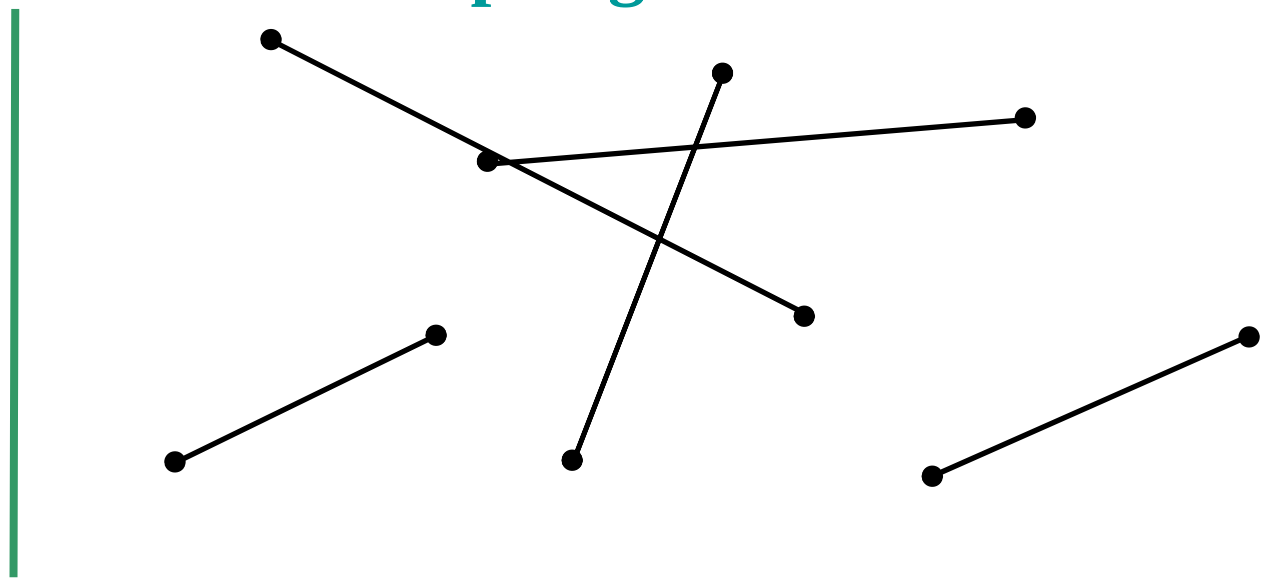
CMPS 6640/4040 Computational Geometry
Spring 2016



Plane Sweep Algorithms

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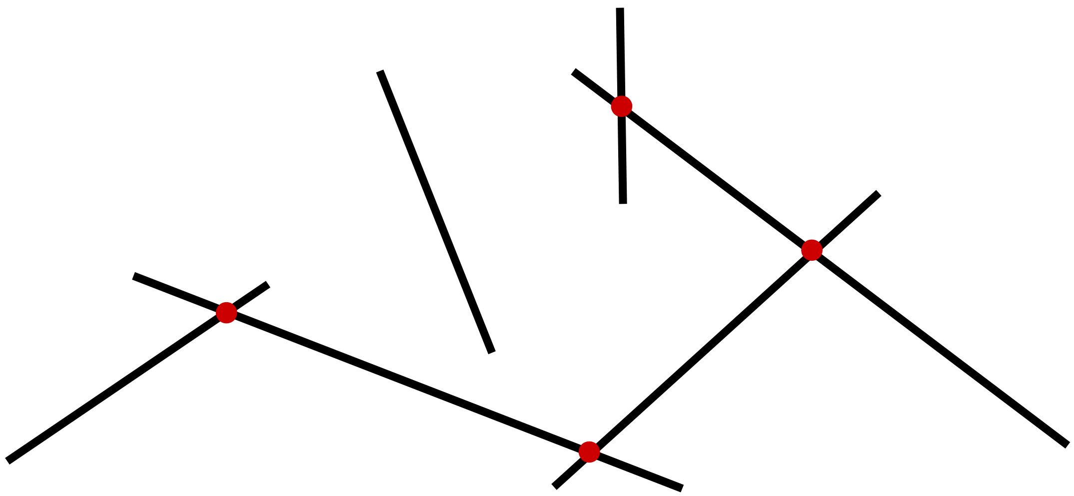
Line Segment Intersection

- Input: A set $S=\{s_1, \ldots, s_n\}$ of (closed) line segments in $\mathbf{R}^2$
- Output: All **intersection points** between segments in S



General position

Assume that "nasty" special cases don't happen:

- No line segment is vertical
- Two segments intersect in at most one point
- No three segments intersect in a common point

Line Segment Intersection

- n line segments can intersect as few as 0 and as many as $\binom{n}{2} = O(n^2)$ times
- Simple algorithm: Try out all pairs of line segments
  → Takes $O(n^2)$ time
  → Is optimal in worst case
- Challenge: Develop an **output-sensitive algorithm**
  - Runtime depends on size k of the output
  - Here: $0 \leq k \leq (n^2+n)/2$
  - Our algorithm will have runtime: $O(\,(n+k) \log n)$
  - Best possible runtime: $O(\,n \log n + k)$
    → $O(n^2)$ in worst case, but better in general

Plane Sweep: An Algorithm Design Technique

- Simulate sweeping a vertical line from left to right across the plane.
- Maintain **cleanliness property**: At any point in time, to the left of sweep line everything is clean, i.e., properly processed.
- **Sweep line status**: Store information along sweep line
- **Events**: Discrete points in time when sweep line status needs to be updated

Algorithm Generic_Plane_Sweep:

Initialize **sweep line status** S at time $x=-\infty$

Store initial events in **event queue** Q, a priority queue ordered by x-coordinate

while $Q \neq \emptyset$

    // extract next event e:
    e = Q.extractMin();
    // handle event:
    Update sweep line status
    Discover new upcoming events and insert them into Q

Plane sweep algorithm

Algorithm Generic_Plane_Sweep:

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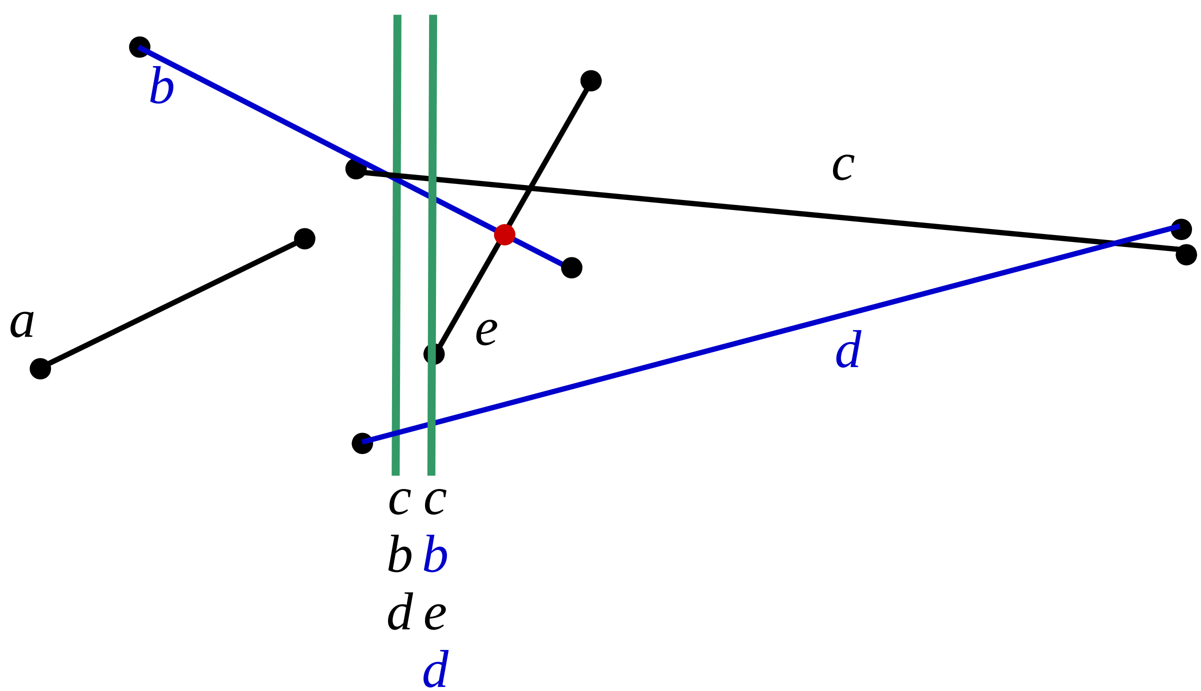
- **Cleanliness property:**
  - All intersections to the left of sweep line l have been reported
- **Sweep line status:**
  - Store segments that intersect the sweep line l, ordered along the intersection with l .
- **Events:**
  - Points in time when sweep line status changes combinatorially (i.e., the order of segments intersecting l changes)

  → Endpoints of segments (insert in beginning)

  → Intersection points (compute on the fly during plane sweep)

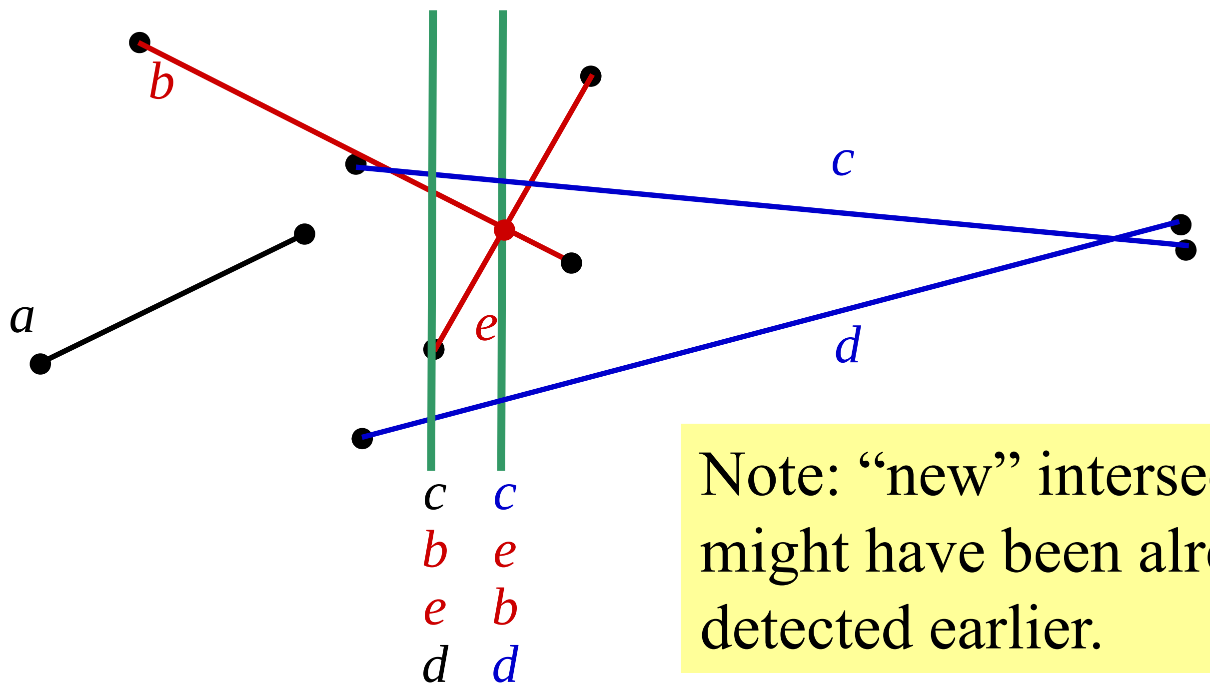
Event Handling

1. Left segment endpoint
   – Add segment to sweep line status
   – Test **adjacent segments** on sweep line *l* for intersection with new segment (see Lemma)
   – Add **new intersection points** to event queue



Event Handling

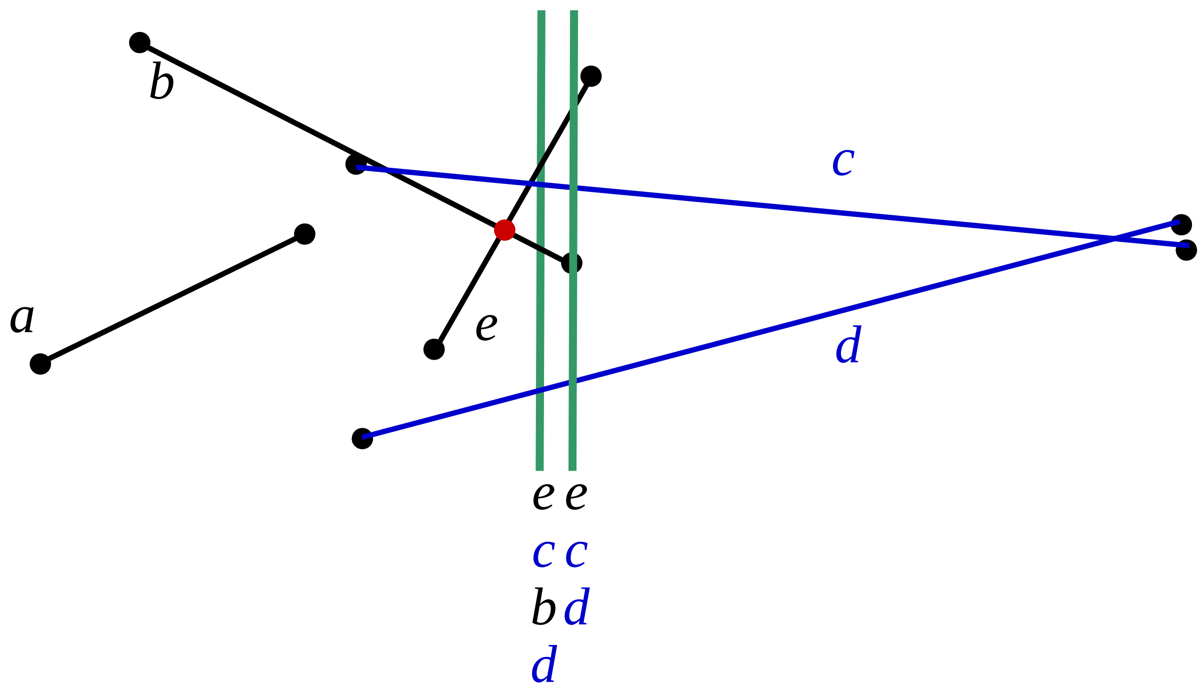
2. Intersection point
   - Report new intersection point
   - Two segments **change order** along l
     → Test new adjacent segments for new intersection points (to insert into event queue)



Note: “new” intersection might have been already detected earlier.

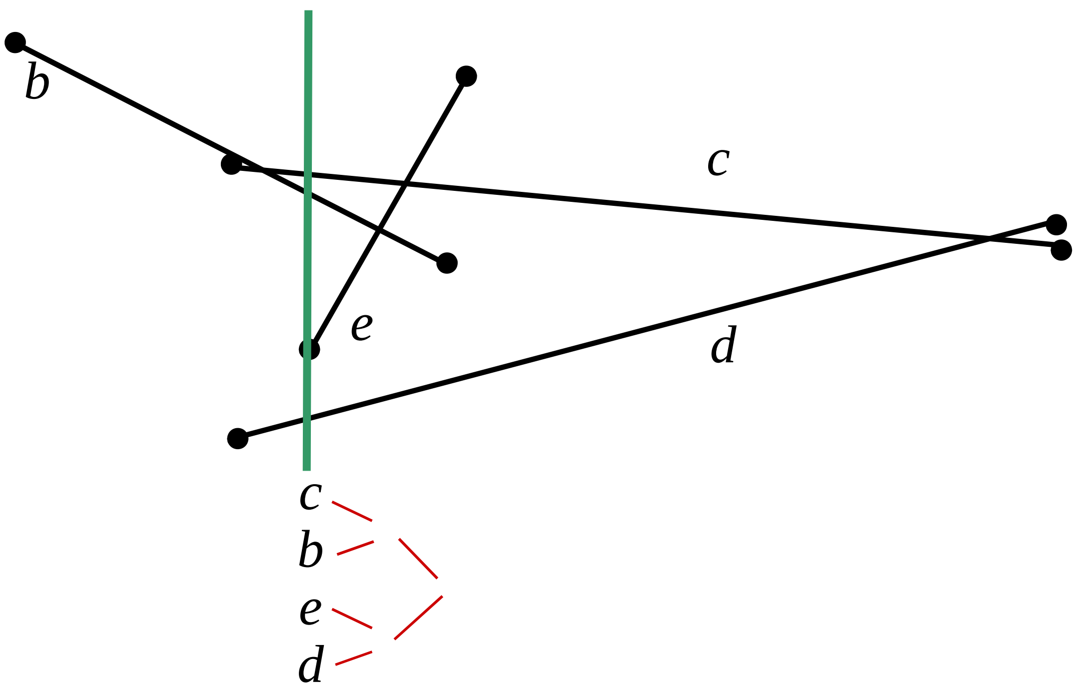
Event Handling

3. Right segment endpoint
   - Delete segment from sweep line status
   - **Two segments become adjacent**. Check for intersection points (to insert in event queue)

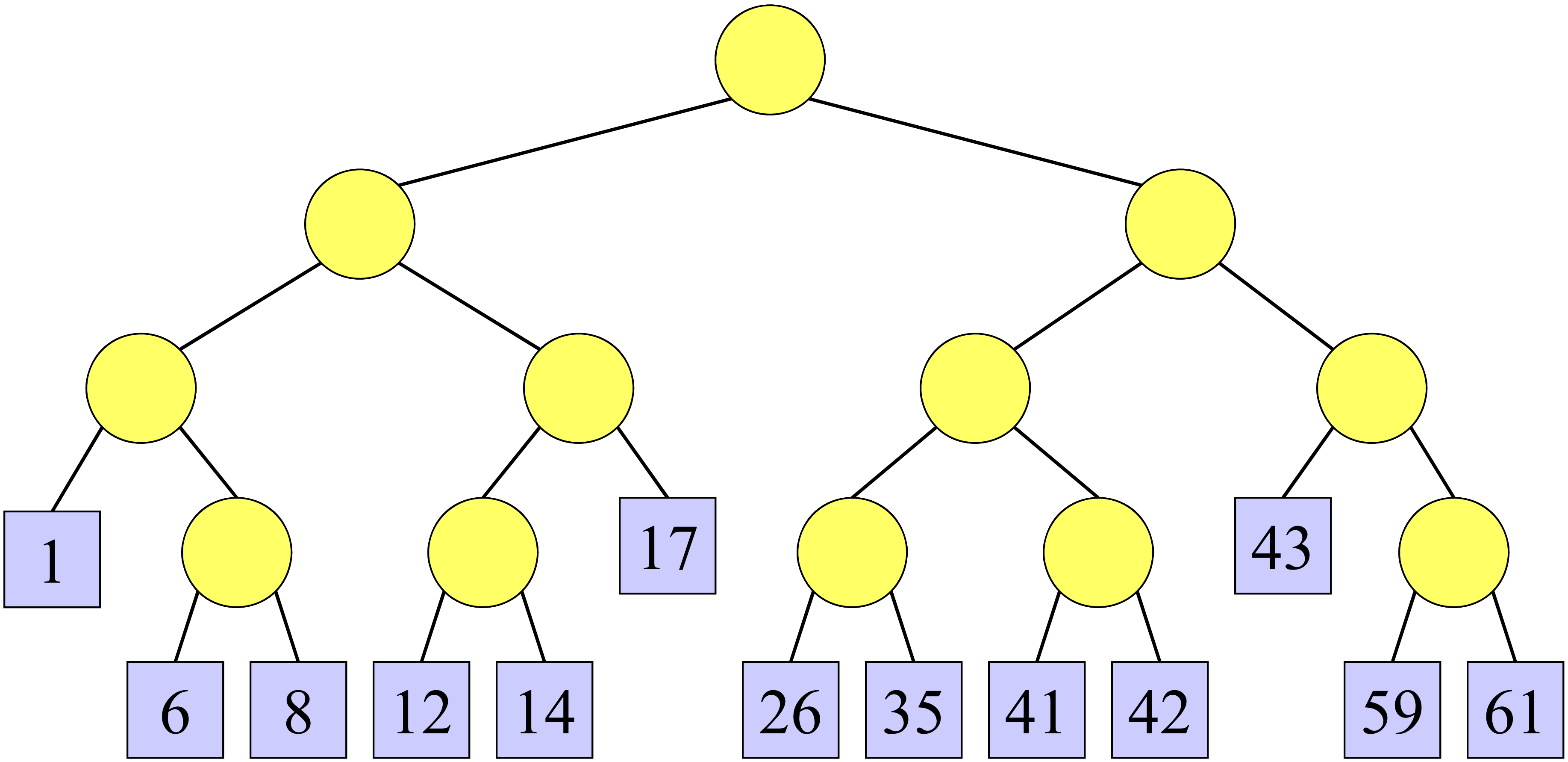


Sweep Line Status

- Store segments that intersect the sweep line l, ordered along the intersection with l .
- Need to insert, delete, and find adjacent neighbor in $O(\log n)$ time
- Use **balanced binary search** tree, storing the order in which segments intersect l in leaves

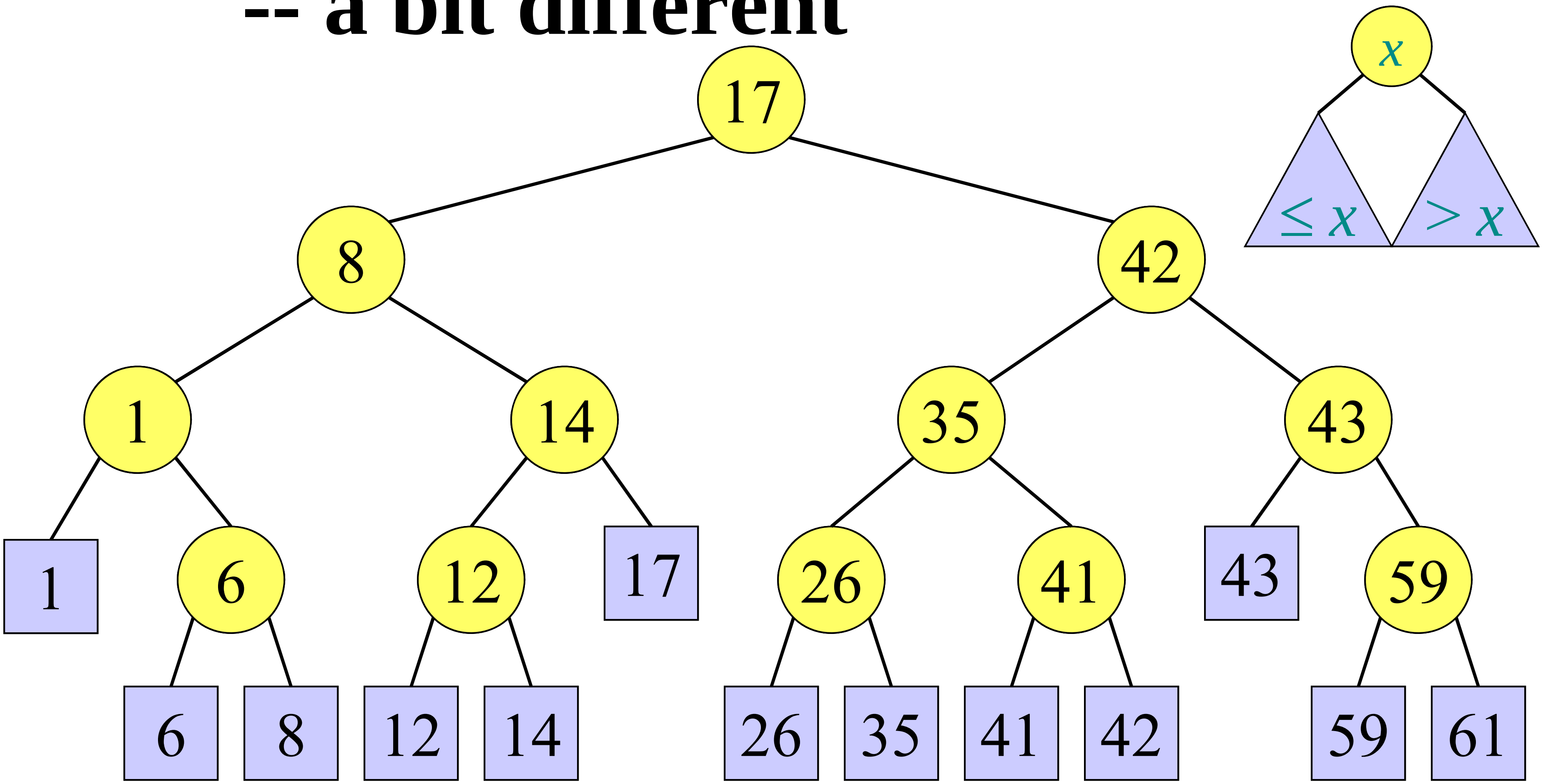


Balanced Binary Search Tree -- a bit different



key[*x*] is the maximum key of any leaf in the left subtree of *x*.

Balanced Binary Search Tree
-- a bit different



key[*x*] is the maximum key of any leaf in the left subtree of *x*.

Event Queue

- Need to keep events sorted:
  - Lexicographic order (first by x-coordinate, and if two events have same x-coordinate then by y-coordinate)
- Need to be able to remove next point, and insert new points in $O(\log n)$ time
- Need to make sure not to process same event twice

$\Rightarrow$ Use a priority queue (heap), and possibly extract multiples

$\Rightarrow$ Or, use balanced binary search tree

Runtime

- Sweep line status updates: $O(\log n)$
- Event queue operations: $O(\log n)$, as the total number of stored events is $\leq 2n + k$, and each operation takes time
$O(\log(2n+k)) = O(\log n^2) = O(\log n)$

$k = O(n^2)$

- There are $O(n+k)$ events. Hence the total runtime is $O((n+k) \log n)$

Plane Sweep: An Algorithm Design Technique

- Plane sweep algorithms (also called sweep line algorithms) are a special kind of incremental algorithms
- Their correctness follows inductively by maintaining the cleanliness property
- *Common* runtimes in the plane are $O(n \log n)$:
  - n events are processed
  - Update of sweep line status takes $O(\log n)$
  - Update of event queue: $O(\log n)$ per event