CMPS 6610 – Fall 2018

Shortest Paths

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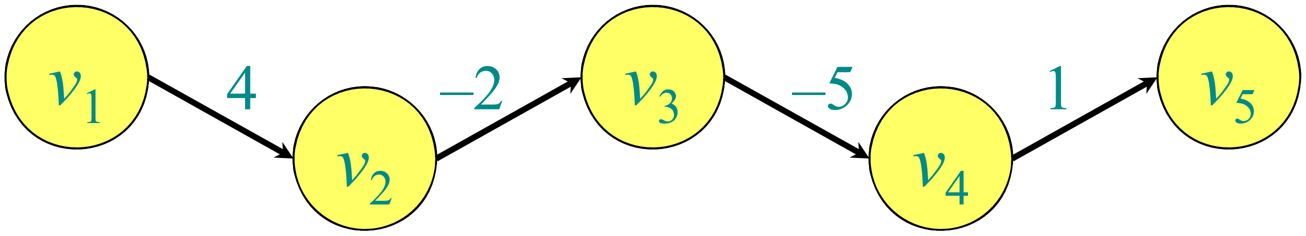
Slides courtesy of Charles Leiserson with changes and additions by Carola Wenk

Paths in graphs

Consider a digraph $G = (V, E)$ with an edge-weight function $w : E \to \mathbb{R}$. The ***weight*** of path $p = v_1 \to v_2 \to \dots \to v_k$ is defined to be

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$

Example:



$w(p) = -2$

Shortest paths

A ***shortest path*** from u to v is a path of minimum weight from u to v.

The ***shortest-path weight*** from u to v is defined as

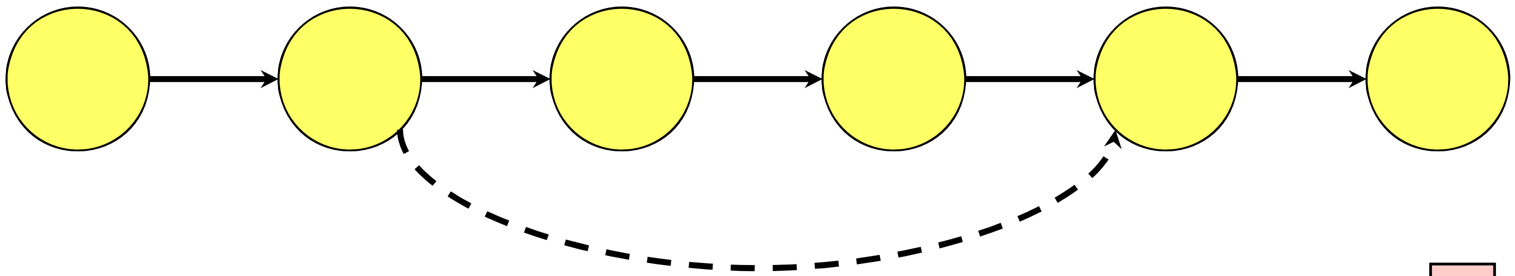
$\delta(u, v) = \min\{w(p) : p$ is a path from u to $v\}$.

Note: $\delta(u, v) = \infty$ if no path from u to v exists.

Optimal substructure

Theorem. A subpath of a shortest path is a shortest path.

Proof. Cut and paste:



□

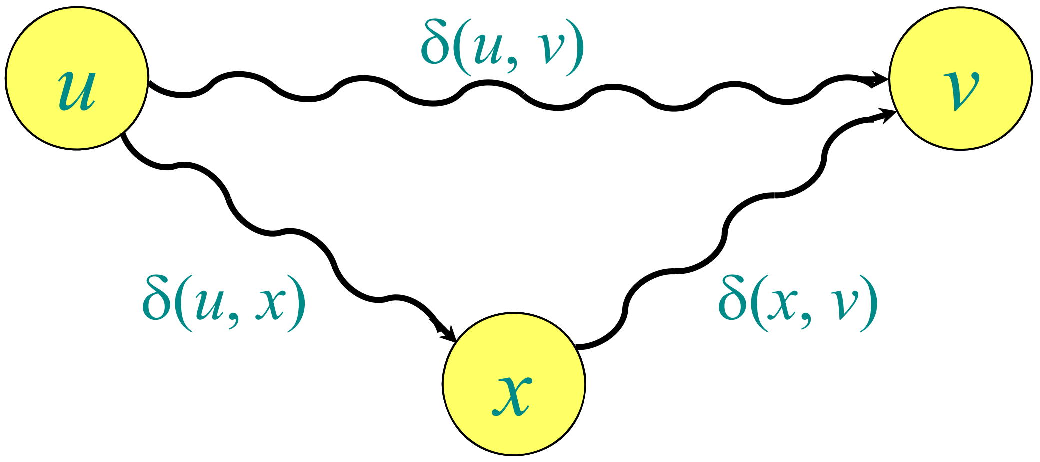
Triangle inequality

Theorem. For all $u, v, x \in V$, we have

$$\delta(u, v) \le \delta(u, x) + \delta(x, v).$$

Proof.

- $\delta(u,v)$ minimizes over **all** paths from u to v
- Concatenating two shortest paths from u to x and from x to v yields **one** specific path from u to v

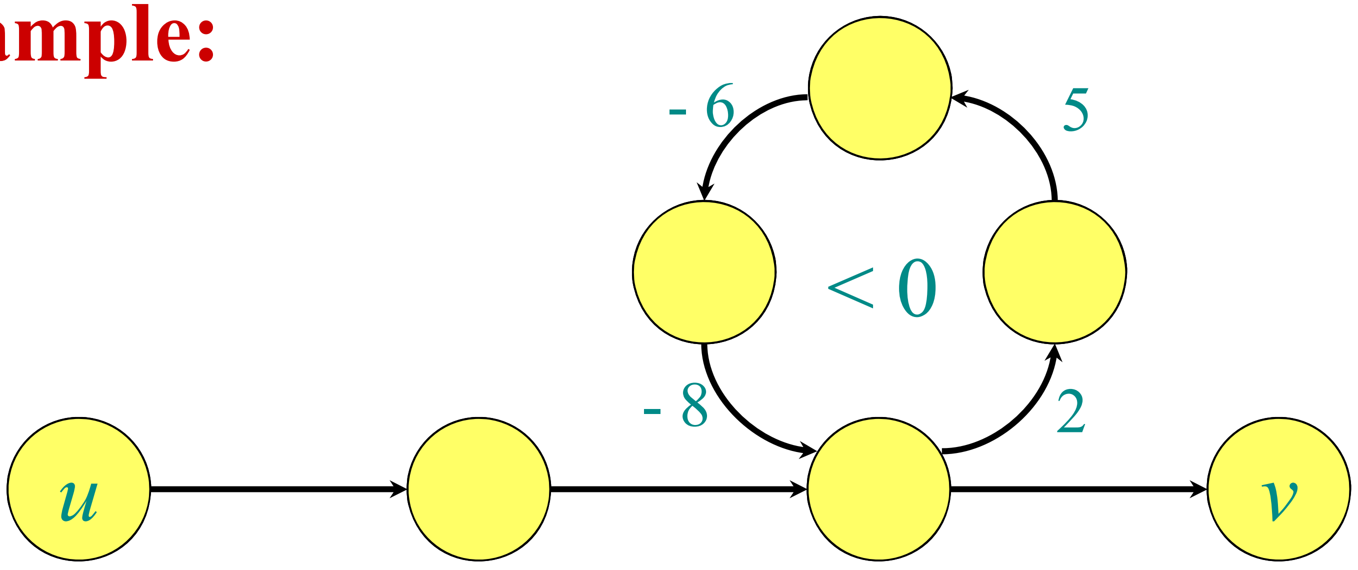


□

Well-definedness of shortest paths

If a graph G contains a negative-weight cycle, then some shortest paths may not exist.

Example:



Single-source shortest paths

Problem. From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.

Assumption:
All edge weights $w(u, v)$ are ***non-negative***.

It follows that all shortest-path weights must exist.

IDEA: Greedy.

1. Maintain a set S of vertices whose shortest-path weights from s are known, i.e., $d[v]=\delta(s,v)$
2. At each step add to S the vertex $u \in V - S$ whose distance estimate $d[u]$ from s is minimal.
3. Update the distance estimates $d[v]$ of vertices v adjacent to u.

Dijkstra's algorithm

```
$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
    do $d[v] \leftarrow \infty$
$S \leftarrow \emptyset$            ⊳ Vertices for which $d[v]=d(s,v)$
$Q \leftarrow V$            ⊳ $Q$ is a priority queue maintaining $V - S$
                    sorted by $d$-values $d[v]$
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
```

relaxation step

implicit DECREASE-KEY in Q

Dijkstra

```
$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
    do $d[v] \leftarrow \infty$
$S \leftarrow \emptyset$                ⊳ Verti
$Q \leftarrow V$                ⊳ $Q$ is
                          sorte
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
```

implicit DECREASE-KEY in Q

relaxation step

```
$Q \leftarrow V$                PRIM's algorithm
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
while $Q \neq \emptyset$
    do $u \leftarrow$ EXTRACT-MIN($Q$)
        for each $v \in Adj[u]$
            do if $v \in Q$ and $w(u, v) < key[v]$
                then $key[v] \leftarrow w(u, v)$
                     $\pi[v] \leftarrow u$
```

Difference to Prim's:
- It suffices to only check $v \in Q$, but it doesn't hurt to check all v
- Add $d[u]$ to the weight

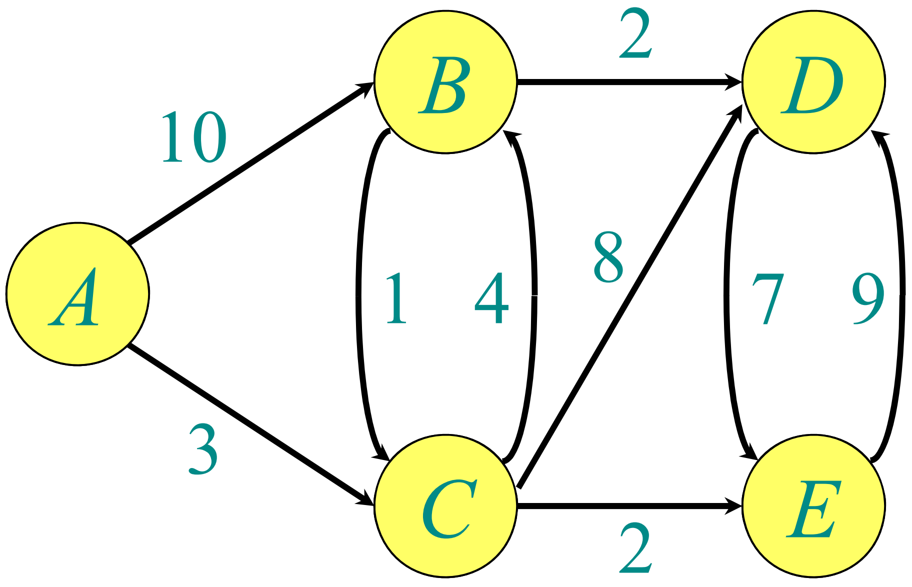
How to find the actual shortest paths?

Store a predecessor tree:

```
$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
    do $d[v] \leftarrow \infty$
$S \leftarrow \varnothing$          ⊳ Vertices for which $d[v]=d(s,v)$
$Q \leftarrow V$          ⊳ $Q$ is a priority queue maintaining $V - S$
                  sorted by $d$-values $d[v]$
while $Q \neq \varnothing$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
      if $d[v] > d[u] + w(u, v)$ then
        $d[v] \leftarrow d[u] + w(u, v)$
        $\pi[v] \leftarrow u$
```

Example of Dijkstra's algorithm

Graph with nonnegative edge weights:

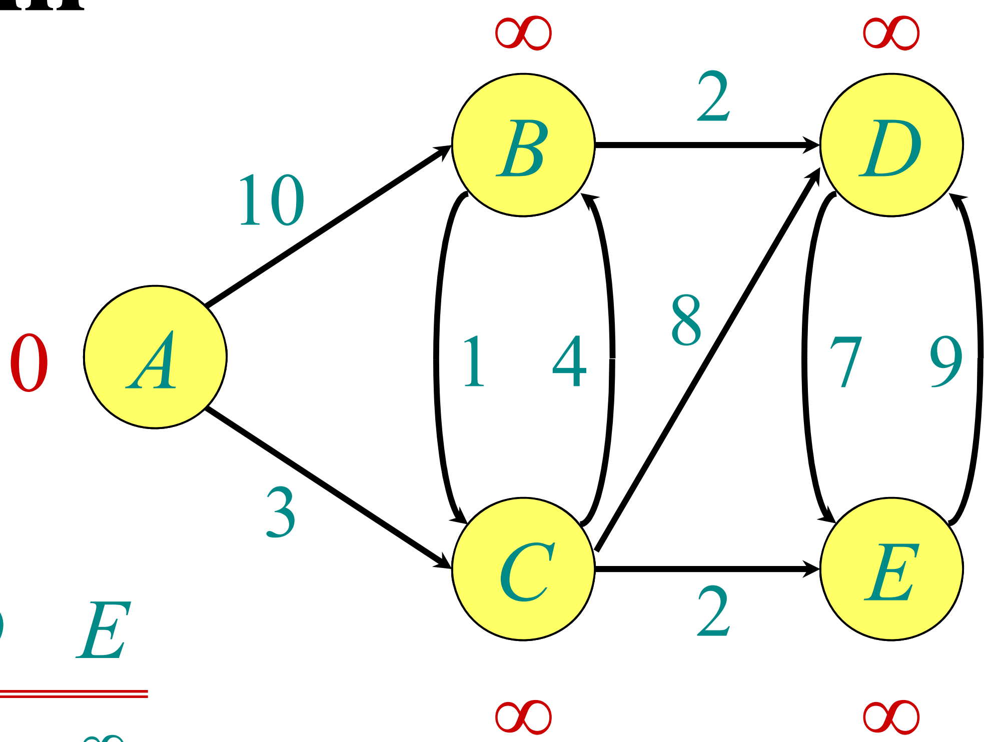


```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN$(Q)$
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

Example of Dijkstra's algorithm

Initialize:

S: {}



Q:	A	B	C	D	E
	0	∞	∞	∞	∞

```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

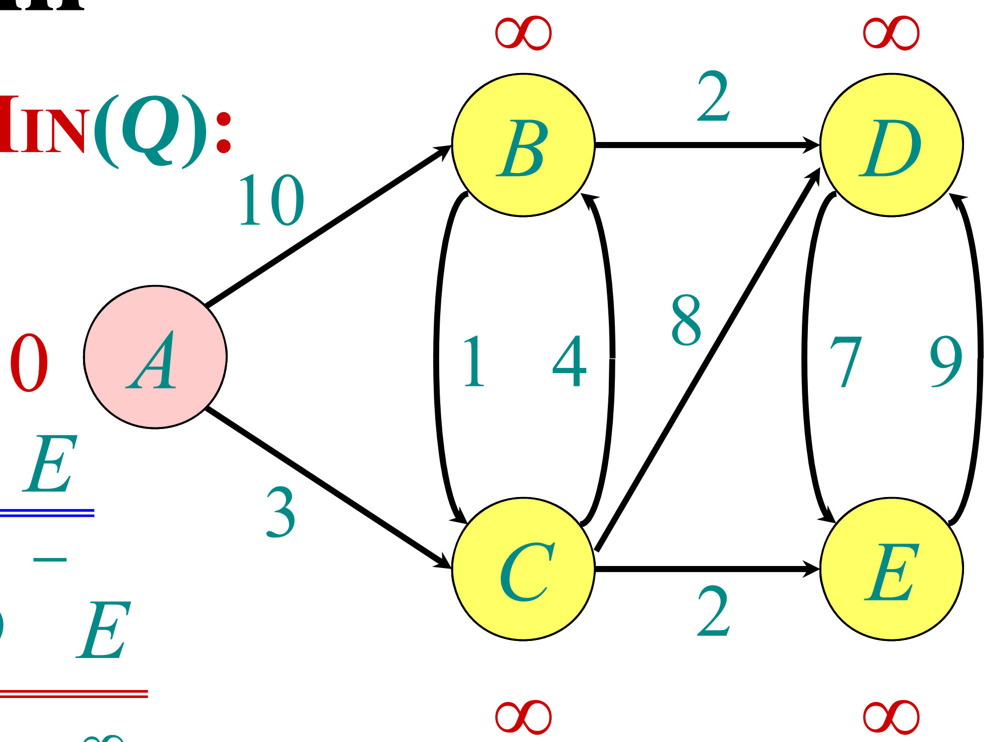
Example of Dijkstra's algorithm

"A" ← EXTRACT-MIN(Q):

S: { A }

π:	A	B	C	D	E
	–	–	–	–	–

Q:	A	B	C	D	E
	0	∞	∞	∞	∞



```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

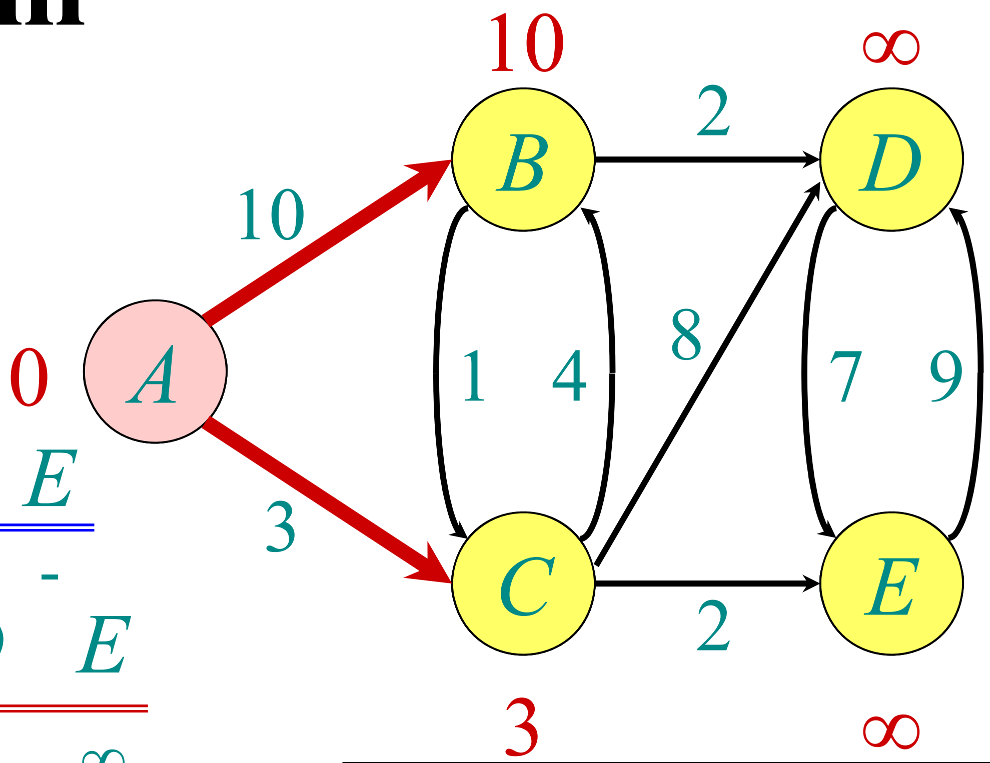
Example of Dijkstra's algorithm

Relax all edges leaving *A*:

S: { A }

π:	A	B	C	D	E
	–	–	–	–	-

Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	–	–



```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

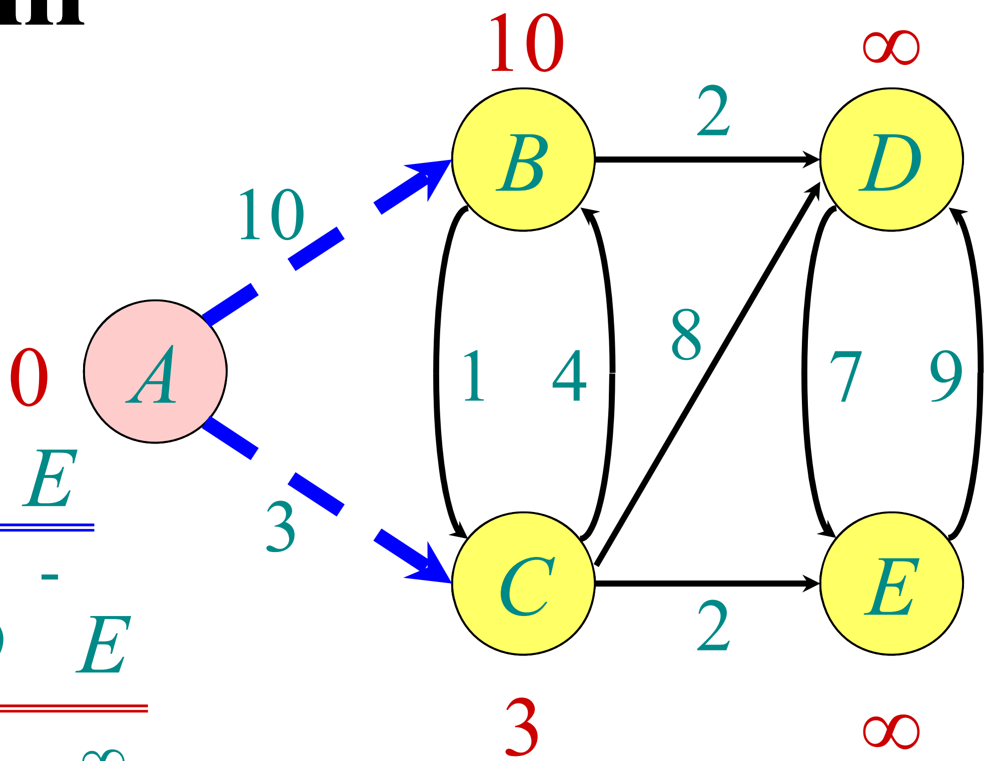
Example of Dijkstra's algorithm

Relax all edges leaving *A*:

S: { *A* }

π:	*A*	*B*	*C*	*D*	*E*
	–	A	A	–	-

Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	–	–



```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

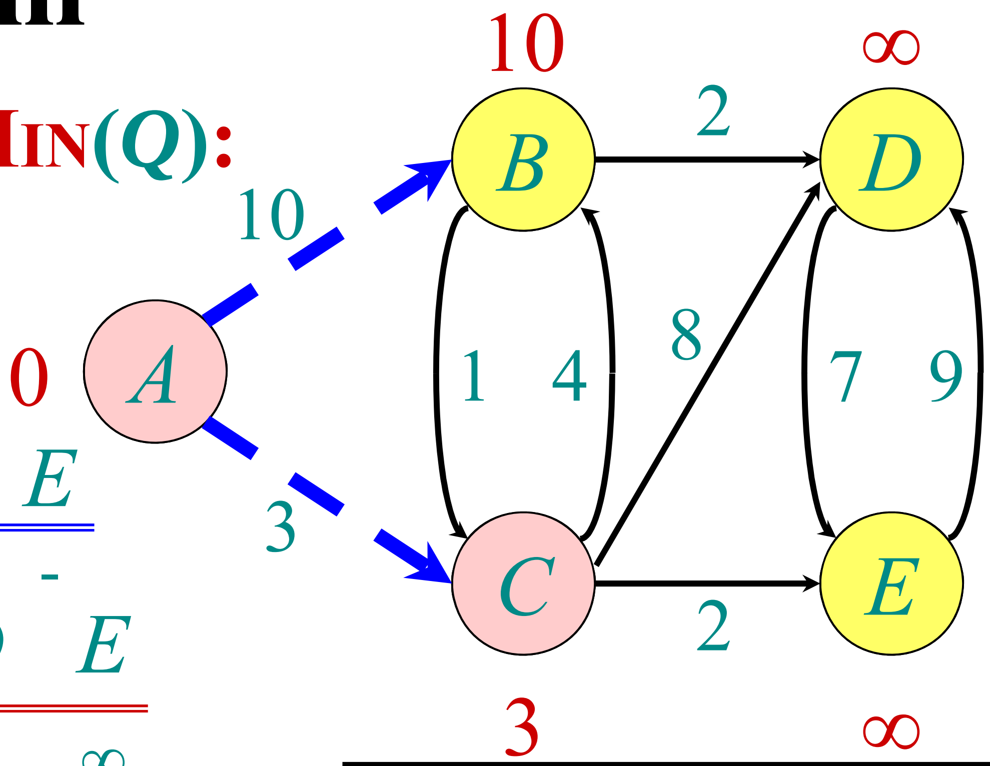
Example of Dijkstra's algorithm

"C" $\leftarrow$ **EXTRACT-MIN(Q):**

S: { A, C }

π:	A	B	C	D	E
	–	A	A	–	-

Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	–	–



```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

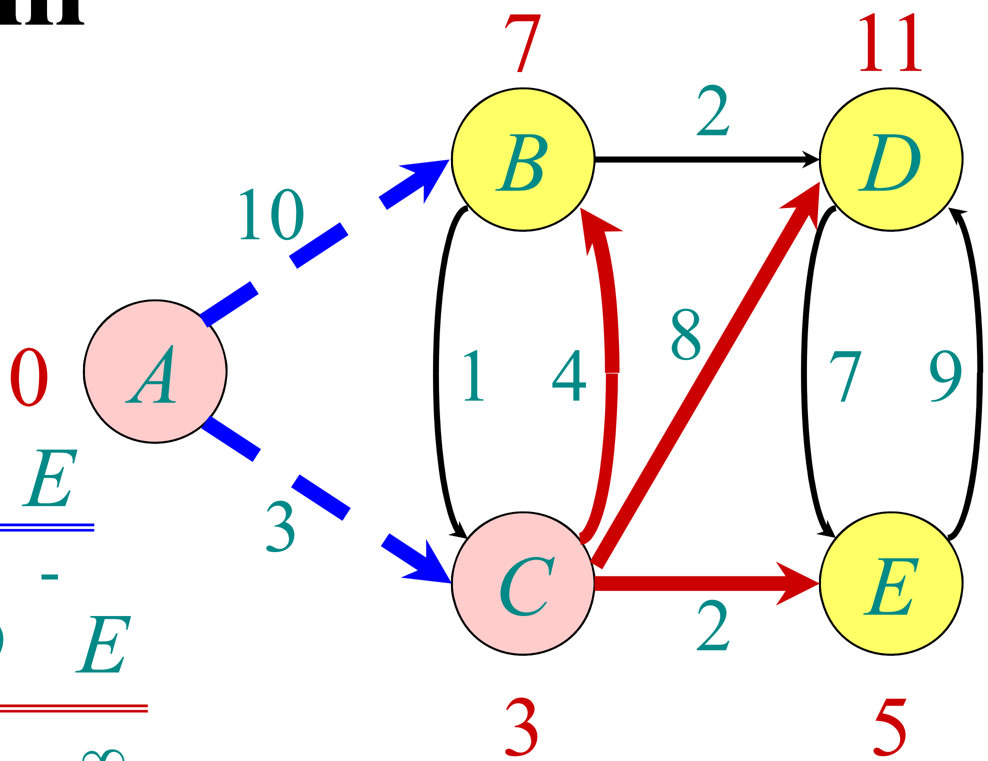
Example of Dijkstra's algorithm

Relax all edges leaving *C*:

S: { *A*, *C* }

π:	*A*	*B*	*C*	*D*	*E*
	–	A	A	–	-

Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	–	–
		7		11	5



```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

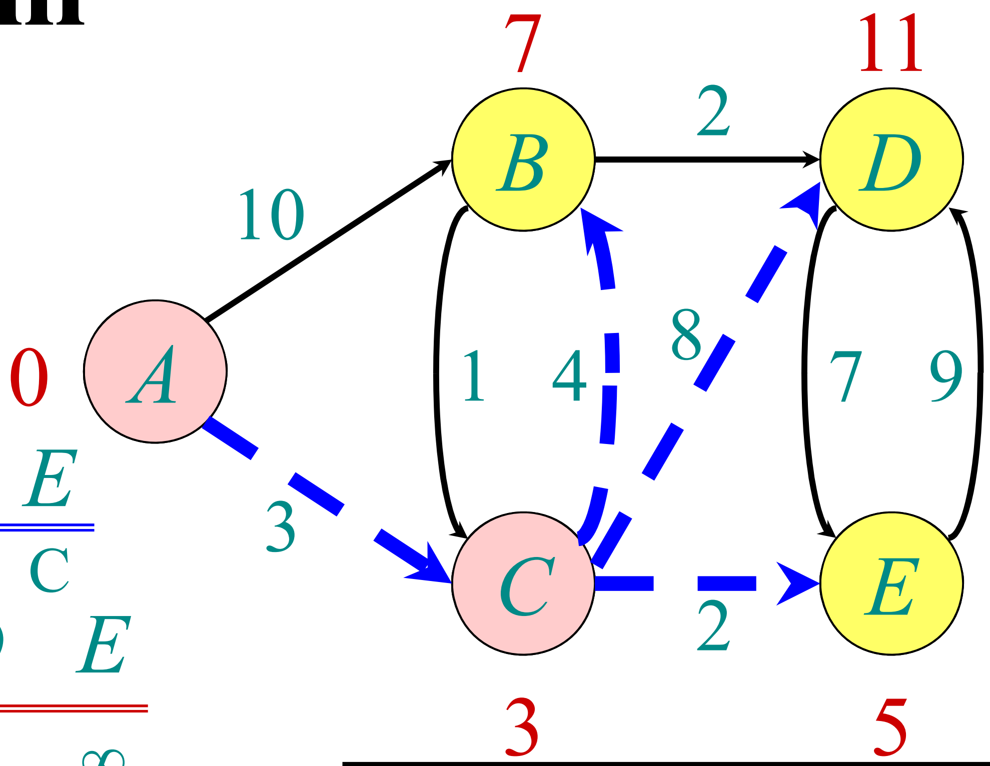
Example of Dijkstra's algorithm

Relax all edges leaving *C*:

S: { *A*, *C* }

π:	*A*	*B*	*C*	*D*	*E*
	–	C	A	C	C

Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	–	–
		7		11	5



```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

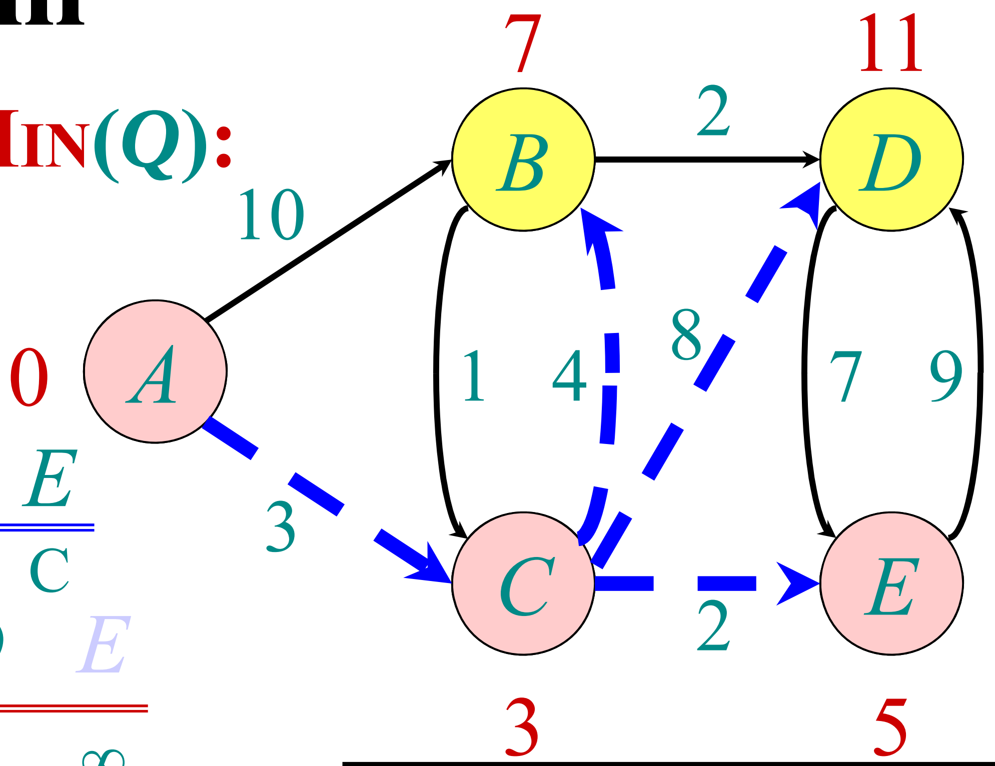
Example of Dijkstra's algorithm

"*E*" ← EXTRACT-MIN(*Q*):

S: { *A*, *C*, *E* }

π:	*A*	*B*	*C*	*D*	*E*
	–	C	A	C	C

Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	–	–
		7		11	5



```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

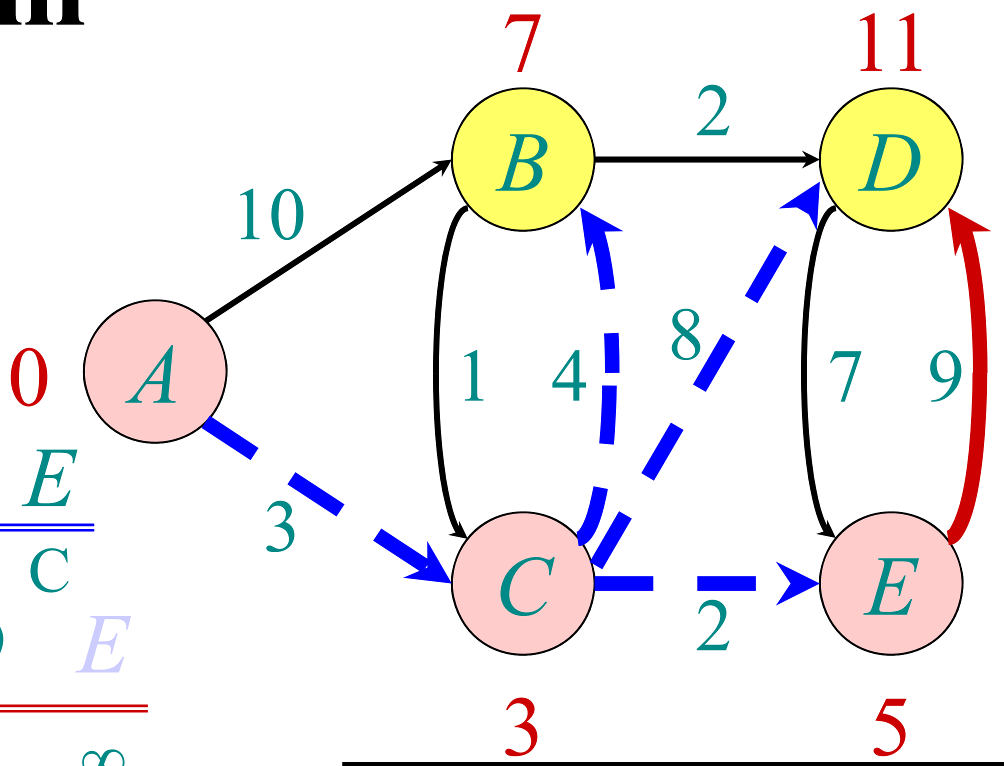
Example of Dijkstra's algorithm

Relax all edges leaving *E*:

S: { *A*, *C*, *E* }

π:	*A*	*B*	*C*	*D*	*E*
	–	C	A	C	C

Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	



```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN$(Q)$
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

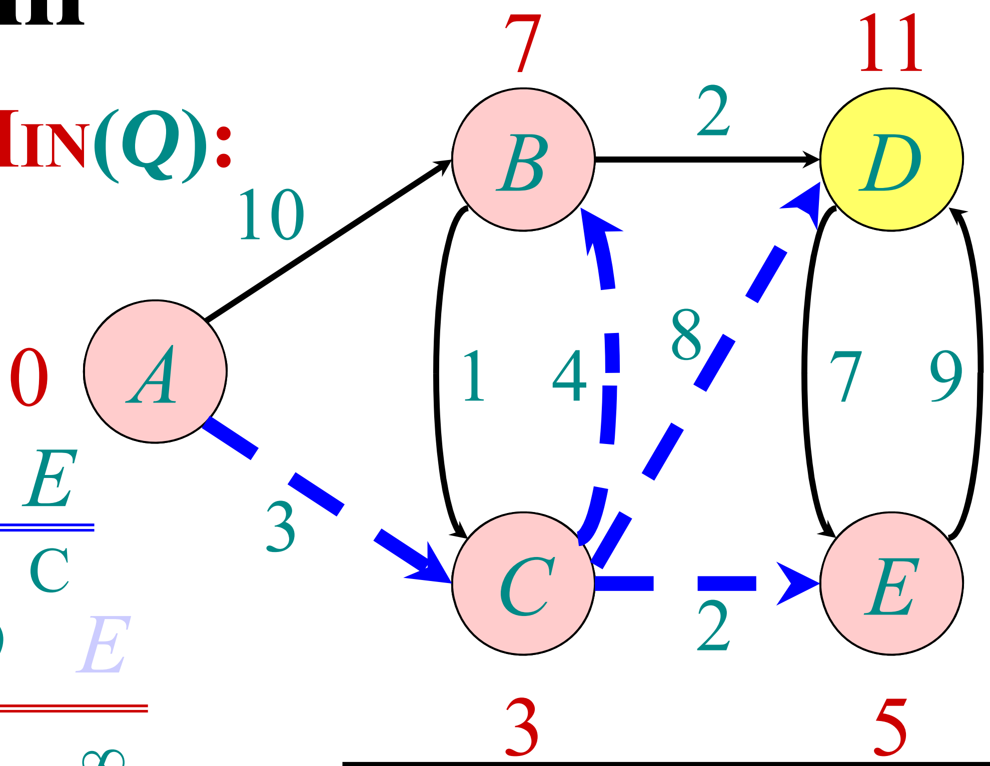
Example of Dijkstra's algorithm

"*B*" ← EXTRACT-MIN(*Q*):

S: { *A*, *C*, *E*, *B* }

π:	*A*	*B*	*C*	*D*	*E*
	–	C	A	C	C

Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

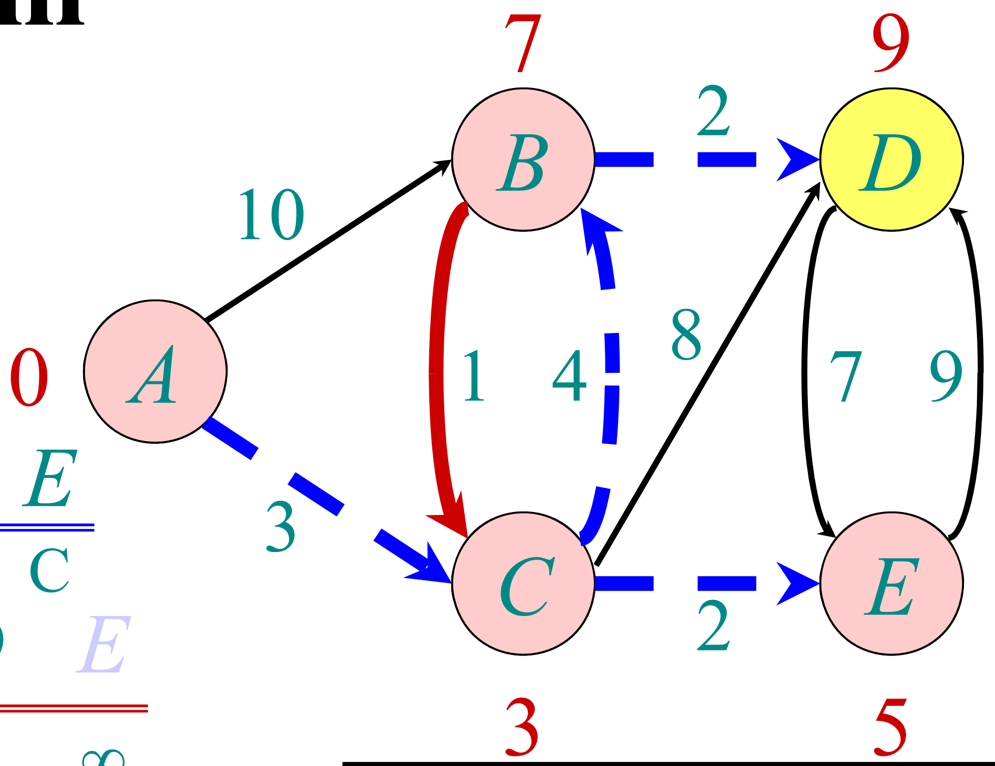


```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

Example of Dijkstra's algorithm

Relax all edges leaving *B*:

S: { *A*, *C*, *E*, *B* }



π:	*A*	*B*	*C*	*D*	*E*
	–	C	A	B	C

Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

Example of Dijkstra’s algorithm

“D” ← **EXTRACT-MIN(Q):**

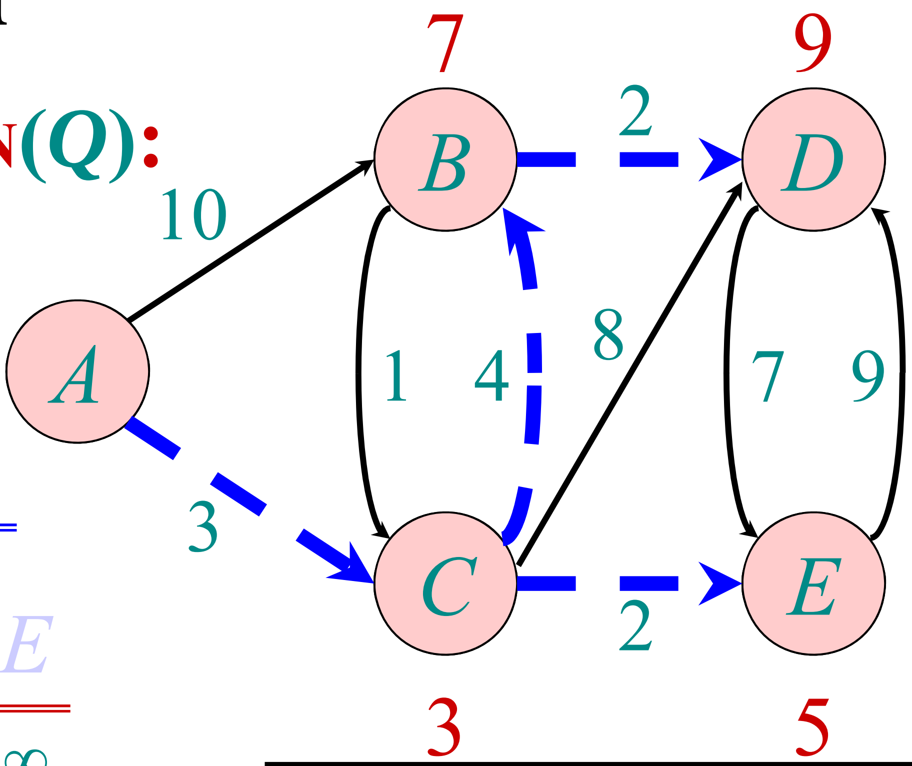
S: { A, C, E, B, D }

π:

A	B	C	D	E
–	C	A	B	C

Q:

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	
			9	



```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
```

Analysis of Dijkstra

```
while $Q \neq \emptyset$ do
    $u \leftarrow \textsc{Extract-Min}(Q)$
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
```

$|V|$ times

$degree(u)$ times

Handshaking Lemma $\Rightarrow \Theta(|E|)$ implicit DECREASE-KEY's.

Time $= \Theta(|V|) \cdot T_{\textsc{Extract-Min}} + \Theta(|E|) \cdot T_{\textsc{Decrease-Key}}$

Analysis of Dijkstra (continued)

Time = $\Theta(|V|)\cdot T_{\text{EXTRACT-MIN}} + \Theta(|E|)\cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(\|V\|)$	$O(1)$	$O(\|V\|^2)$
binary heap	$O(\log \|V\|)$	$O(\log \|V\|)$	$O(\|E\| \log \|V\|)$
Fibonacci heap	$O(\log \|V\|)$ amortized	$O(1)$ amortized	$O(\|E\| + \|V\| \log \|V\|)$ worst case

Correctness

Theorem. (i) For all $v \in S$: $d[v] = \delta(s, v)$
(ii) For all $v \notin S$: $d[v]$ = weight of shortest path from s to v that uses only (besides v itself) vertices in S.

Corollary. Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.

Correctness

Theorem. (i) For all $v \in S$: $d[v] = \delta(s, v)$
(ii) For all $v \notin S$: $d[v]$ = weight of shortest path from s to v that uses only (besides v itself) vertices in S.

Proof. By induction.

- Base: Before the while loop, $d[s]=0$ and $d[v]=\infty$ for all $v \neq s$, so (i) and (ii) are true.
- Step: Assume (i) and (ii) are true before an iteration; now we need to show they remain true after another iteration. Let u be the vertex added to S, so $d[u] \le d[v]$ for all other $v \notin S$.

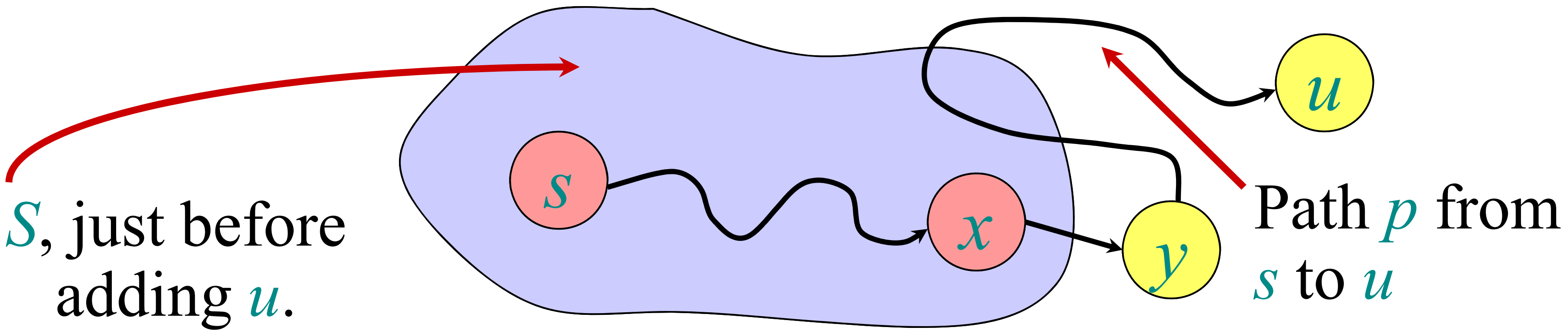
Correctness

Theorem. (i) For all $v \in S$: $d[v] = \delta(s, v)$
(ii) For all $v \notin S$: $d[v] =$ weight of shortest path from s to v that uses only (besides v itself) vertices in S.

- (i) Need to show that $d[u] = \delta(s, u)$. Assume the contrary.

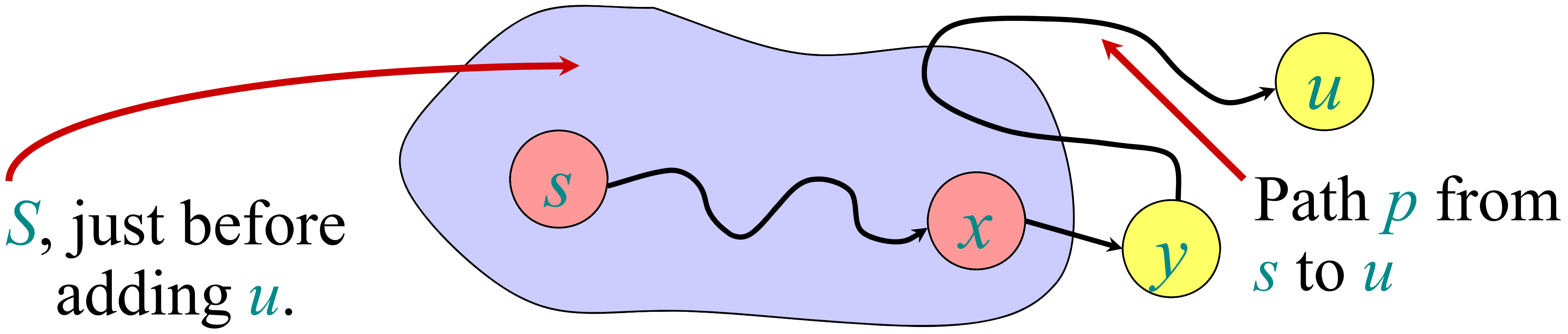
$\Rightarrow$ There is a path p from s to u with $w(p) < d[u]$. Because of (ii) that path uses vertices $\notin S$, in addition to u.

$\Rightarrow$ Let y be first vertex on p such that $y \notin S$.



Correctness

Theorem. (i) For all $v \in S$: $d[v] = \delta(s, v)$
(ii) For all $v \notin S$: $d[v] =$ weight of shortest path from s to v that uses only (besides v itself) vertices in S.



$\Rightarrow \underbrace{d[y] \leq w(p)}_{\text{weights are nonnegative}} \underbrace{< d[u]}_{\text{assumption about path}}$. Contradiction to the choice of u.

Correctness

Theorem. (i) For all $v \in S$: $d[v] = \delta(s, v)$
(ii) For all $v \notin S$: $d[v]$ = weight of shortest path from s to v that uses only (besides v itself) vertices in S.

- (ii) Let $v \notin S$. Let p be a shortest path from s to v that uses only (besides v itself) vertices in S.
  - p does not contain u: (ii) true by inductive hypothesis
  - p contains u: p consists of vertices in $S \backslash \{u\}$ and ends with an edge from u to v.
    $\Rightarrow$ $w(p)=d[u]+w(u,v)$, which is the value of $d[v]$ after adding u. So (ii) is true.

Unweighted graphs

Suppose $w(u, v) = 1$ for all $(u, v) \in E$. Can the code for Dijkstra be improved?

- Use a simple FIFO queue instead of a priority queue.
- ***Breadth-first search***

```
while $Q \neq \emptyset$
    do $u \leftarrow$ DEQUEUE($Q$)
        for each $v \in Adj[u]$
            do if $d[v] = \infty$
                then $d[v] \leftarrow d[u] + 1$
                     ENQUEUE($Q, v$)
```

```
while $Q \neq \emptyset$ do
    $u \leftarrow$ EXTRACT-MIN($Q$)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$
```

Analysis: Time $= O(|V| + |E|)$.

Correctness of BFS

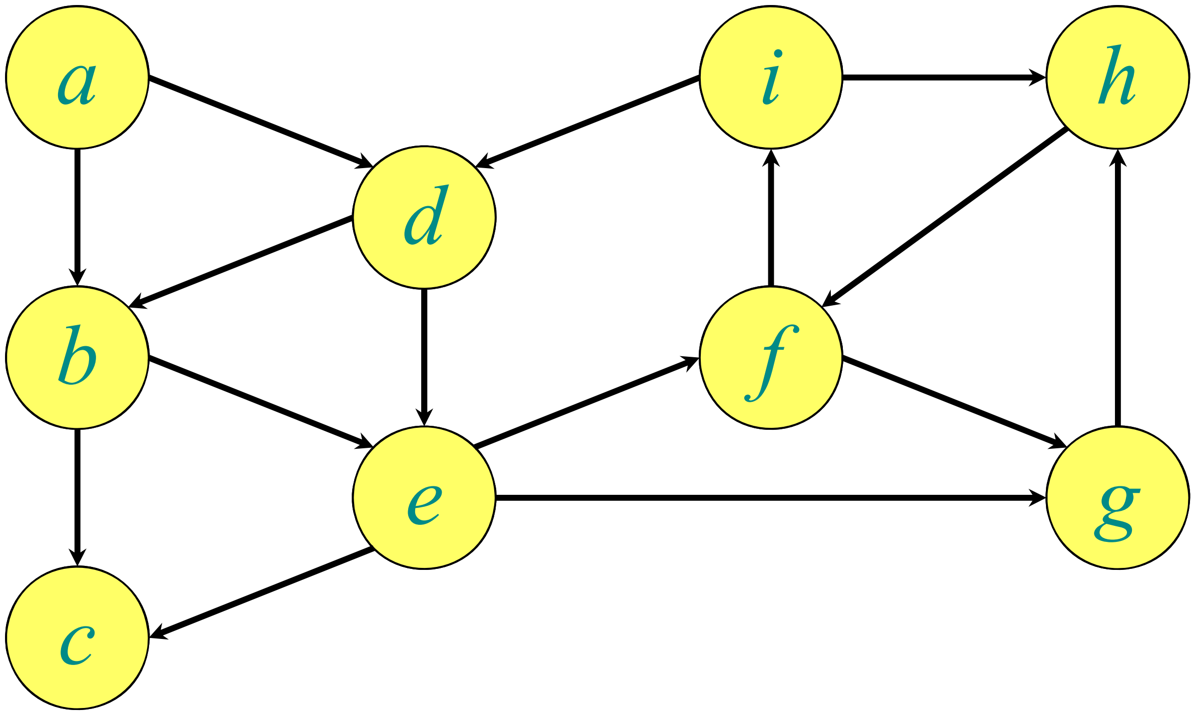
```
while $Q \neq \emptyset$
    do $u \leftarrow$ DEQUEUE$(Q)$
        for each $v \in Adj[u]$
            do if $d[v] = \infty$
                then $d[v] \leftarrow d[u] + 1$
                     ENQUEUE$(Q, v)$
```

Key idea:

The FIFO Q in breadth-first search mimics the priority queue Q in Dijkstra.

- **Invariant:** v comes after u in Q implies that $d[v] = d[u]$ or $d[v] = d[u] + 1$.

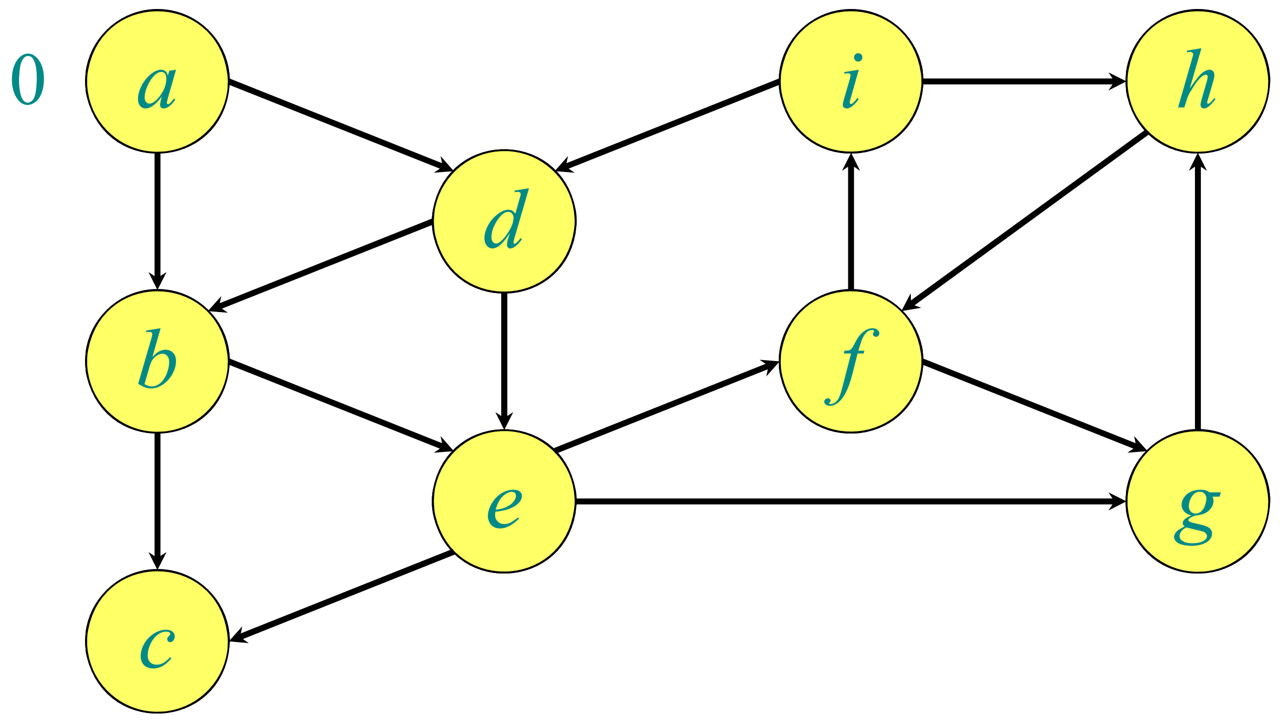
Example of breadth-first search



Q:

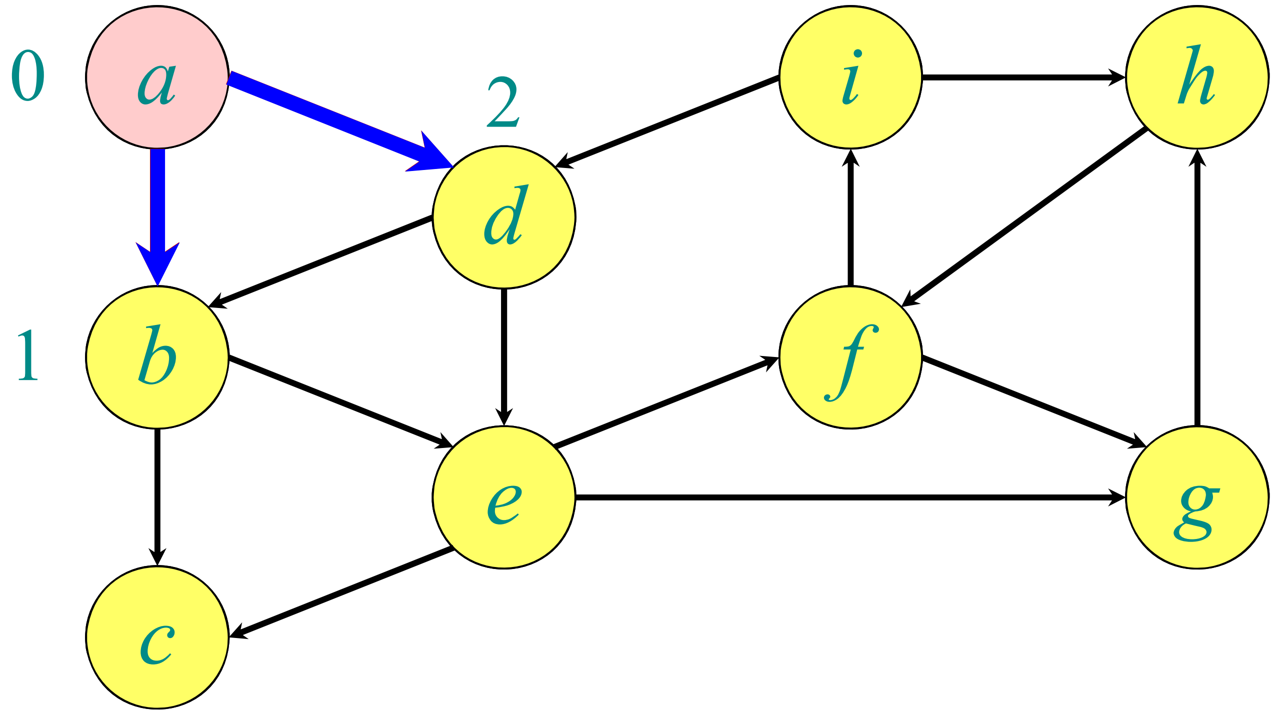
d[*v*]

Example of breadth-first search



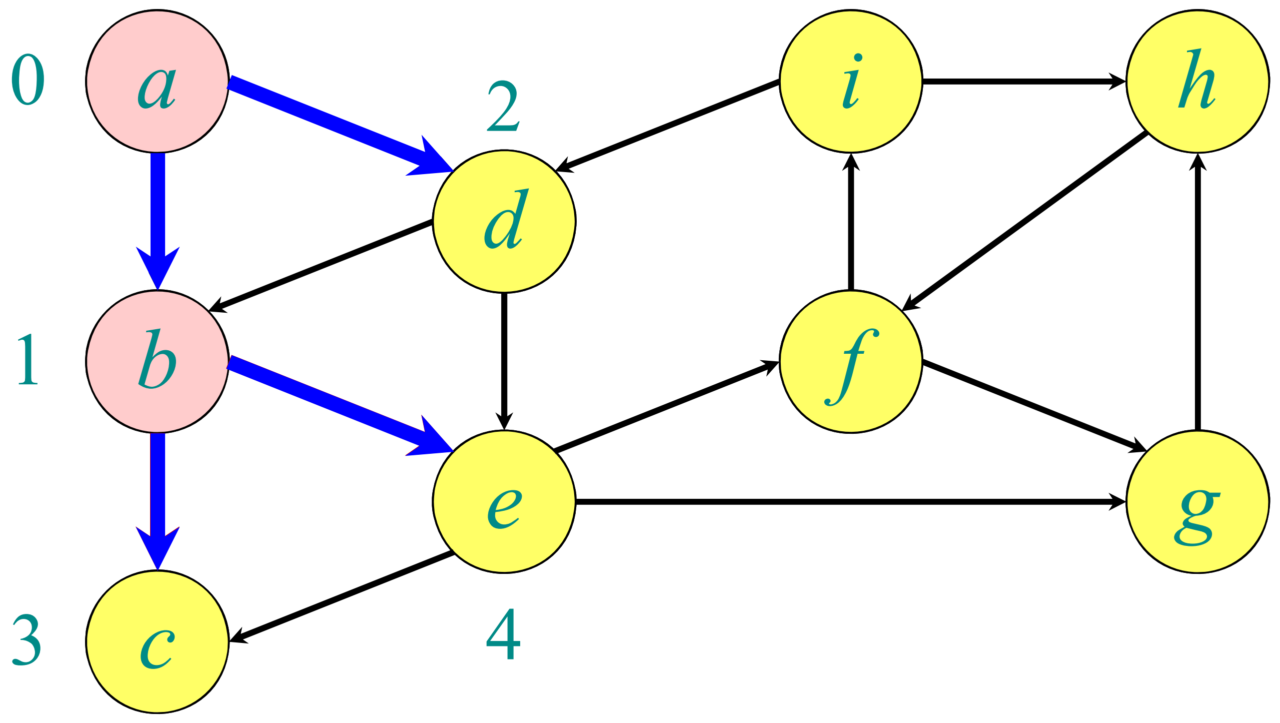
$$\begin{array}{rl} & 0 \\ Q\text{:} & a \\ d[v] & 0 \end{array}$$

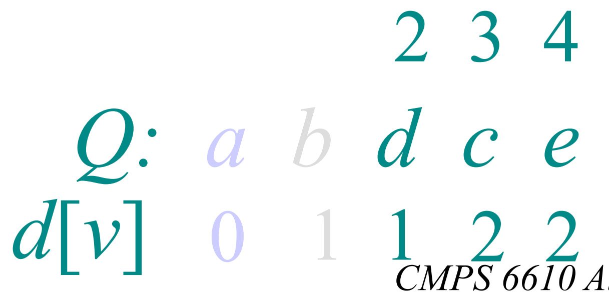
Example of breadth-first search



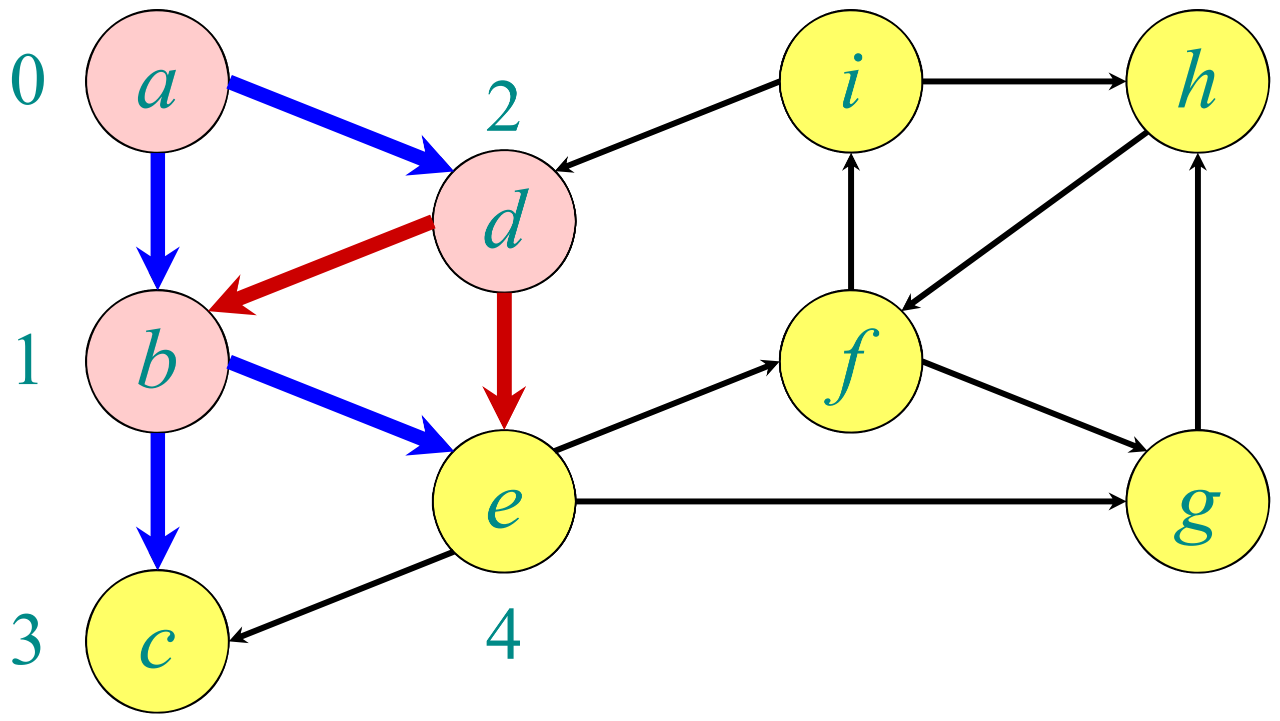
		1	2
Q:	*a*	*b*	*d*
d[*v*]	0	1	1

Example of breadth-first search



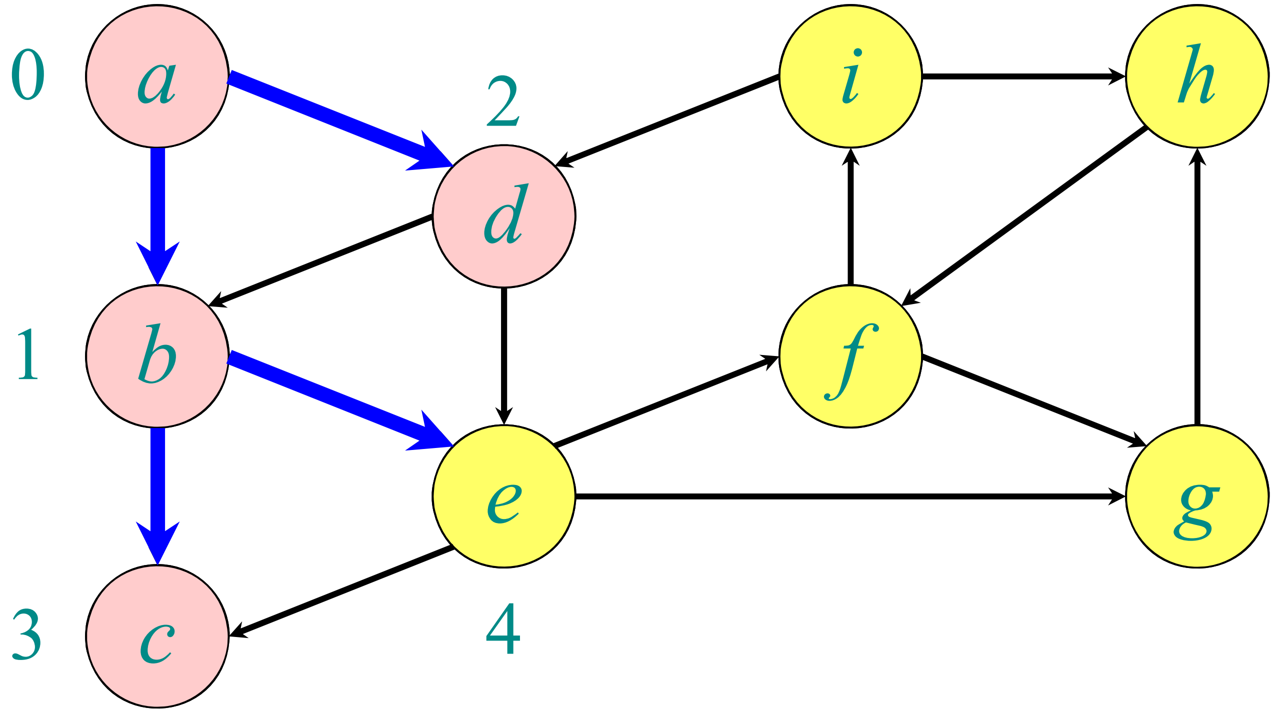


Example of breadth-first search



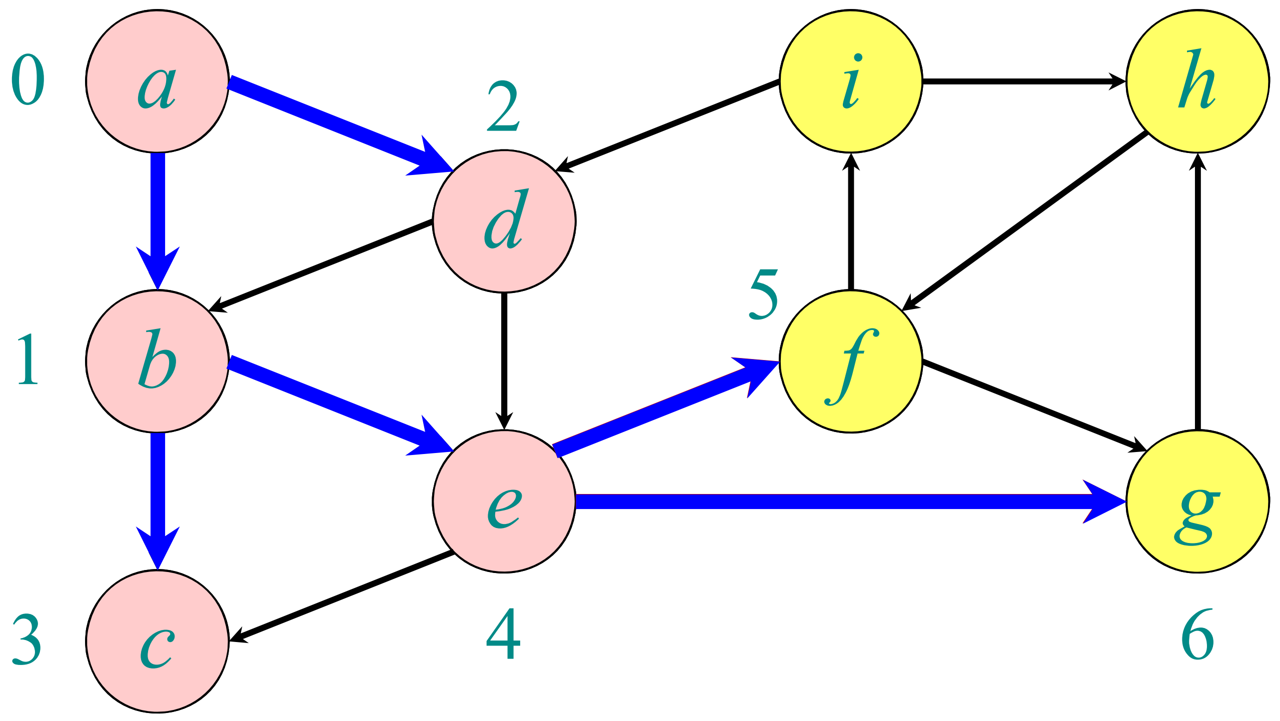
				3	4
Q:	*a*	*b*	*d*	*c*	*e*
d[*v*]	0	1	1	2	2

Example of breadth-first search



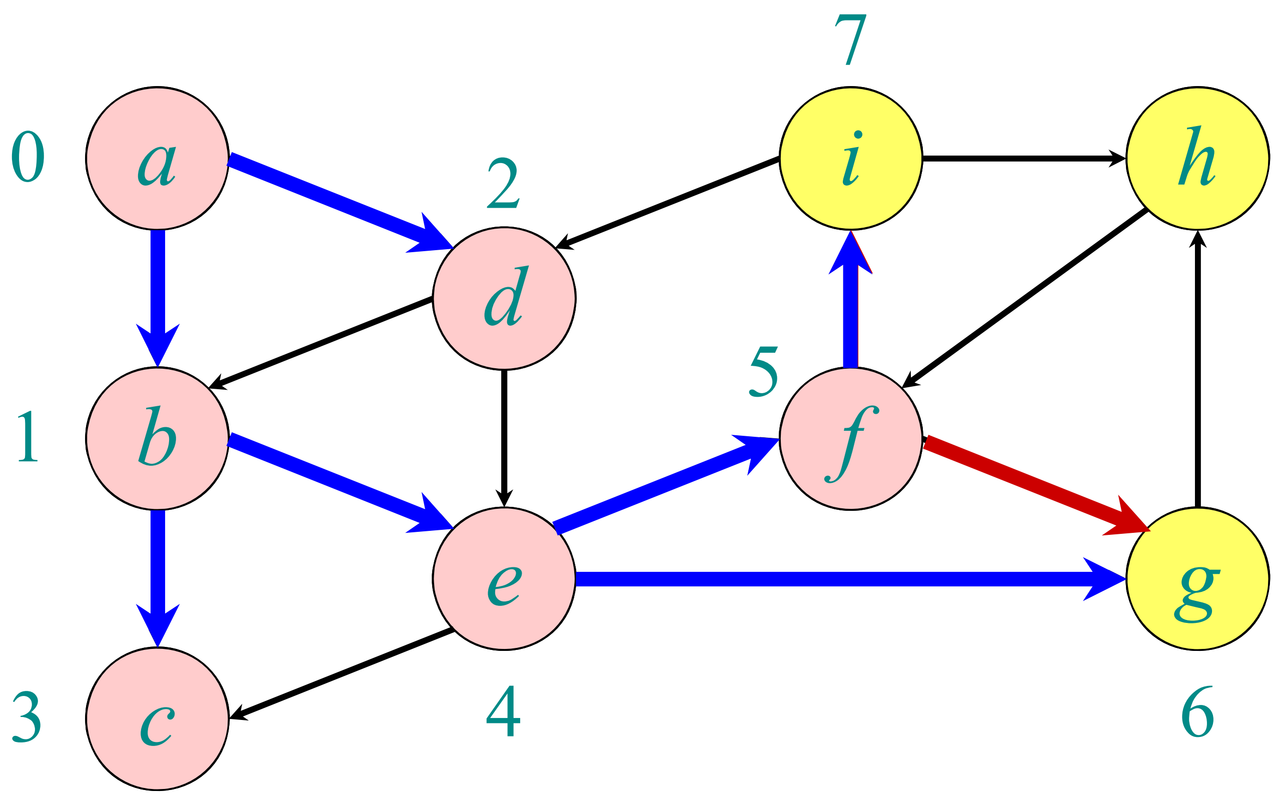
					4
Q:	*a*	*b*	*d*	*c*	*e*
d[*v*]	0	1	1	2	2

Example of breadth-first search



						5	6
Q:	*a*	*b*	*d*	*c*	*e*	*f*	*g*
d[*v*]	0	1	1	2	2	3	3

Example of breadth-first search

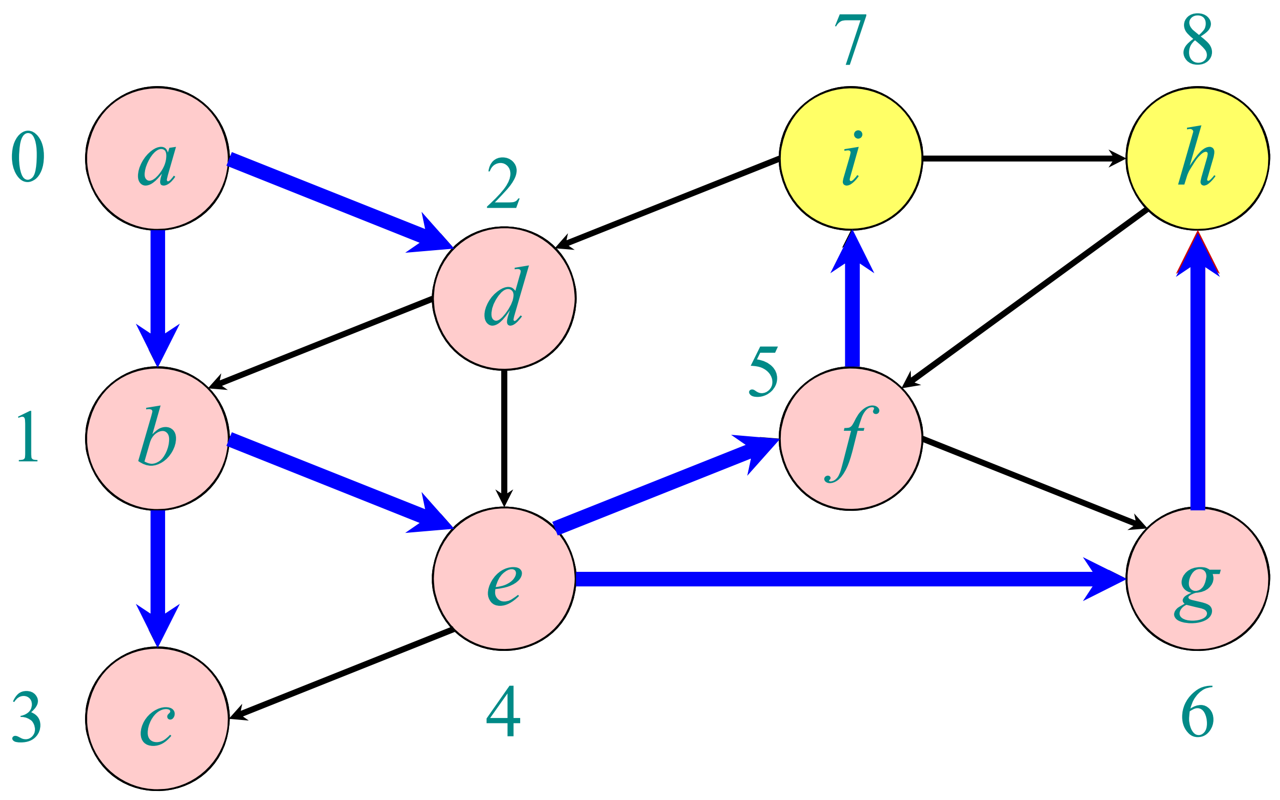


6 7

Q: *a b d c e f g i*

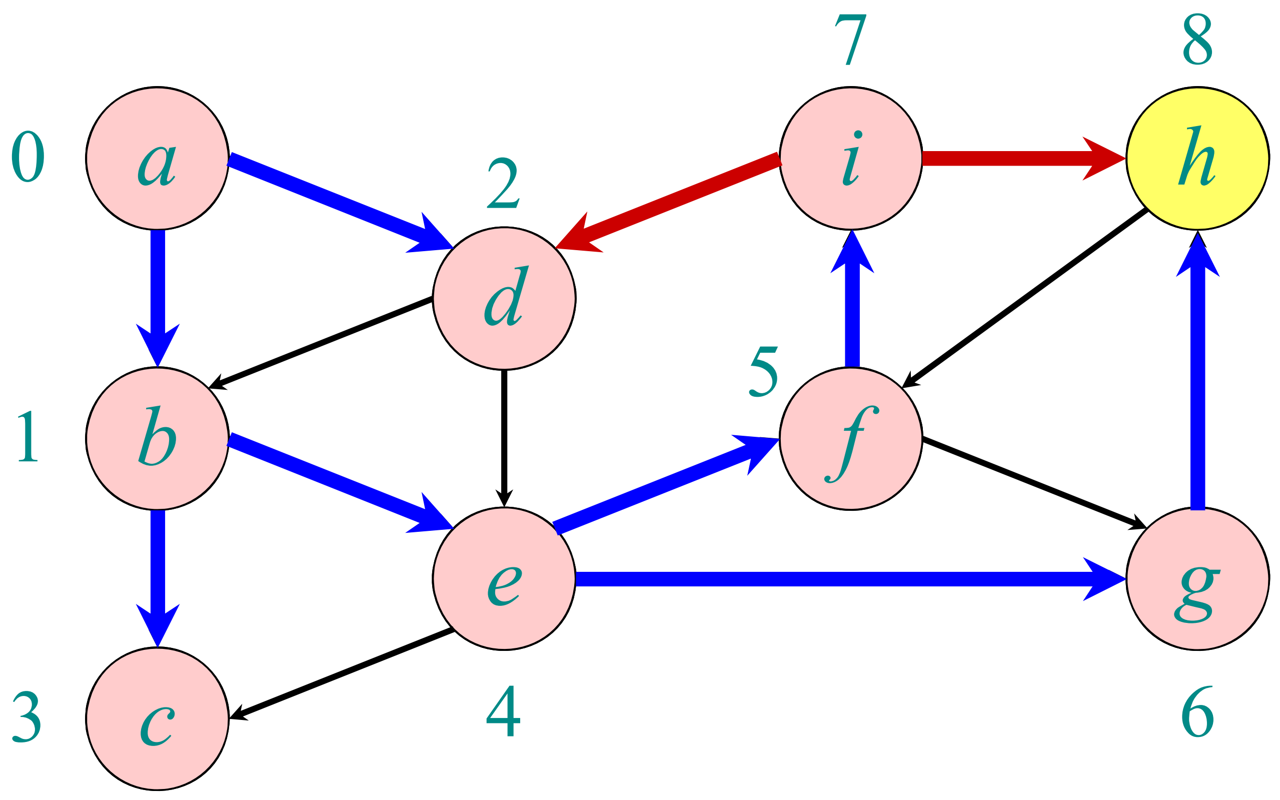
d[*v*] 0 1 1 2 2 3 3 4

Example of breadth-first search



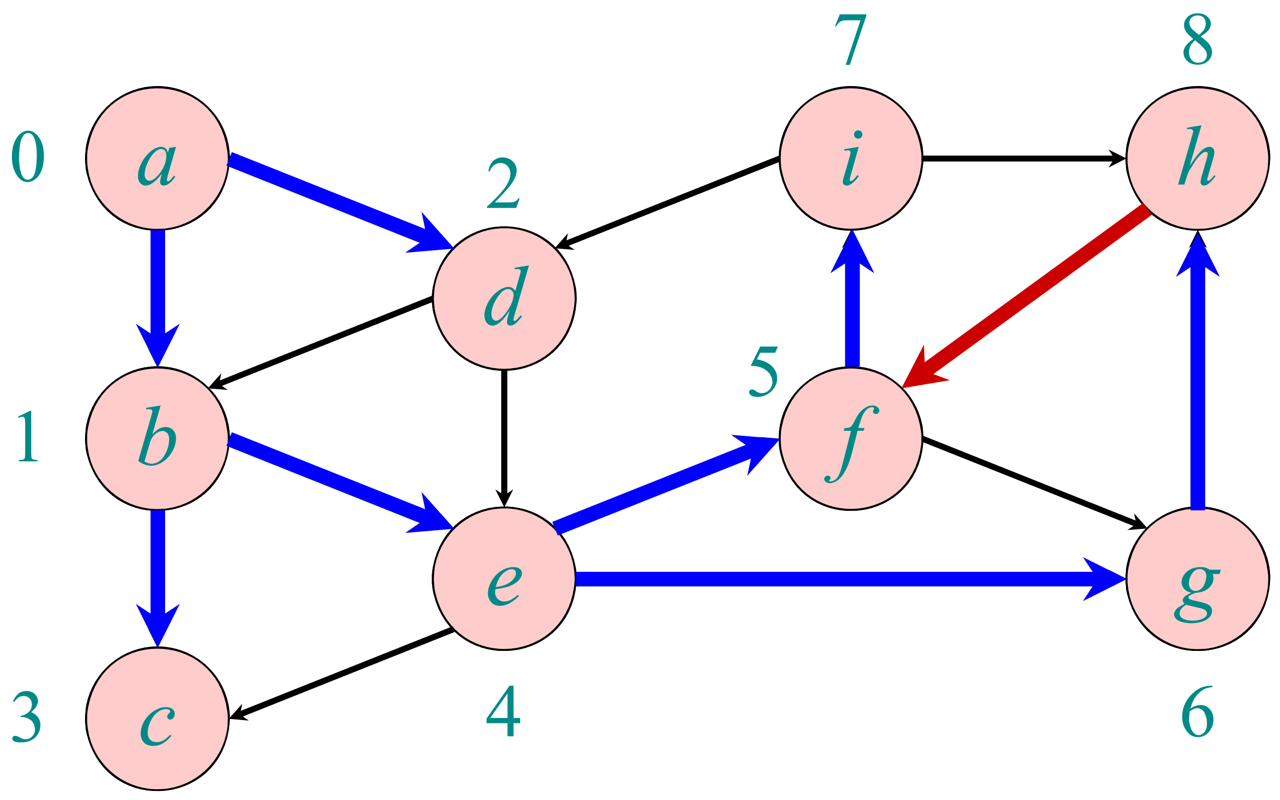
								7	8
Q:	*a*	*b*	*d*	*c*	*e*	*f*	*g*	*i*	*h*
d[*v*]	0	1	1	2	2	3	3	4	4

Example of breadth-first search



									8
Q:	a	b	d	c	e	f	g	i	h
$d[v]$	0	1	1	2	2	3	3	4	4

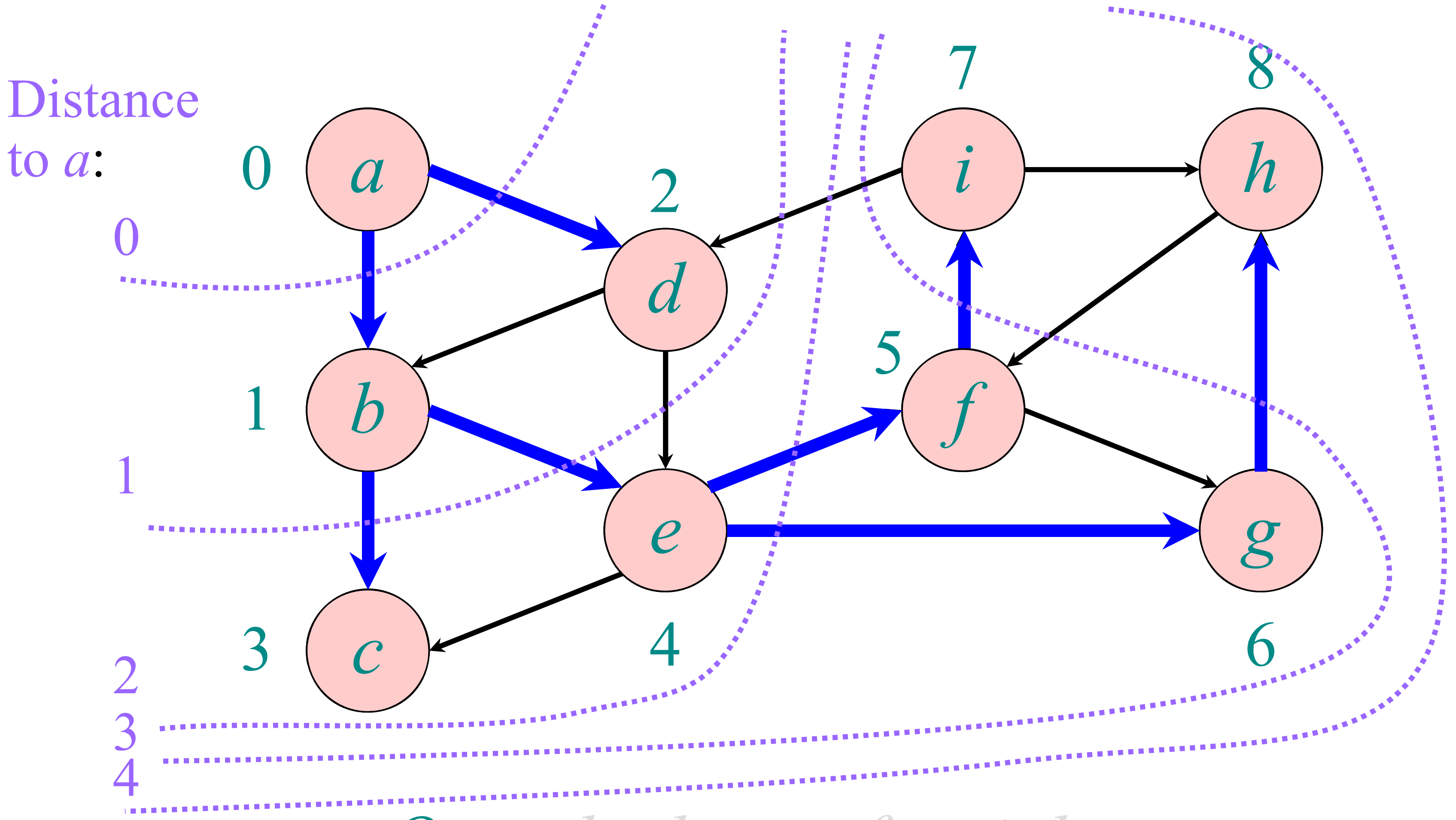
Example of breadth-first search



Q: a b d c e f g i h

$d[v]$ 0 1 1 2 2 3 3 4 4

Example of breadth-first search



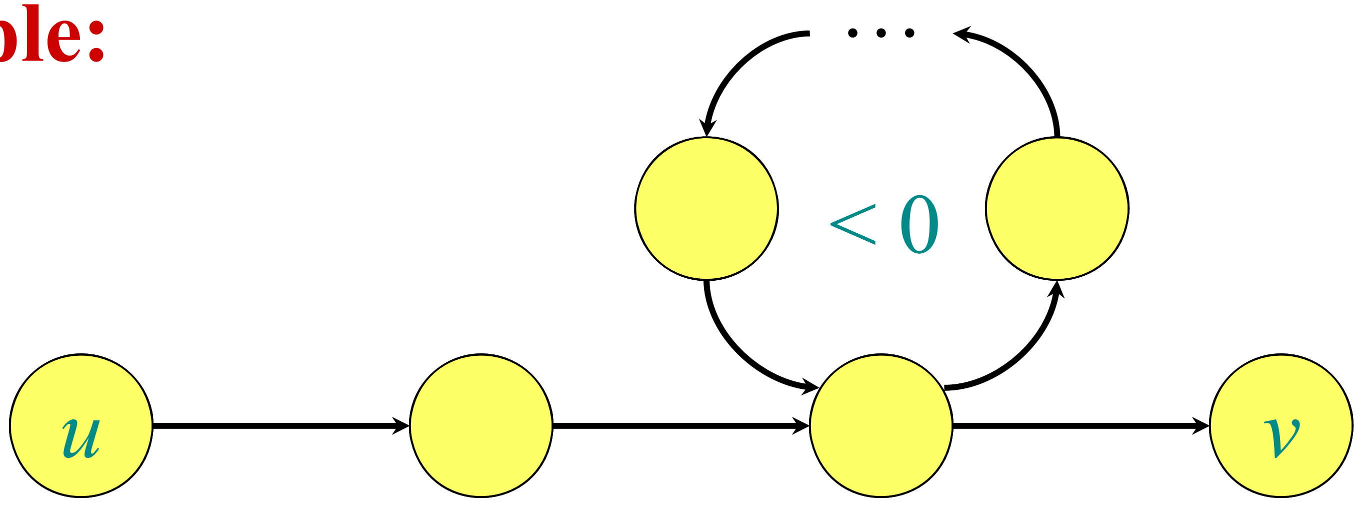
Q: *a b d c e f g i h*

d[*v*] 0 1 1 2 2 3 3 4 4

Negative-weight cycles

Recall: If a graph $G = (V, E)$ contains a negative-weight cycle, then some shortest paths may not exist.

Example:



Bellman-Ford algorithm: Finds all shortest-path weights from a ***source*** $s \in V$ to all $v \in V$ or determines that a negative-weight cycle exists.

Bellman-Ford algorithm

```
$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
    do $d[v] \leftarrow \infty$
```
$\}$ initialization

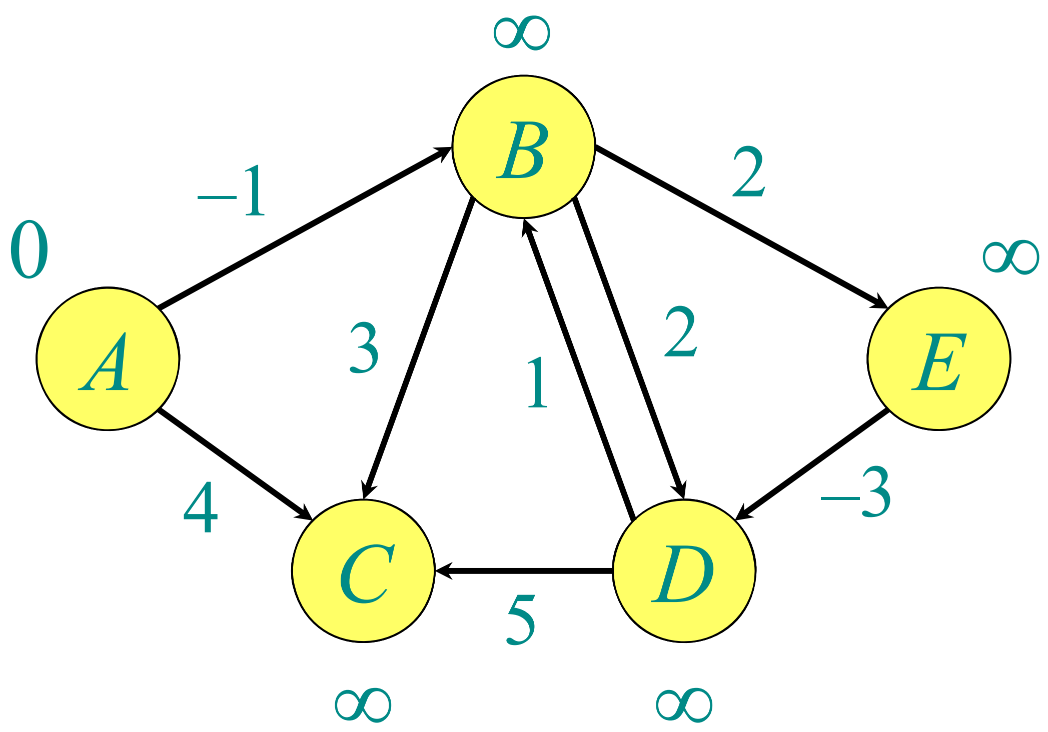
```
for $i \leftarrow 1$ to $|V| - 1$ do
  for each edge $(u, v) \in E$ do
      if $d[v] > d[u] + w(u, v)$ then
          $d[v] \leftarrow d[u] + w(u, v)$
          $\pi[v] \leftarrow u$
```
$\}$ ***relaxation step***

```
for each edge $(u, v) \in E$
    do if $d[v] > d[u] + w(u, v)$
           then report that a negative-weight cycle exists
```

At the end, $d[v] = \delta(s, v)$. Time $= O(|V||E|)$.

Example of Bellman-Ford

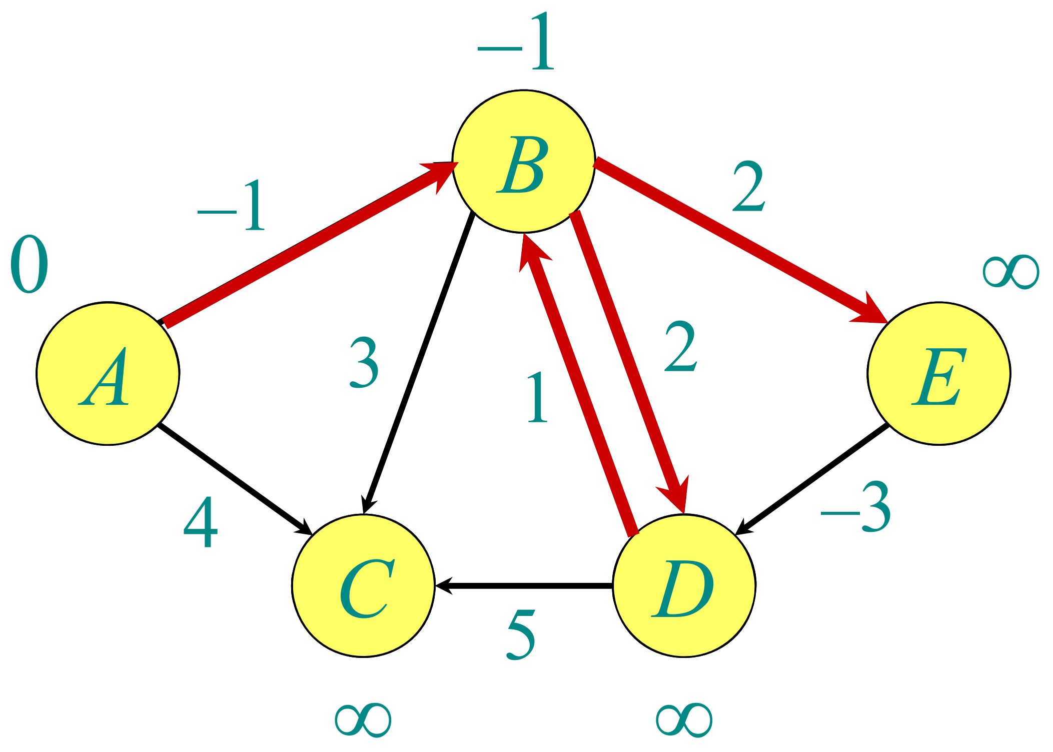
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*



A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞

Example of Bellman-Ford

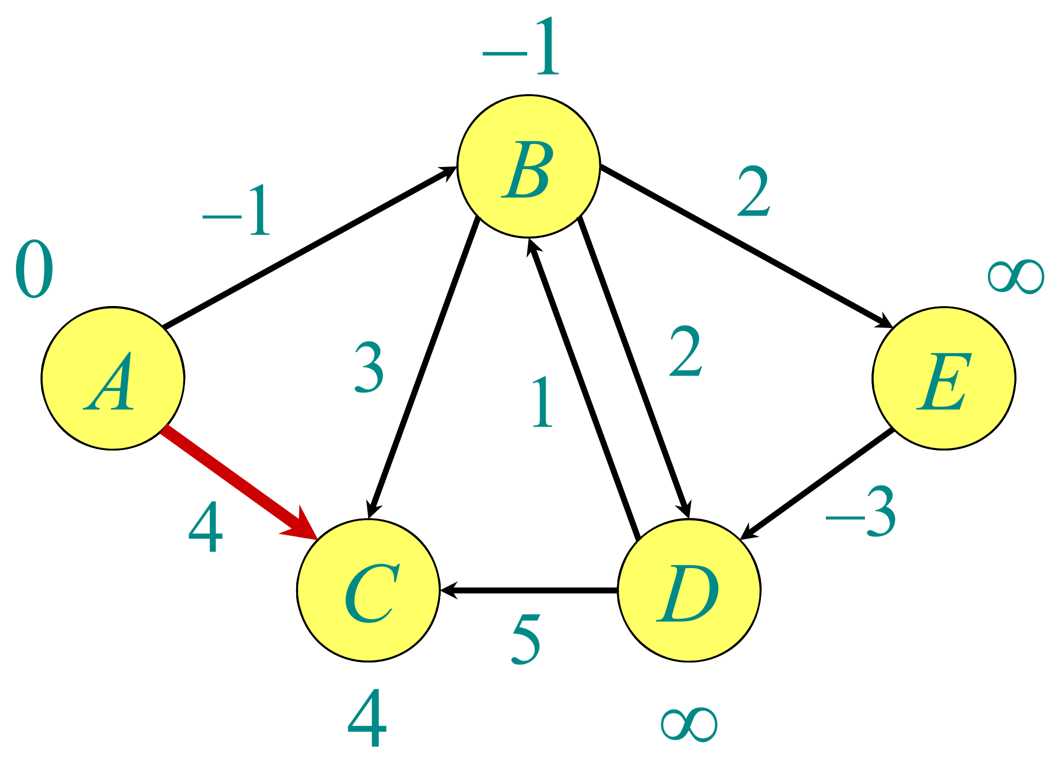
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*



A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞

Example of Bellman-Ford

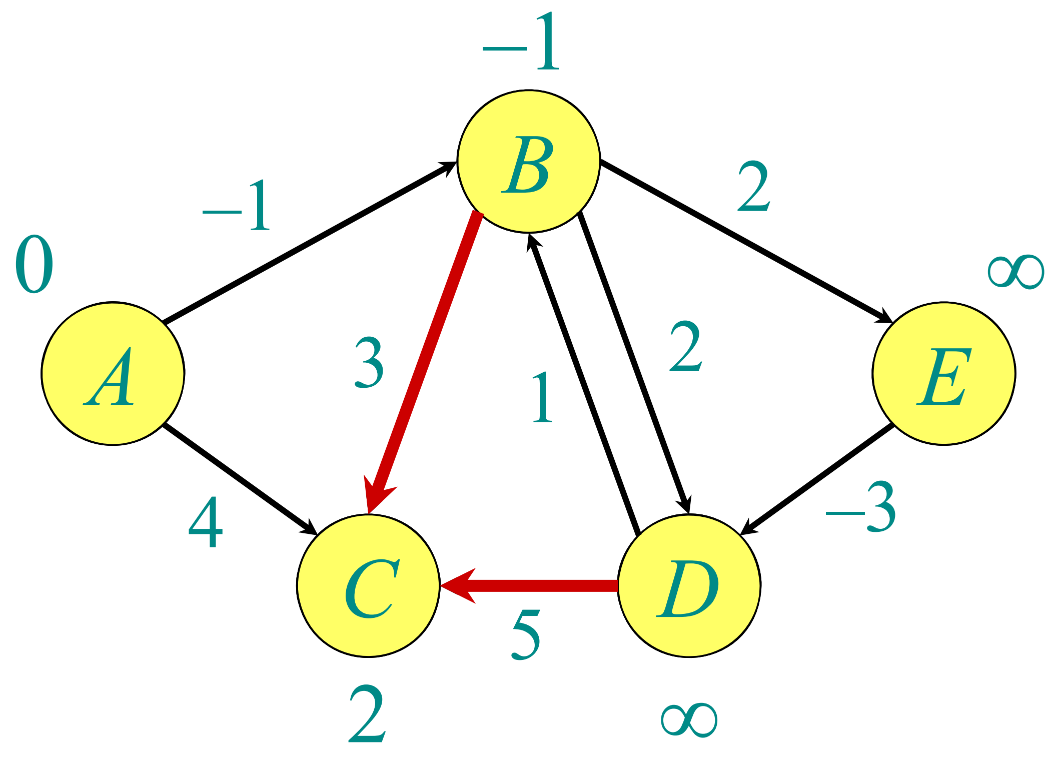
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*



A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞

Example of Bellman-Ford

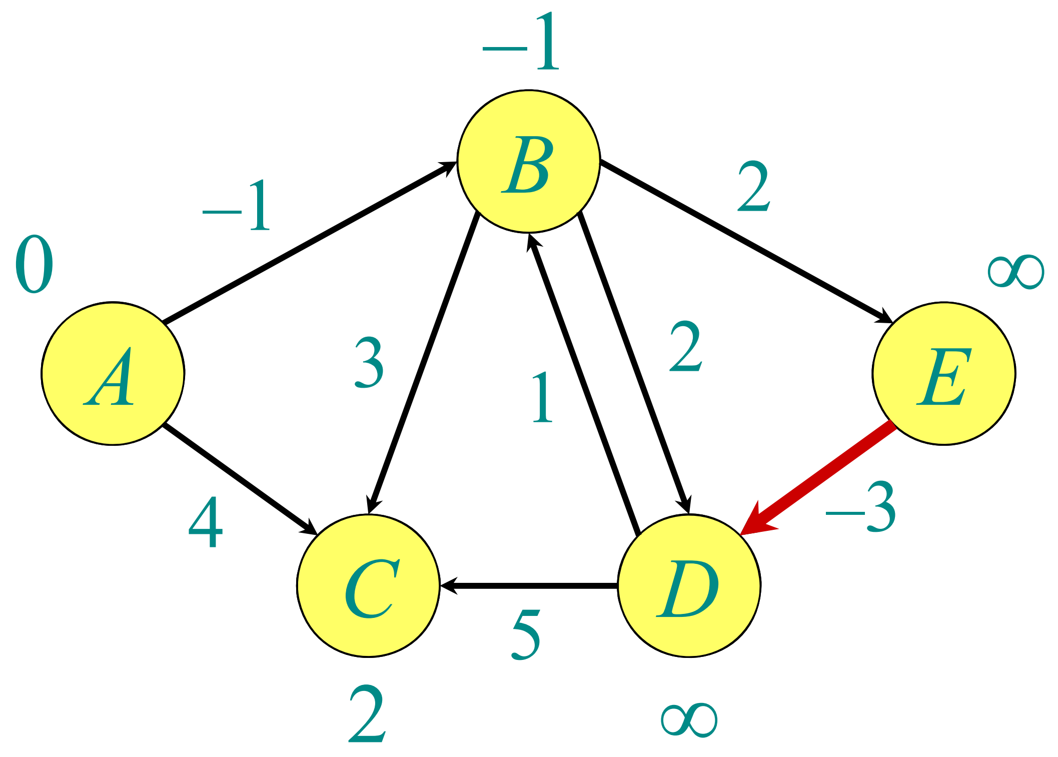
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*



A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞

Example of Bellman-Ford

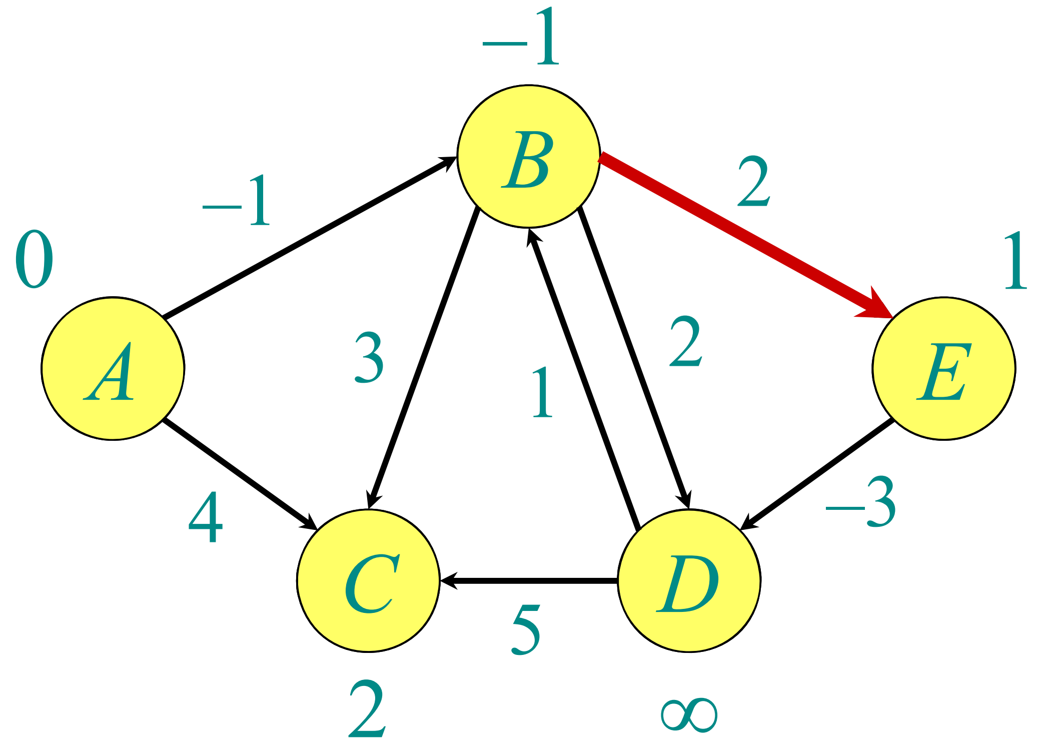
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*



A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞

Example of Bellman-Ford

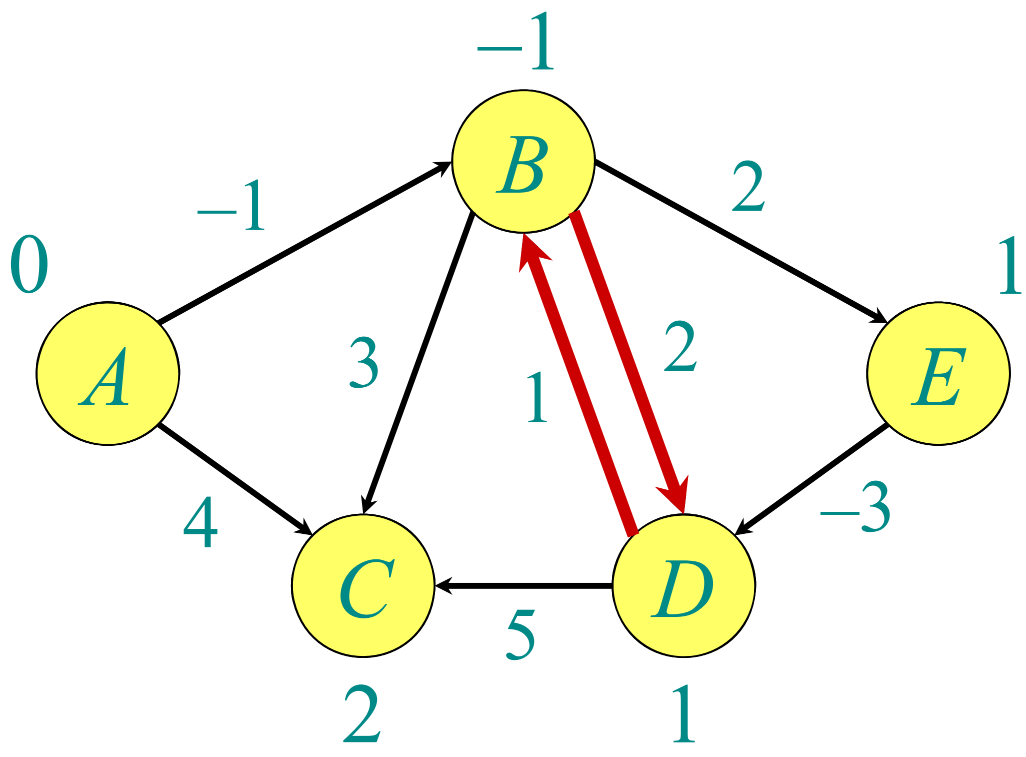
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*



A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞
0	–1	2	∞	1

Example of Bellman-Ford

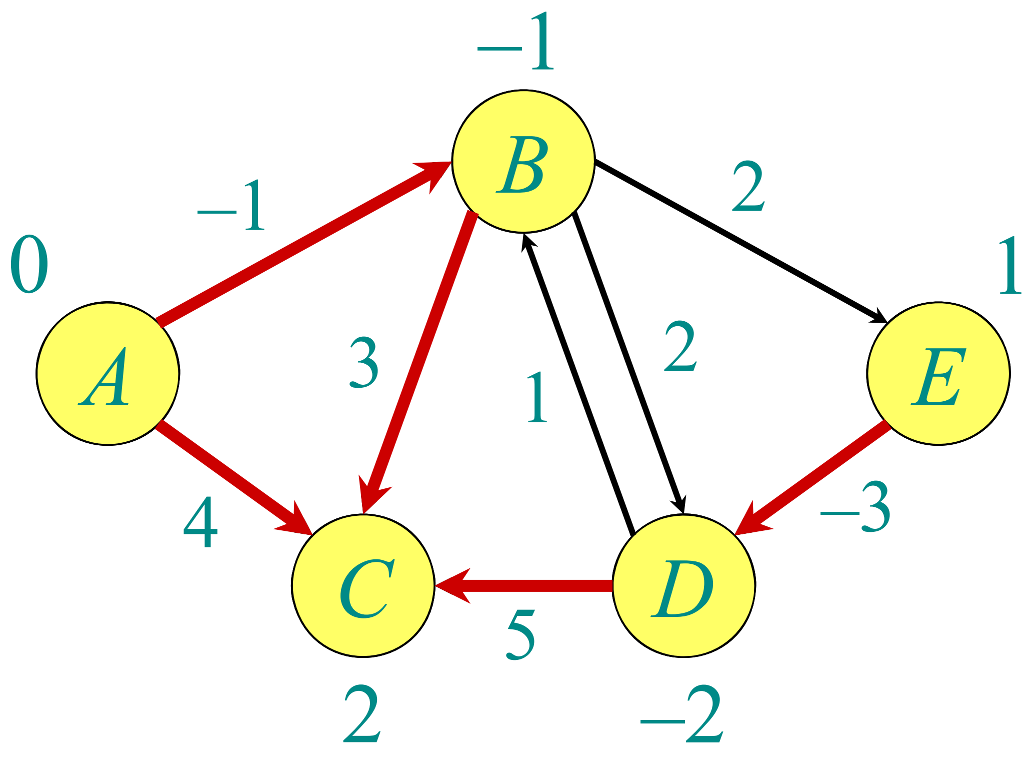
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*



A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞
0	–1	2	∞	1
0	–1	2	1	1

Example of Bellman-Ford

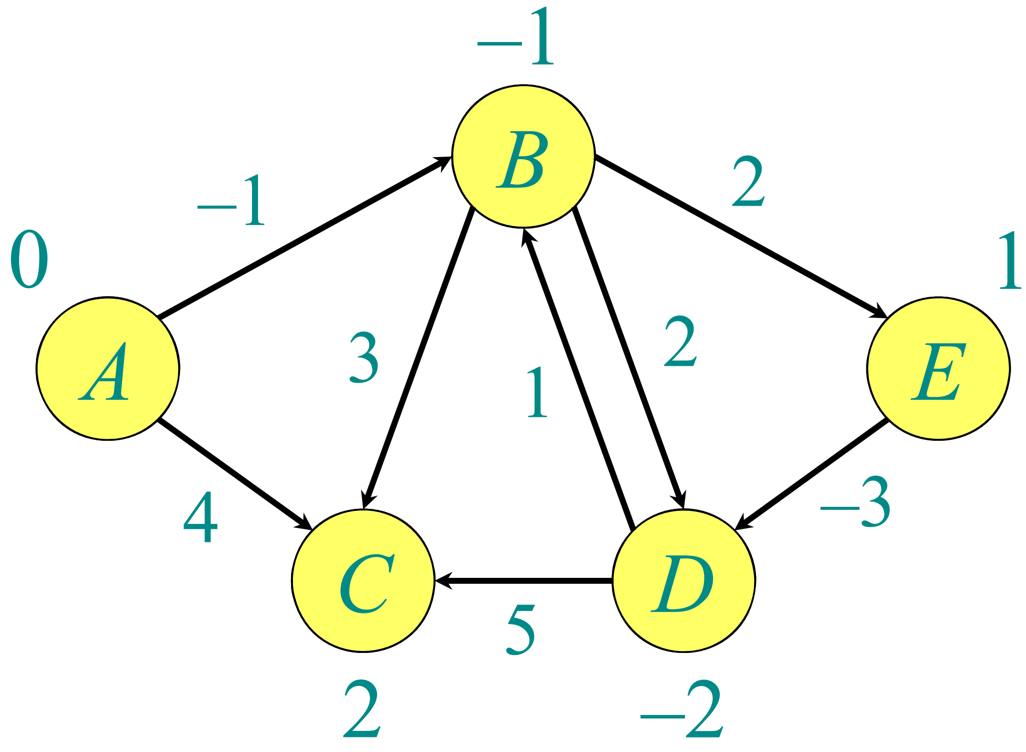
Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*



A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞
0	–1	2	∞	1
0	–1	2	1	1
0	–1	2	–2	1

Example of Bellman-Ford

Order of edges: *(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)*



A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞
0	–1	2	∞	1
0	–1	2	1	1
0	–1	2	–2	1

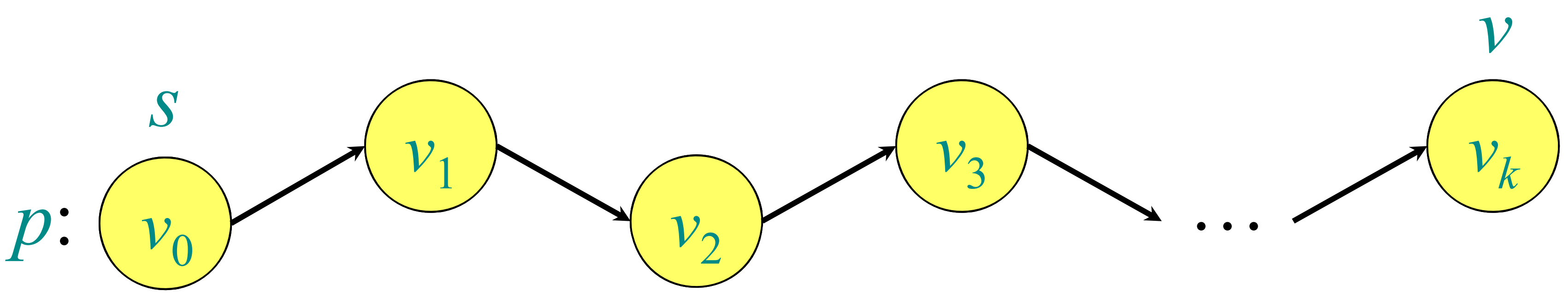
… and 2 more iterations

Note: *d*-values decrease monotonically.

Correctness

Theorem. If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.

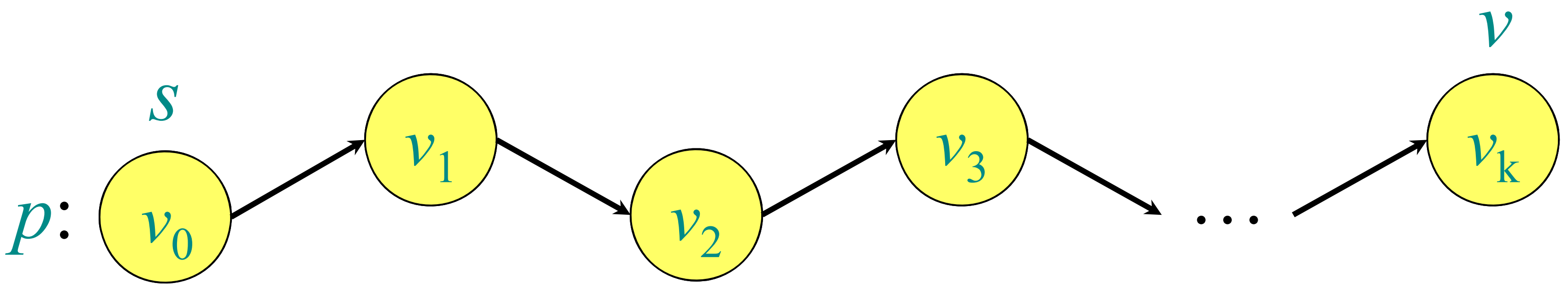
Proof. Let $v \in V$ be any vertex, and consider a shortest path p from s to v with the minimum number of edges.



Since p is a shortest path, we have

$$\delta(s, v_i) = \delta(s, v_{i-1}) + w(v_{i-1}, v_i) .$$

Correctness (continued)



Initially, $d[v_0] = 0 = \delta(s, v_0)$, and $d[s]$ is unchanged by subsequent relaxations.

- After 1 pass through E, we have $d[v_1] = \delta(s, v_1)$.
- After 2 passes through E, we have $d[v_2] = \delta(s, v_2)$.

  …
- After k passes through E, we have $d[v_k] = \delta(s, v_k)$.

Since G contains no negative-weight cycles, p is simple. Longest simple path has $\leq |V| - 1$ edges. □

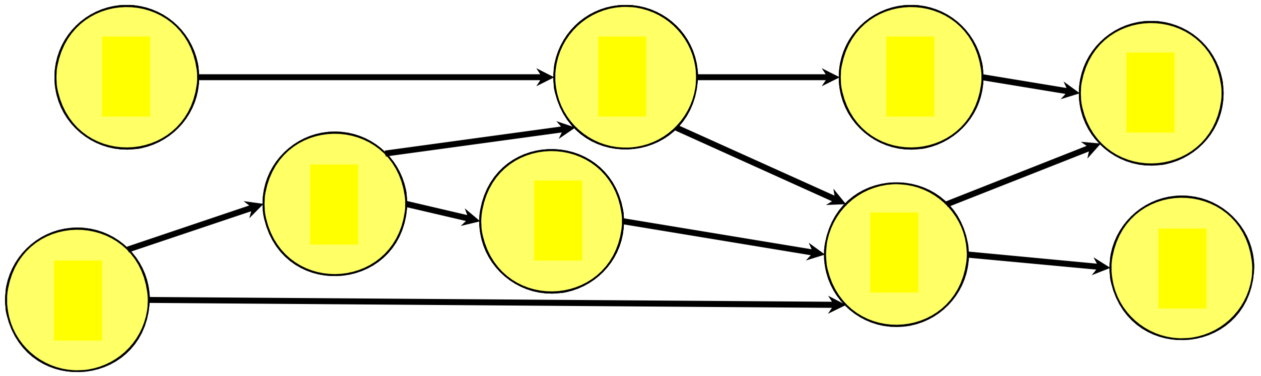
Detection of negative-weight cycles

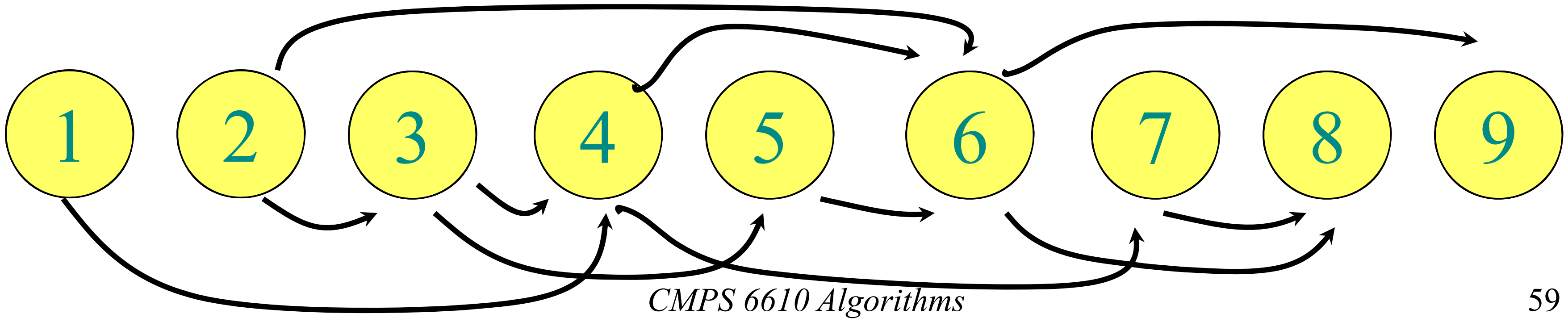
Corollary. If a value $d[v]$ fails to converge after $|V| - 1$ passes, there exists a negative-weight cycle in G reachable from s. $\square$

DAG shortest paths

If the graph is a ***directed acyclic graph*** (***DAG***), we first ***topologically sort*** the vertices.

- Determine $f : V \to \{1, 2, \ldots, |V|\}$ such that $(u, v) \in E$ $\Rightarrow f(u) < f(v)$.

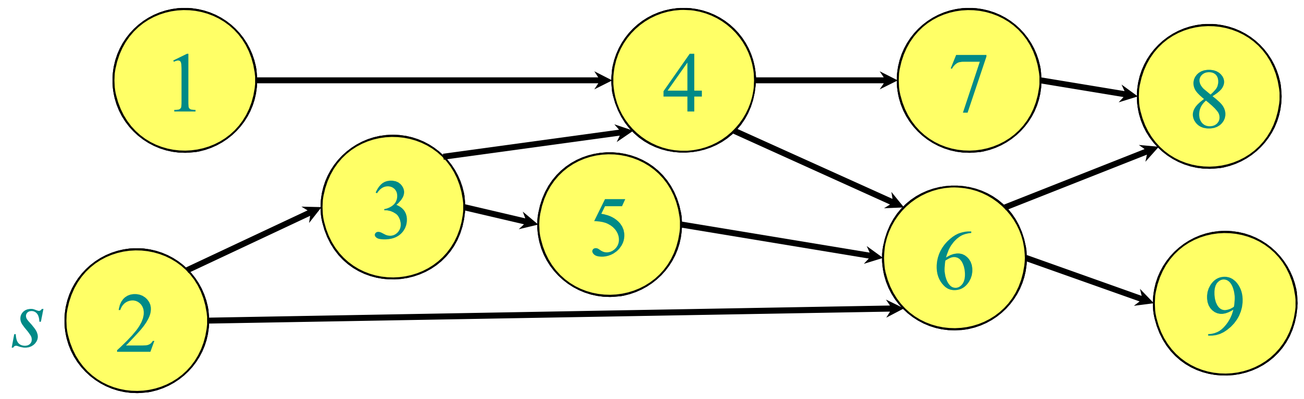




DAG shortest paths

If the graph is a ***directed acyclic graph*** (***DAG***), we first ***topologically sort*** the vertices.

- Determine $f : V \to \{1, 2, \ldots, |V|\}$ such that $(u, v) \in E \Rightarrow f(u) < f(v)$.
- $O(|V| + |E|)$ time



- Walk through the vertices $u \in V$ in this order, relaxing the edges in $Adj[u]$, thereby obtaining the shortest paths from s in a total of $O(|V| + |E|)$ time.

Shortest paths

Single-source shortest paths

- Nonnegative edge weights
  - Dijkstra's algorithm: $O(|E| + |V| \log |V|)$
- General: Bellman-Ford: $O(|V||E|)$
- DAG: One pass of Bellman-Ford: $O(|V| + |E|)$

All-pairs shortest paths

All-pairs shortest paths

Input: Digraph $G = (V, E)$, where $|V| = n$, with edge-weight function $w : E \to \mathbb{R}$.
Output: $n \times n$ matrix of shortest-path lengths $\delta(i, j)$ for all $i, j \in V$.

Algorithm #1:
- Run Bellman-Ford once from each vertex.
- Time $= O(|V|^2 |E|)$.
- But: Dense graph $\Rightarrow$ $O(|V|^4)$ time.

Shortest paths

Single-source shortest paths

- Nonnegative edge weights
  - Dijkstra's algorithm: $O(|E| + |V| \log |V|)$
- General: Bellman-Ford: $O(|V||E|)$
- DAG: One pass of Bellman-Ford: $O(|V| + |E|)$

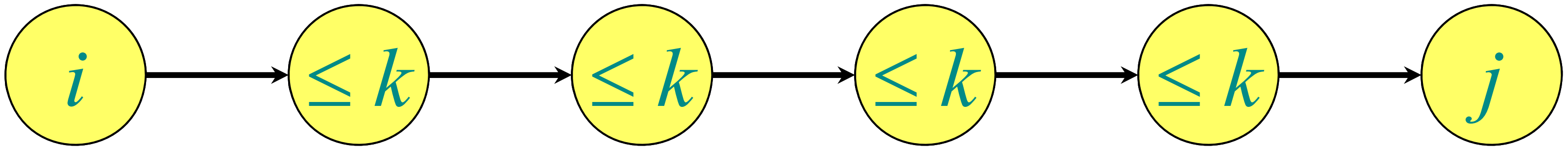
All-pairs shortest paths

- Nonnegative edge weights
  - Dijkstra's algorithm $|V|$ times: $O(|V||E|+|V|^2 \log |V|)$
- General
  - Bellman-Ford $|V|$ times: $\mathrm{O}(|V|^2 |E|)$
  - Floyd-Warshall: $\mathrm{O}(|V|^3)$

Floyd-Warshall algorithm

- Dynamic programming algorithm.
- Assume $V=\{1, 2, \ldots, n\}$, and assume G is given in an **adjacency matrix** $A=(a_{ij})_{1\leq i,j\leq n}$ where a_{ij} is the weight of the edge from i to j.

Define $c_{ij}^{(k)}$ = weight of a shortest path from i to j with intermediate vertices belonging to the set $\{1, 2, \ldots, k\}$.



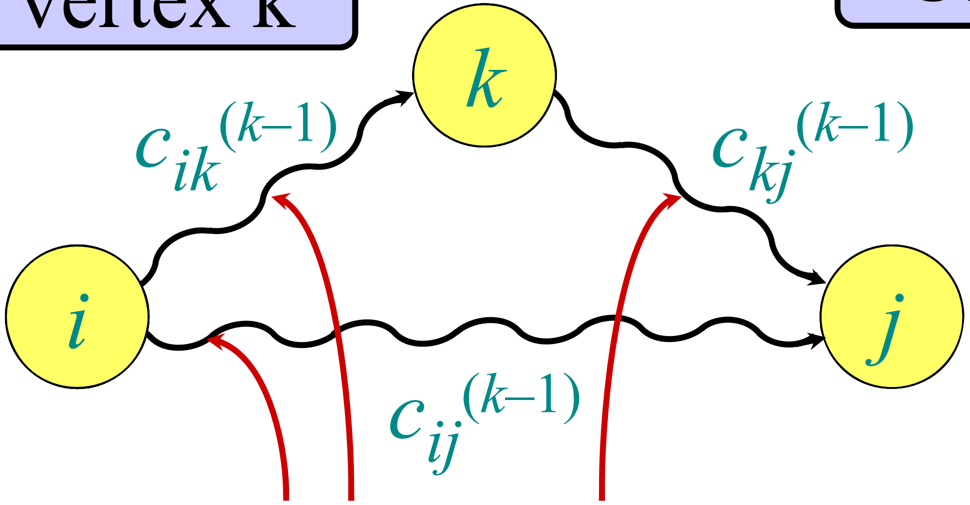
Thus, $\delta(i, j) = c_{ij}^{(n)}$. Also, $c_{ij}^{(0)} = a_{ij}$.

Floyd-Warshall recurrence

$$c_{ij}^{(k)} = \min\{c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)}\}$$

Do not use vertex k

Use vertex k



intermediate vertices in $\{1, 2, \ldots, k\text{-}1\}$

Pseudocode for Floyd-Warshall

```
for $k \leftarrow 1$ to $n$ do
  for $i \leftarrow 1$ to $n$ do
    for $j \leftarrow 1$ to $n$ do
      if $c_{ij}^{(k-1)} > c_{ik}^{(k-1)} + c_{kj}^{(k-1)}$ then
          $c_{ij}^{(k)} \leftarrow c_{ik}^{(k-1)} + c_{kj}^{(k-1)}$      } relaxation
      else
          $c_{ij}^{(k)} \leftarrow c_{ij}^{(k-1)}$
```

- Runs in $\Theta(n^3)$ time and space
- Simple to code.
- Efficient in practice.

Transitive Closure of a Directed Graph

Compute $t_{ij} = \begin{cases} 1 & \text{if there exists a path from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$

IDEA: Use Floyd-Warshall, but with $(\vee, \wedge)$ instead of $(\min, +)$:

$$t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}).$$

Time $= \Theta(n^3)$.

Floyd-Warshall recurrence

$$c_{ij}^{(k)} = \min \{c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)}\}$$

Shortest paths

Single-source shortest paths

- Nonnegative edge weights
  - Dijkstra's algorithm: $O(|E| + |V| \log |V|)$
- General: Bellman-Ford: $O(|V||E|)$
- DAG: One pass of Bellman-Ford: $O(|V| + |E|)$

$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$ adj. list

All-pairs shortest paths

- Nonnegative edge weights adj. list
  - Dijkstra's algorithm $|V|$ times: $O(|V||E|+|V|^2 \log |V|)$
- General
  - Bellman-Ford $|V|$ times: $O(|V|^2 |E|)$ adj. list
  - Floyd-Warshall: $O(|V|^3)$ adj. matrix

Graph reweighting

Theorem. Given a label $h(v)$ for each $v \in V$, ***reweight*** each edge $(u, v) \in E$ by

$$\hat{w}(u, v) = w(u, v) + h(u) - h(v).$$

Then, all paths between the same two vertices are reweighted by the same amount.

Proof. Let $p = v_1 \to v_2 \to 6 \quad \to v_k$ be a path in the graph.

Then, we have

$$\begin{aligned}
\hat{w}(p) &= \sum_{i=1}^{k-1} \hat{w}(v_i, v_{i+1}) \\
&= \sum_{i=1}^{k-1} \big(w(v_i, v_{i+1}) + h(v_i) - h(v_{i+1})\big) \\
&= \sum_{i=1}^{k-1} w(v_i, v_{i+1}) + h(v_1) - h(v_k) \\
&= w(p) + h(v_1) - h(v_k).
\end{aligned}$$

$\square$

Johnson's algorithm

1. Find a vertex labeling h, by running Bellman-Ford on $G \cup \{\text{super-source } s\}$. Set $h(v)=\delta(s,v)$ or determine that a negative-weight cycle exists.
   By triangle inequality $h(v) \le h(u) + w(u, v)$, and hence $\hat{w}(u, v) = w(u, v) + h(u) - h(v) \ge 0$.
   - Time $= O(|V||E|)$
2. Run Dijkstra's algorithm from each vertex using $\hat{w}$.
   - Time $= O(|V||E|+|V|^2 \log |V|)$.
3. Reweight each shortest-path weight $\hat{\delta}(u,v)$ to compute the shortest-path weight $\delta(u,v) = \hat{\delta}(u,v) - h(u) + h(v)$ of the original graph G.
   - Time $= O(|V|^2)$

Total time $= O(|V||E|+|V|^2 \log |V|)$.

Shortest paths

Single-source shortest paths

- Nonnegative edge weights
  - Dijkstra's algorithm: $O(|E| + |V| \log |V|)$
- General: Bellman-Ford: $O(|V||E|)$
- DAG: One pass of Bellman-Ford: $O(|V| + |E|)$

$\left.\right\}$ adj. list

All-pairs shortest paths

- Nonnegative edge weights — adj. list
  - Dijkstra's algorithm $|V|$ times: $O(|V||E|+|V|^2 \log |V|)$
- General
  - Bellman-Ford $|V|$ times: $O(|V|^2 |E|)$ — adj. list
  - Floyd-Warshall: $O(|V|^3)$ — adj. matrix
  - Johnson's algorithm: $O(|V||E|+|V|^2 \log |V|)$ adj. list