4/14/16

6. Homework

Due **4/28/16** at noon in the department office

1. **Quadtree range queries (12 points)**
Quadtrees can be used to perform range queries. Consider a quadtree of height h on a set P of n points in the plane. Describe an algorithm for querying the quadtree with a query region R. Analyze the worst-case query time for the case where R is a rectangle, and for the case where R is a half-plane bounded by a vertical line.

2. **WSPD bounds (12 points)**
   (a) (6 points) Show that there is a set P of n points in the plane such that $\sum_{i=1}^{m}(|P_i| + |Q_i|) \in \Omega(n^2)$ for any WSPD $\{\{P_1, Q_1\}, \ldots, \{P_m, Q_m\}\}$ for P and any $s > 0$.
   (b) (6 points) Show that if the WSPD $\{\{P_1, Q_1\}, \ldots, \{P_m, Q_m\}\}$ is obtained using a quadtree then $\sum_{i=1}^{m} \min(|P_i|, |Q_i|) \in O(n \log n)$.

3. **Spanner (12 points)**
A desirable property of spanners is that every vertex should have constant degree.
   (a) (6 points) Show that there exists a set of points in the plane such that the WSPD-based spanner construction given in class results in at least one vertex of degree $\Omega(n)$. (Note: The representatives for nodes in the quadtree were chosen arbitrarily. In order to obtain the worst case, you will need to select representatives carefully.)
   (b) (6 points) Show that it is possible to modify the WSPD-based spanner construction so that the number of edges is the same, but each vertex has constant degree (where the constant depends on the stretch factor). For this, you may make the simplifying assumption that the quadtree is not compressed and that every internal node has at least two non-empty children.
   (Hint: Show that in such a tree it is possible to distribute representatives so that each point occurs as the representative for at most a constant number of nodes in the tree.)

4. **Approximate Distance Counting (14 points)**
Let P be a set of n points in $\mathbb{R}^d$ and let $\varepsilon > 0$. We define an *exact distance query* to take an input number $\delta > 0$ and to return the number of pairs of points $(p, q) \in P \times P$ such that their distance $||p - q|| \geq \delta$. An *ε-approximate distance query* returns the number of pairs of points (p, q) such that $||p - q|| \geq \delta(1 + \varepsilon)$. Note that pairs of points whose distances lie between those two bounds may or may not be counted, at the discretion of the algorithm.

   Explain how to preprocess P into a data structure such that ε-approximate distance counting queries can be answered in $O(n/\varepsilon^d)$ time and $O(n/\varepsilon^d)$ space.
   (Hint: Use a WSPD. Explain what separation factor is used and any needed modification to the WSPD.)