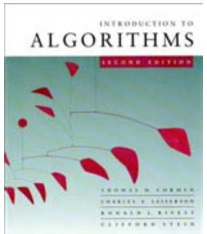
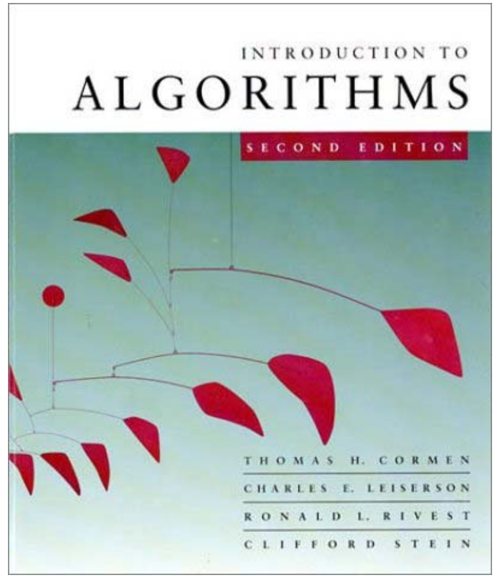


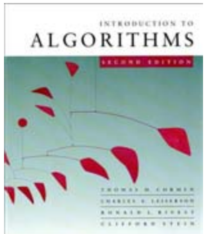
CS 3343 – Fall 2007



Graphs

Carola Wenk

Slides courtesy of Charles Leiserson with changes and additions by Carola Wenk



Graphs

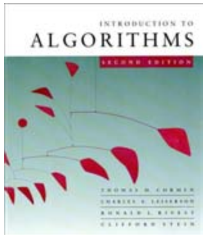
Definition. A ***directed graph (digraph)*** $G = (V, E)$ is an ordered pair consisting of

- a set V of ***vertices*** (singular: ***vertex***),
- a set $E \subseteq V \times V$ of ***edges***.

In an ***undirected graph*** $G = (V, E)$, the edge set E consists of *unordered* pairs of vertices.

In either case, we have $|E| = O(|V|^2)$.

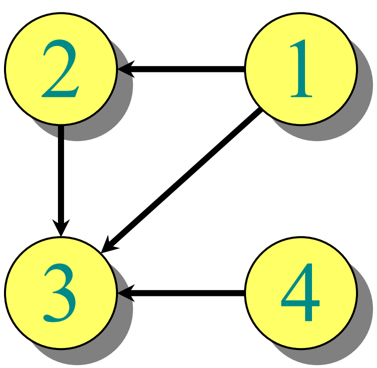
(Review CLRS, Appendix B.4 and B.5.)



Adjacency-matrix representation

The ***adjacency matrix*** of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1 \,.\,.\, n, 1 \,.\,.\, n]$ given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E. \end{cases}$$

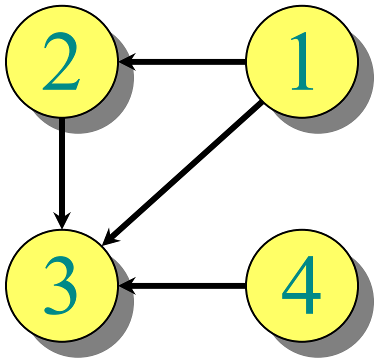


A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

$\Theta(|V|^2)$ storage $\Rightarrow$ ***dense*** representation.

Adjacency-list representation

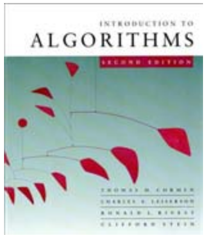
An ***adjacency list*** of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to v.



$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3\}$

For undirected graphs, $|Adj[v]| = degree(v)$.

For digraphs, $|Adj[v]| = out\text{-}degree(v)$.



Adjacency-list representation

Handshaking Lemma:

Every edge is counted twice

- For undirected graphs:

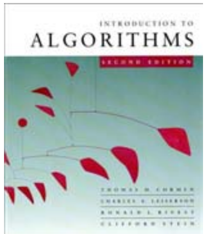
  $\sum_{v \in V} degree(v) = 2|E|$

- For digraphs:

  $\sum_{v \in V} in\text{-}degree(v) + \sum_{v \in V} out\text{-}degree(v) = 2|E|$

⇒ adjacency lists use $\Theta(|V| + |E|)$ storage

⇒ a ***sparse*** representation



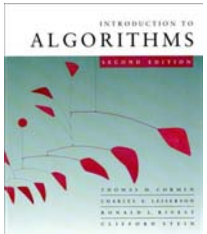
Graph Traversal

Let $G=(V,E)$ be a (directed or undirected) graph, given in adjacency list representation.

$|V| = n$, $|E| = m$

A graph traversal visits every vertex:

- Breadth-first search (BFS)
- Depth-first search (DFS)



Depth-First Search (DFS)

$O(n)$

$O(n)$ without DFS_rec

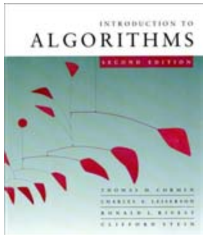
DFS($G=(V,E)$)
  Mark all vertices in G as "unvisited" // time=0
  for each vertex $v \in V$ **do**
    if v is unvisited
      DFS_rec(G,v)

$O(1)$

$O(deg(v))$ without recursive call

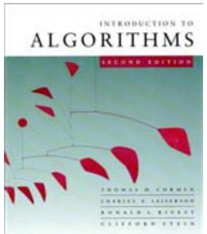
DFS_rec(G, v)
  visit v // time++
  for each w adjacent to v **do**
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)

$\Rightarrow$ With Handshaking Lemma, all recursive calls are O(m), for a total of $O(n + m)$ runtime



DFS runtime

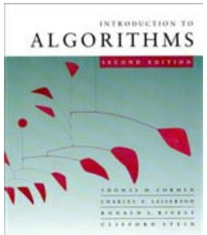
- Each vertex is visited at most once $\Rightarrow$ O(n) time
- The body of the **for** loops (except the recursive call) take constant time per graph edge
- All **for** loops take O(m) time
- Total runtime is O(n+m) = O(|V| + |E|)



Breadth-First Search (BFS)

```
BFS(G=(V,E))
    Mark all vertices in G as "unvisited" // time=0          O(n)
    Initialize empty queue Q                                  O(1)
    for each vertex v ∈ V do
        if v is unvisited                      } O(n) without BFS_iter
            visit v // time++
            Q.enqueue(v)
            BFS_iter(G)
```

```
BFS_iter(G)
    while Q is non-empty do                    } O(m)
        v = Q.dequeue()
        for each w adjacent to v do
            if w is unvisited                  } O(deg(v))
                visit w   // time++
                Add edge (v,w) to T
                Q.enqueue(w)
```



BFS runtime

- Each vertex is marked as unvisited in the beginning $\Rightarrow$ O(n) time
- Each vertex is marked at most once, enqueued at most once, and therefore dequeued at most once
- The time to process a vertex is proportional to the size of its adjacency list (its degree), since the graph is given in adjacency list representation
$\Rightarrow$ O(m) time
- Total runtime is O(n+m) = O(|V| + |E|)