construct an input $\underbrace{G'=(V',E'), K'}_{f(x')=x}$ to VC, such that

  G has an IS of size ≥K iff G' has VC of size ≤K'.
- Construction: V'=V, E'=E, K'=|V|-K
- Reason: S is an IS in G iff V-S is a VC in G.