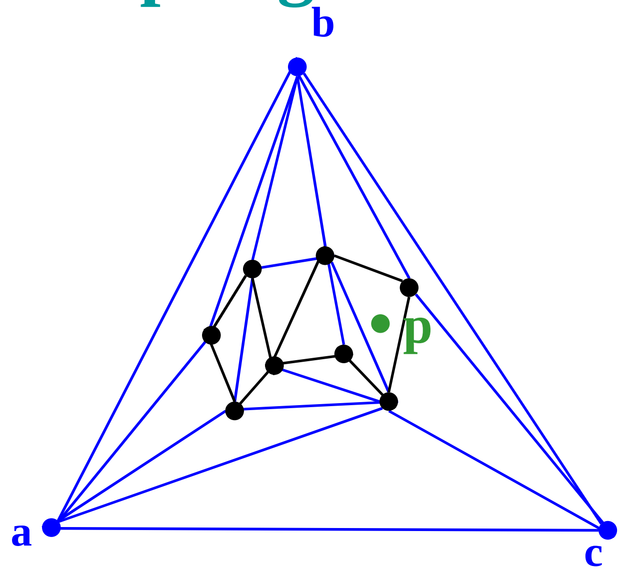
CMPS 6640/4040 Computational Geometry
Spring 2016



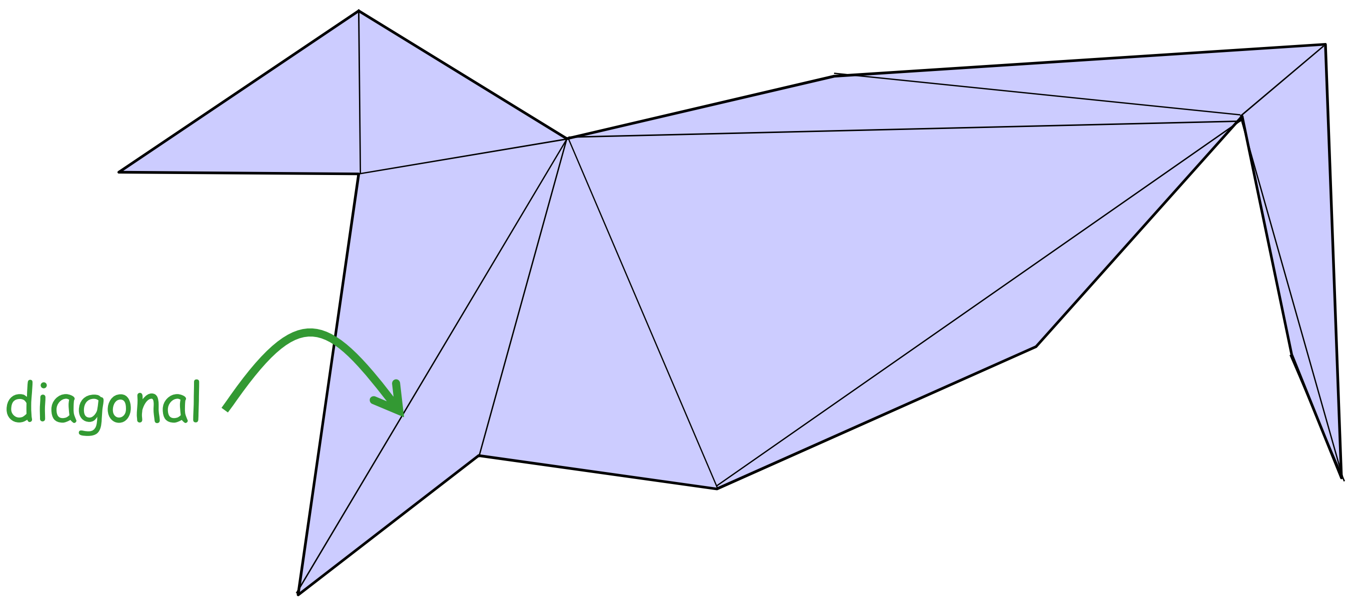
Triangulations, Planar Subdivisions and Point Location

Carola Wenk

Based on:
Computational Geometry: Algorithms and Applications
and David Mount's lecture notes

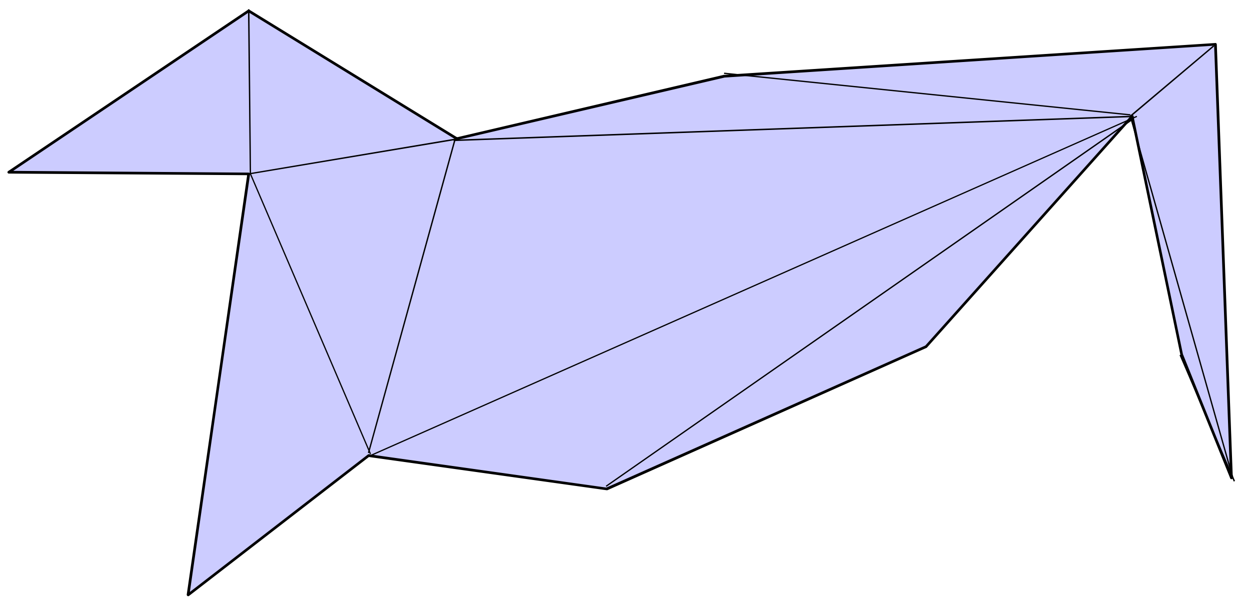
Polygons and Triangulations

- A **simple polygon** *P* in the plane is the region enclosed by a simple polygonal chain that does not self-intersect.
- A **triangulation** of a polygon *P* is a decomposition of *P* into triangles whose vertices are vertices of *P*. In other words, a triangulation is a maximal set of non-crossing diagonals.



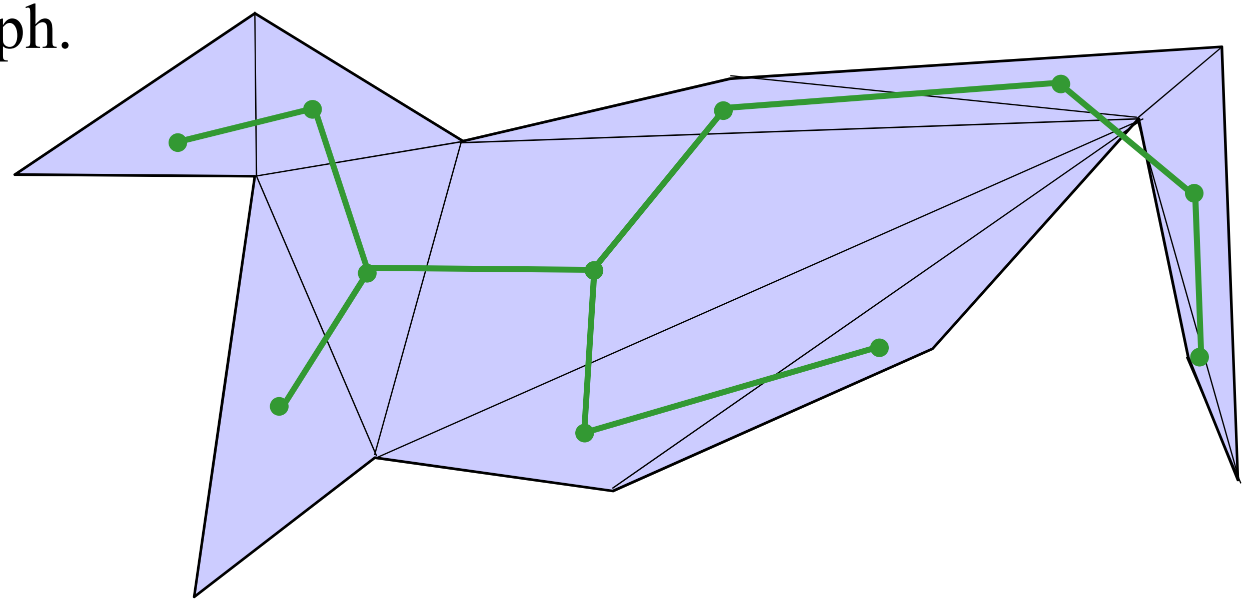
Polygons and Triangulations

- A polygon can be triangulated in many different ways.



Dual graph

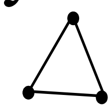
- The **dual graph** of a triangulation (or of a planar subdivision in general) has a vertex for each triangle (face) and an edge for each edge between triangles (faces)
- The dual graph of a triangulated polygon is a tree (connected acyclic graph): Removing an edge corresponds to removing a diagonal in the polygon which disconnects the polygon and with that the graph.

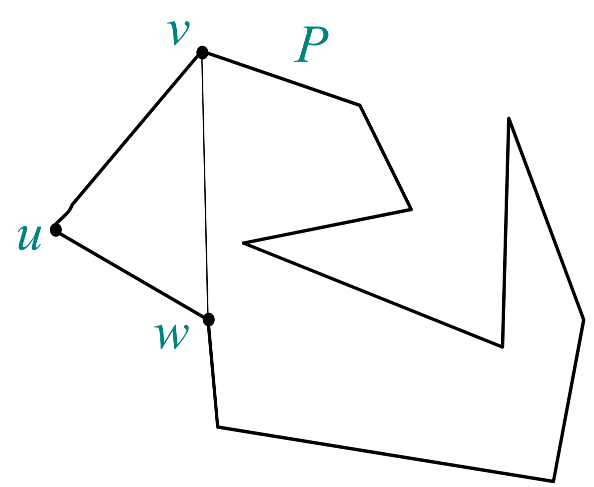


Triangulations of Simple Polygons

Theorem 1: Every simple polygon admits a triangulation, and any triangulation of a simple polygon with n vertices consists of exactly n-2 triangles.

Proof: By induction.

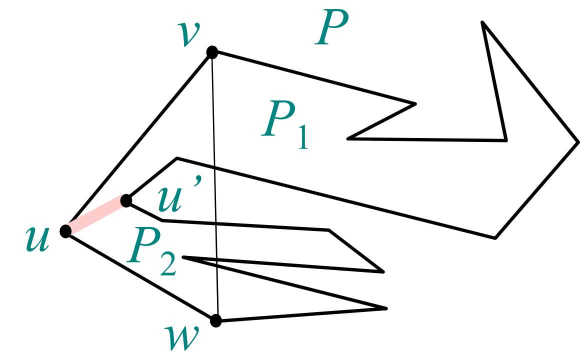
- $n=3$: 
- $n>3$: Let u be leftmost vertex, and v and w adjacent to v. If $\overline{vw}$ does not intersect boundary of P: #triangles = 1 for new triangle + (n-1)-2 for remaining polygon = n-2



Triangulations of Simple Polygons

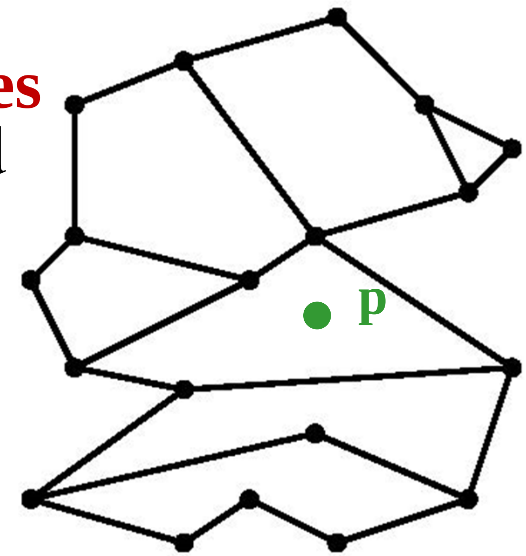
Theorem 1: Every simple polygon admits a triangulation, and any triangulation of a simple polygon with n vertices consists of exactly n-2 triangles.

If $\overleftrightarrow{vw}$ intersects boundary of P: Let $u' \neq u$ be the the vertex furthest to the left of $\overleftrightarrow{vw}$. Take $\overline{uu'}$ as diagonal, which splits P into P_1 and P_2. #triangles in P = #triangles in P_1 + #triangles in P_2 = $|P_1|$-2 + $|P_2|$-2 = $|P_1|$+$|P_2|$-4 = n+2-4 = n-2



Point Location

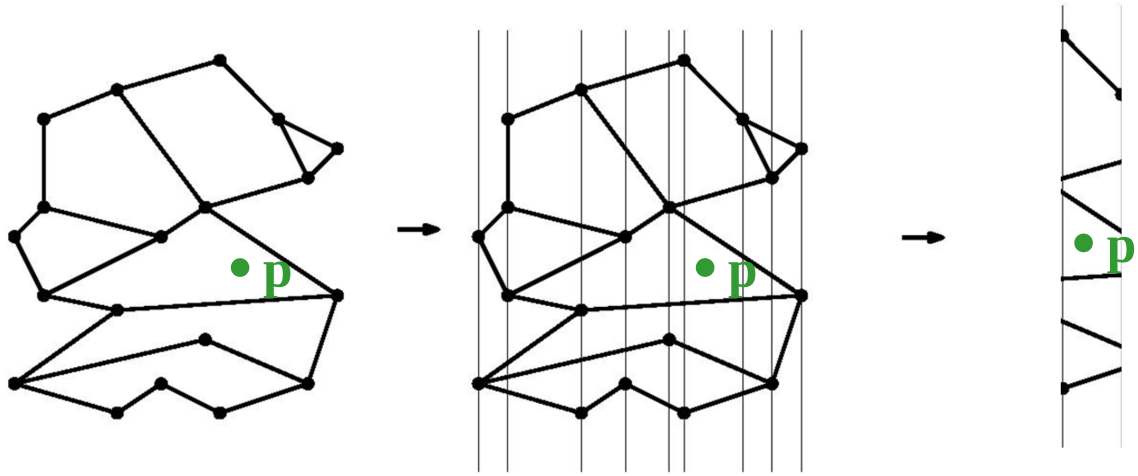
- **Point location task:**
  Preprocess a planar subdivision to efficiently answer **point-location queries** of the type: Given a point $p=(p_x,p_y)$, find the face it lies in.



- **Important metrics:**
  – Time complexity for preprocessing = time to construct the data structure
  – Space needed to store the data structure
  – Time complexity for querying the data structure

Slab Method

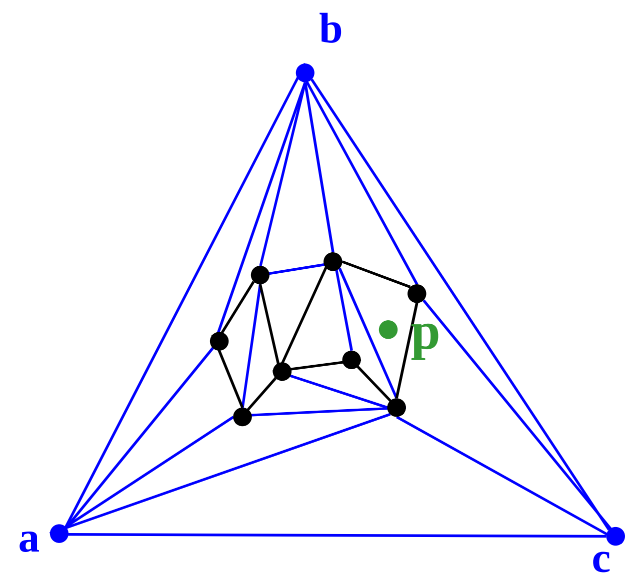
- **Slab method:**
  Draw a vertical line through each vertex. This decomposes the plane into slabs.



- In each slab, the vertical order of the line segments remains constant.
- If we know in which slab ***p*** lies, we can perform binary search, using the sorted order of the segments in the slab.
- Find slab that contains ***p*** by binary search on x among slab boundaries.
- A second binary search in slab determines the face containing ***p***.
- Search complexity $O(\log n)$, but space complexity $\Theta(n^2)$.

Kirkpatrick's Algorithm

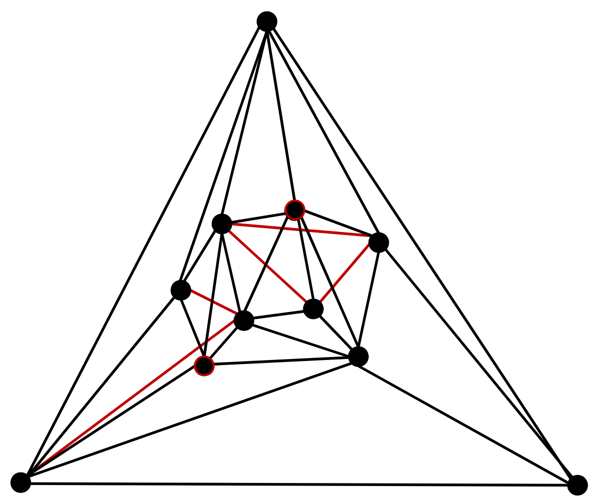
- Needs a triangulation as input.
- Can convert a planar subdivision with *n* vertices into a triangulation:
  - Triangulate each face, keep same label as original face.
  - If the outer face is not a triangle:
    - Compute the convex hull of the subdivision.
    - Triangulate pockets between the subdivision and the convex hull.
    - Add a large triangle (new vertices **a**, **b**, **c**) around the convex hull, and triangulate the space in-between.



- The size of the triangulated planar subdivision is still $O(n)$, by Euler's formula.
- The conversion can be done in $O(n \log n)$ time.
- Given ***p***, if we find a triangle containing ***p*** we also know the (label of) the original subdivision face containing ***p***.

Kirkpatrick's Hierarchy

- Compute a sequence T_0, T_1, ..., T_k of increasingly coarser triangulations such that the last one has constant complexity.
- The sequence T_0, T_1, ..., T_k should have the following properties:
  - T_0 is the input triangulation, T_k is the outer triangle
  - $k \in O(\log n)$
  - Each triangle in T_{i+1} overlaps O(1) triangles in T_i
- How to build such a sequence?
  - Need to delete vertices from T_i .
  - Vertex deletion creates holes, which need to be re-triangulated.
- How do we go from T_0 of size $O(n)$ to T_k of size O(1) in $k=O(\log n)$ steps?
  - In each step, delete a constant fraction of vertices from T_i .

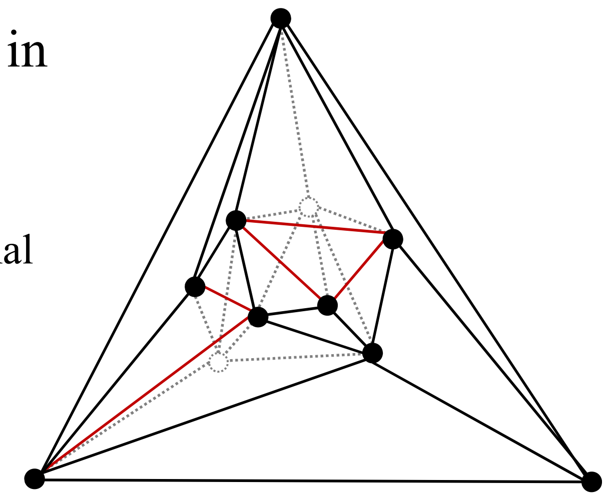


- We also need to ensure that each new triangle in T_{i+1} overlaps with only O(1) triangles in T_i .

Vertex Deletion and Independent Sets

When creating T_{i+1} from T_i , delete vertices from T_i that have the following properties:

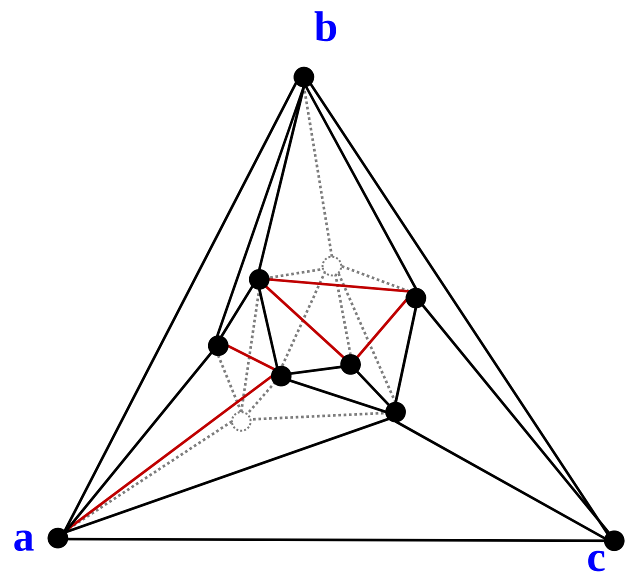
- **Constant degree:**
  Each vertex **v** to be deleted has O(1) degree in the graph T_i .
  - If **v** has degree d, the resulting hole can be re-triangulated with d-2 triangles
  - Each new triangle in T_{i+1} overlaps at most d original triangles in T_i



- **Independent sets:**
  No two deleted vertices are adjacent.
  - Each hole can be re-triangulated independently.

Independent Set Lemma

Lemma: Every planar graph on n vertices contains an independent vertex set of size $n/18$ in which each vertex has degree at most 8. Such a set can be computed in O(n) time.



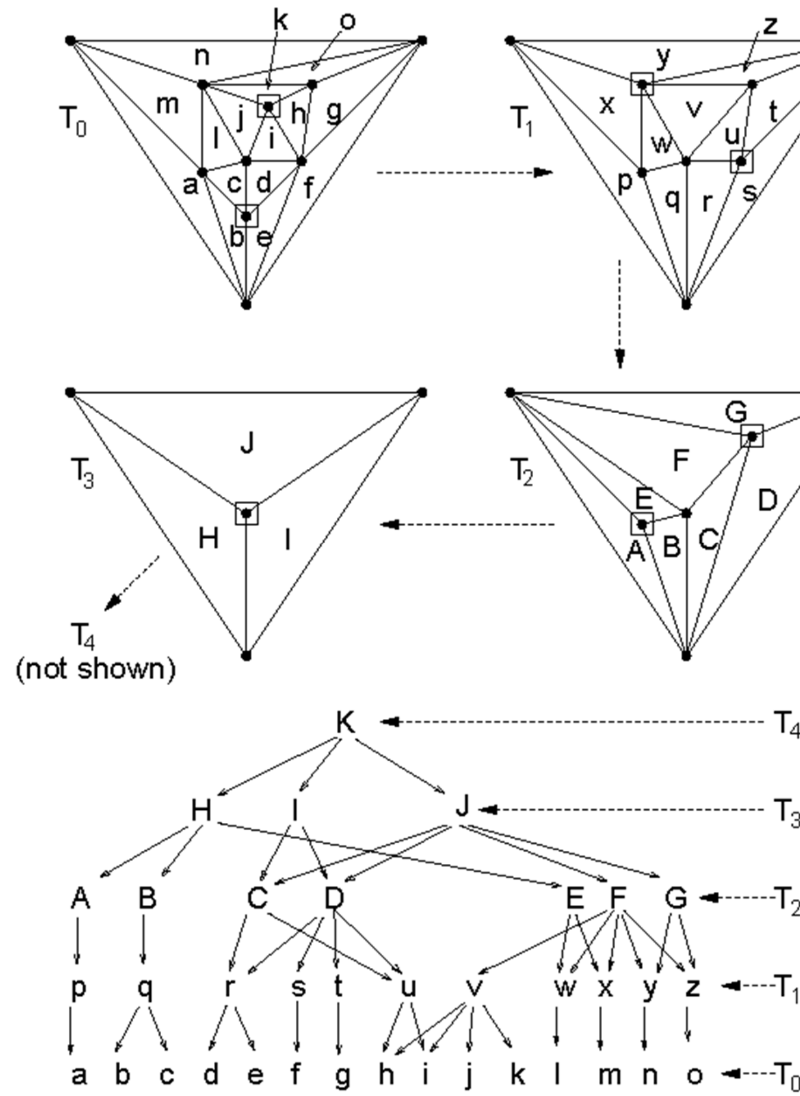
Use this lemma to construct Kirkpatrick's hierarchy:

- Start with T_0, and select an independent set S of size $n/18$ in which each vertex has maximum degree 8. [Never pick the outer triangle vertices **a**, **b**, **c**.]
- Remove vertices of S, and re-triangulate holes.
- The resulting triangulation, T_1, has at most $17/18n$ vertices.
- Repeat the process to build the hierarchy, until T_k equals the outer triangle with vertices **a**, **b**, **c**.
- The depth of the hierarchy is $k = \log_{18/17} n$

Hierarchy Example

Use this lemma to construct Kirkpatrick's hierarchy:

- Start with T_0, and select an independent set S of size $n/18$ in which each vertex has maximum degree 8. [Never pick the outer triangle vertices **a**, **b**, **c**.]
- Remove vertices of S, and re-triangulate holes.
- The resulting triangulation, T_1, has at most $17/18n$ vertices.
- Repeat the process to build the hierarchy, until T_k equals the outer triangle with vertices **a**, **b**, **c**.
- The depth of the hierarchy is $k = \log_{18/17} n$



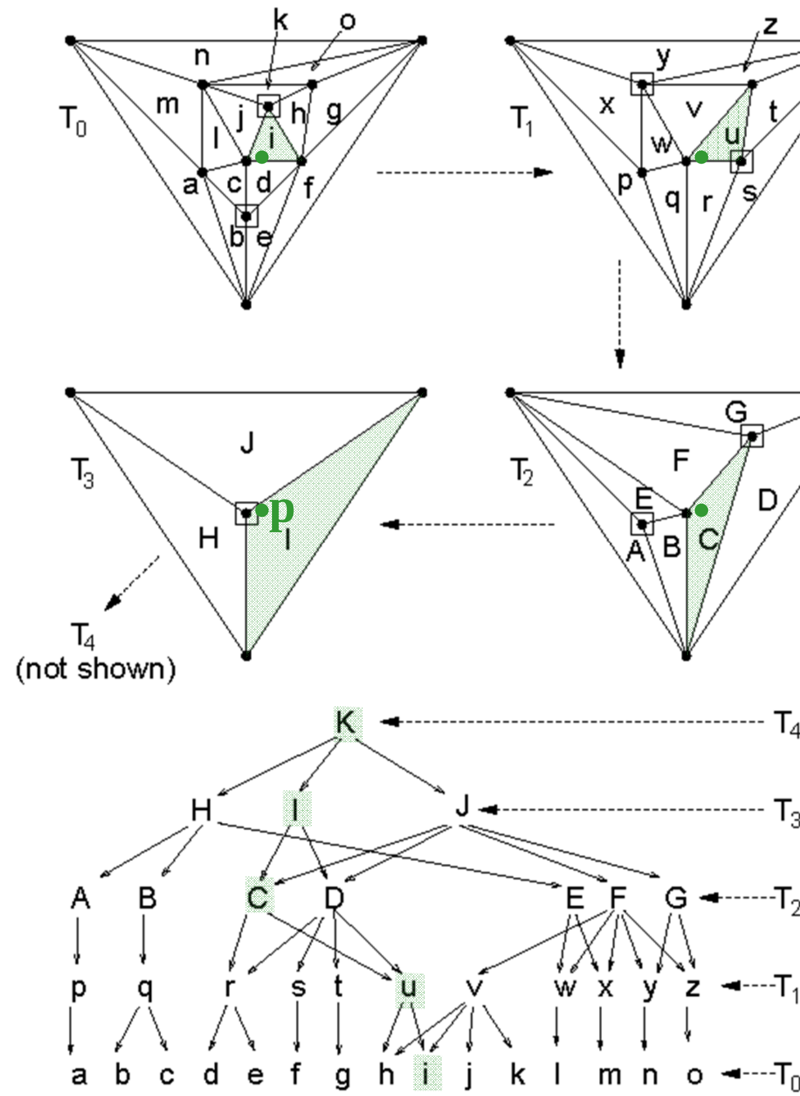
Hierarchy Data Structure

Store the hierarchy as a DAG:

- The root is T_k .
- Nodes in each level correspond to triangles T_i .
- Each node for a triangle in T_{i+1} stores pointers to all triangles of T_i that it overlaps.

How to locate point ***p*** in the DAG:

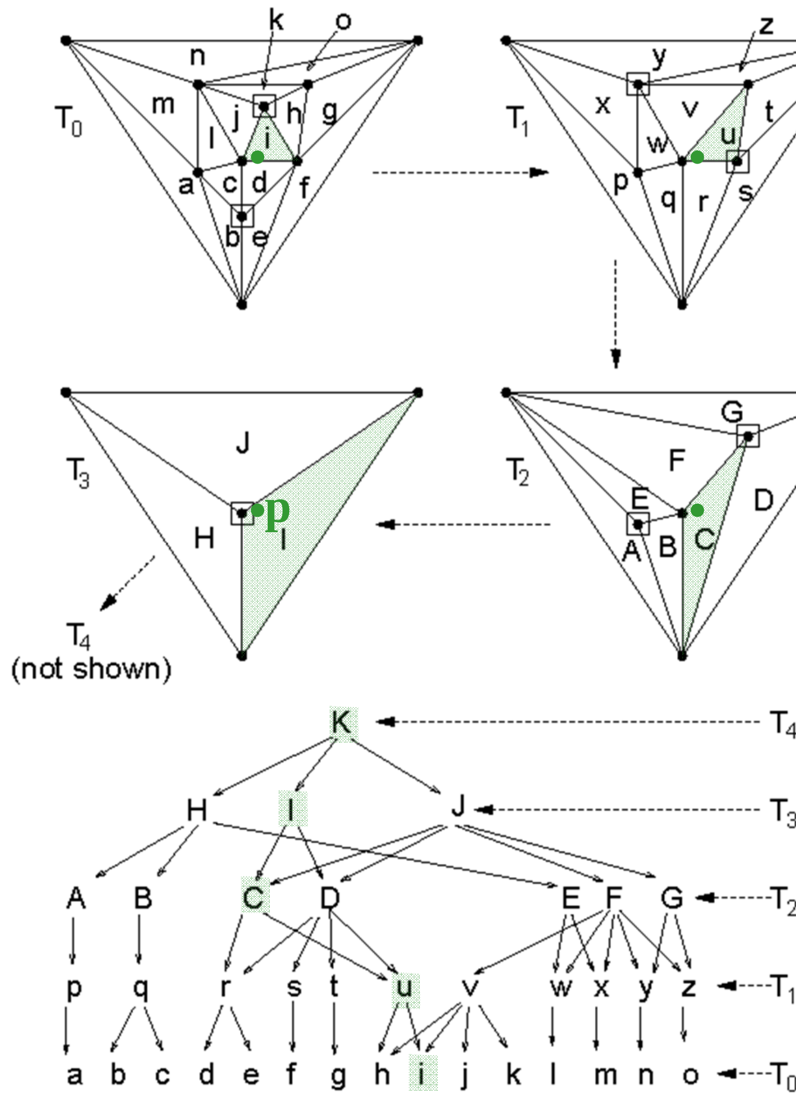
- Start at the root. If ***p*** is outside of T_k then ***p*** is in exterior face; done.
- Else, set Δ to be the triangle at the current level that contains ***p***.
- Check each of the at most 6 triangles of T_{k-1} that overlap with Δ, whether they contain ***p***. Update Δ and descend in the hierarchy until reaching T_0 .
- Output Δ .



Analysis

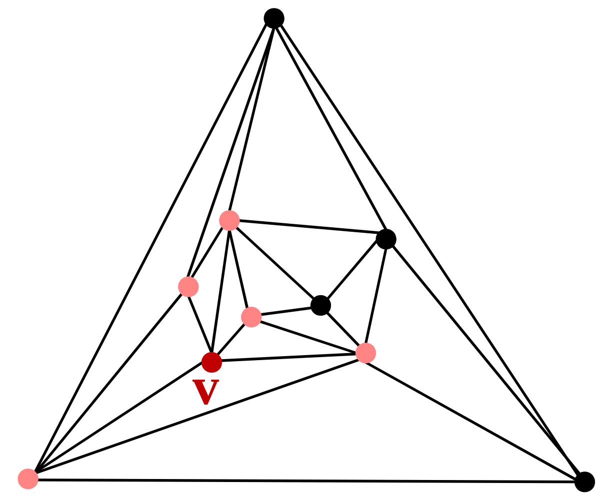
- **Query time** is $O(\log n)$: There are $O(\log n)$ levels and it takes constant time to move between levels.
- **Space complexity** is $O(n)$:
  - Sum up sizes of all triangulations in hierarchy.
  - Because of Euler's formula, it suffices to sum up the number of vertices.
  - Total number of vertices:

    $n + 17/18\, n + (17/18)^2\, n + (17/18)^3\, n + \ldots$
    $\leq 1/(1-17/18)\, n = 18\, n$
- **Preprocessing time** is $O(n \log n)$:
  - Triangulating the subdivision takes $O(n \log n)$ time.
  - The time to build the DAG is proportional to its size.



Independent Set Lemma

Lemma: Every planar graph on n vertices contains an independent vertex set of size $n/18$ in which each vertex has degree at most 8. Such a set can be computed in $O(n)$ time.



Proof:

Algorithm to construct independent set:

- Mark all vertices of degree ≥ 9
- While there is an unmarked vertex
  - Let **v** be an unmarked vertex
  - Add **v** to the independent set
  - Mark **v** and all its neighbors
- Can be implemented in $O(n)$ time: Keep list of unmarked vertices, and store the triangulation in a data structure that allows finding neighbors in $O(1)$ time.

Independent Set Lemma

Still need to prove existence of large independent set.

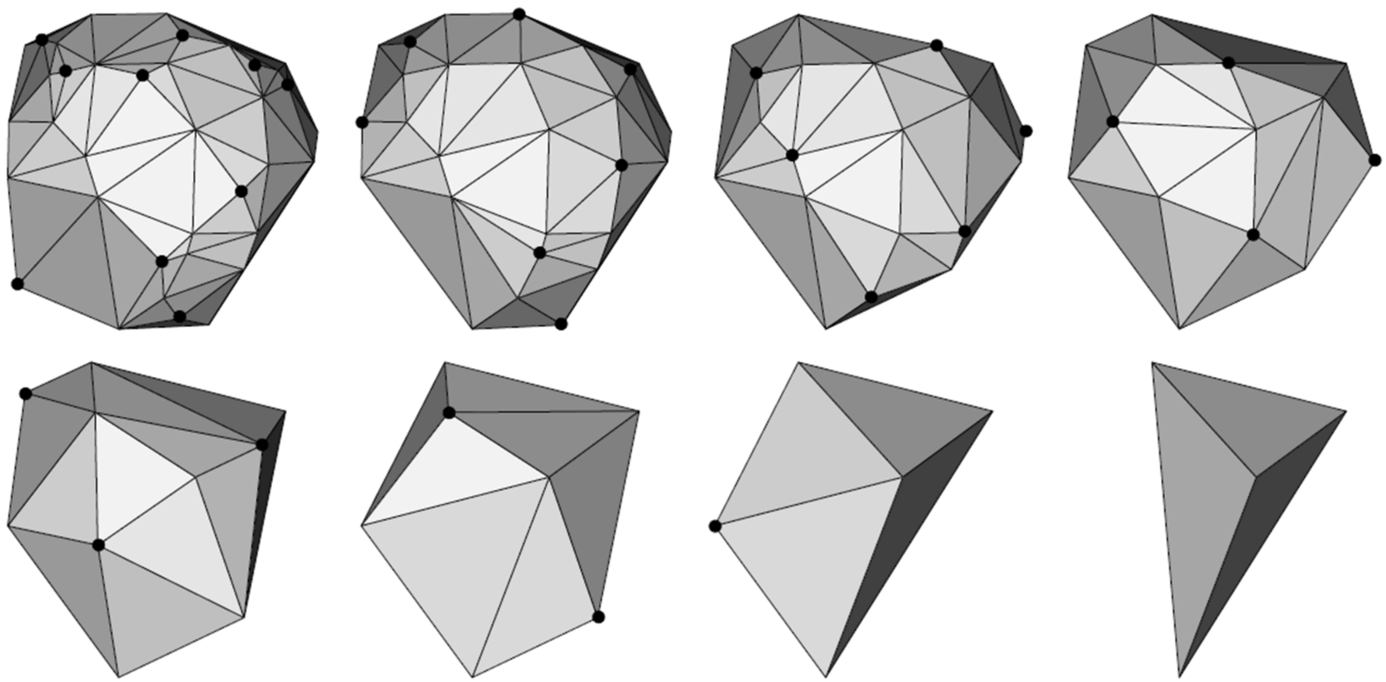
- Euler's formula for a triangulated planar graph on n vertices:
  #edges $= 3n - 6$
- Sum over vertex degrees:
  $\sum_v \deg(v) = 2$ #edges $= 6n - 12 < 6n$
- **Claim:** At least $n/2$ vertices have degree ≤ 8.
  Proof: By contradiction. So, suppose otherwise.
  $\rightarrow$ $n/2$ vertices have degree ≥ 9. The remaining have degree ≥ 3.
  $\rightarrow$ The sum of the degrees is $\geq 9\ n/2 + 3\ n/2 = 6n$. Contradiction.

- In the beginning of the algorithm, at least $n/2$ nodes are unmarked. Each picked vertex **v** marks ≤ 8 other vertices, so including itself 9.
- Therefore, the while loop can be repeated at least $n/18$ times.
- This shows that there is an independent set of size at least $n/18$ in which each node has degree ≤ 8.

□

Kirkpatrick's Hierarchy Summary

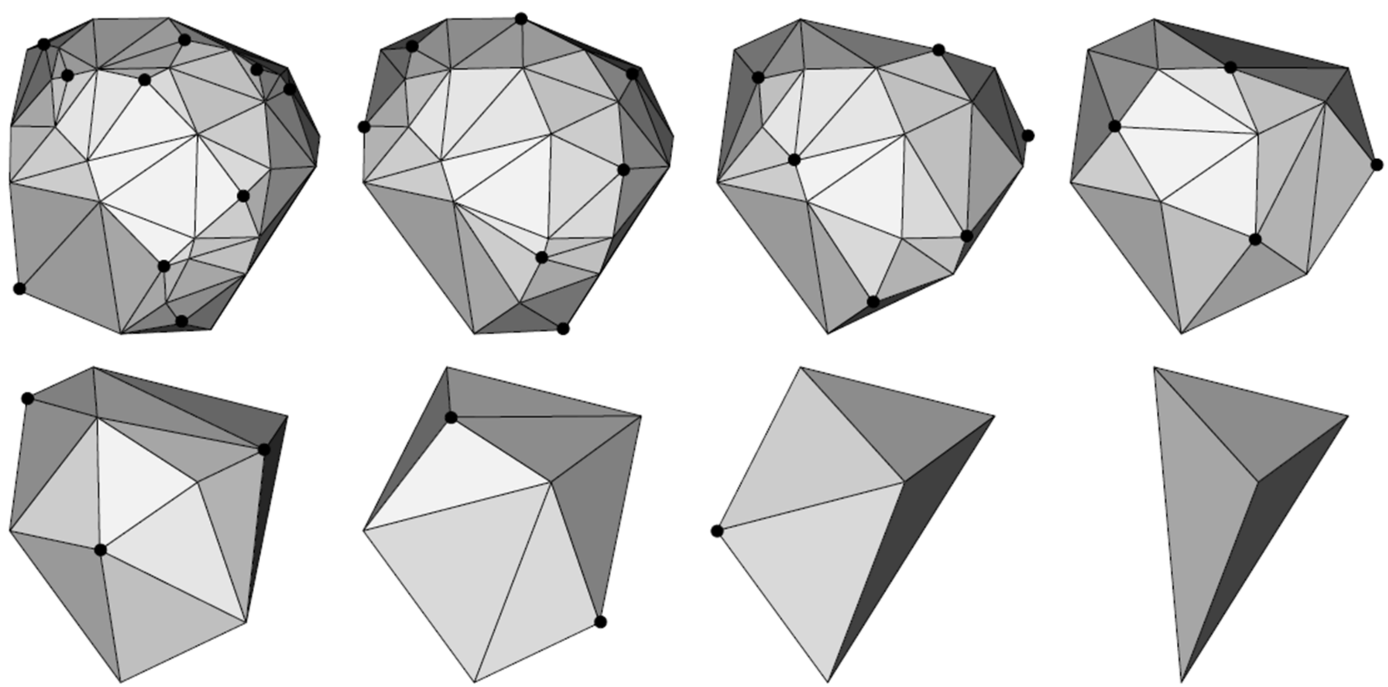
- Kirkpatrick's point location data structure needs $O(n \log n)$ preprocessing time, $O(n)$ space, and has $O(\log n)$ query time. It involves rather high constant factors though.
- It can also be used to create a hierarchy of polytopes: The Dobkin-Kirkpatrick decomposition



Use of Dobkin-Kirkpatrick's Hierarchy for Polytopes/Polyhedra

Efficiently answer the following types of queries:

- Find an extreme point in a given direction.
- Locate a point on the polytope closest to a query point.
- Compute the intersection of two polytopes (→ collision detection)



Extreme Points

Let's start with 2D:

Given a convex polygon (as a list of n vertices in counter-clockwise order around the polygon), how fast can one find a point with maximum y-coordinate?

Answer: In $O(\log n)$ time using a variant of binary search.

What about a convex polytope in 3D? How fast can one find a point on it with maximum z-coordinate?

Answer 1: Trivially in $O(n)$ time by checking each vertex.

Answer 2: Preprocess the polytope using Dobkin-Kirkpatrick's hierarchy in $O(n \log n)$ time and $O(n)$ space. Then develop an $O(\log n)$ time query algorithm.