The master method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\ ,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

Slides courtesy of Charles Leiserson with small changes by Carola Wenk

Master theorem

$T(n) = a\,T(n/b) + f(n)$

CASE 1: $f(n) \in O(n^{\log_b a - \varepsilon})$
$\Rightarrow T(n) \in \Theta(n^{\log_b a})$.

CASE 2: $f(n) \in \Theta(n^{\log_b a})$
$\Rightarrow T(n) \in \Theta(n^{\log_b a} \log n)$.

CASE 3: $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ and $a f(n/b) \leq c f(n)$ for some $c<1$
$\Rightarrow T(n) \in \Theta(f(n))$.

Merge sort: $a = 2, b = 2 \Rightarrow n^{\log_b a} = n$
$\Rightarrow$ CASE 2 $\Rightarrow T(n) \in \Theta(n \log n)$.

Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) \in O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) \in \Theta(n^{\log_b a})$.
2. $f(n) \in \Theta(n^{\log_b a})$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) \in \Theta(n^{\log_b a} \log n)$.

Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

   and $f(n)$ satisfies the ***regularity condition*** that $a f(n/b) \leq c f(n)$ for some constant $c < 1$.

   Solution: $T(n) \in \Theta(f(n))$.

Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n$.

Case 1: $f(n) \in O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) \in \Theta(n^2)$.

Ex. $T(n) = 4T(n/2) + n^2$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^2$.

Case 2: $f(n) \in \Theta(n^2)$.

$\therefore\ T(n) \in \Theta(n^2 \log n)$.

Examples

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^3$.

Case 3: $f(n) \in \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) \in \Theta(n^3)$.

Ex. $T(n) = 4T(n/2) + n^2/\log n$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^2/\log n$.

Master method does not apply.