CMPS 6610 – Fall 2018

Matrix-chain multiplication

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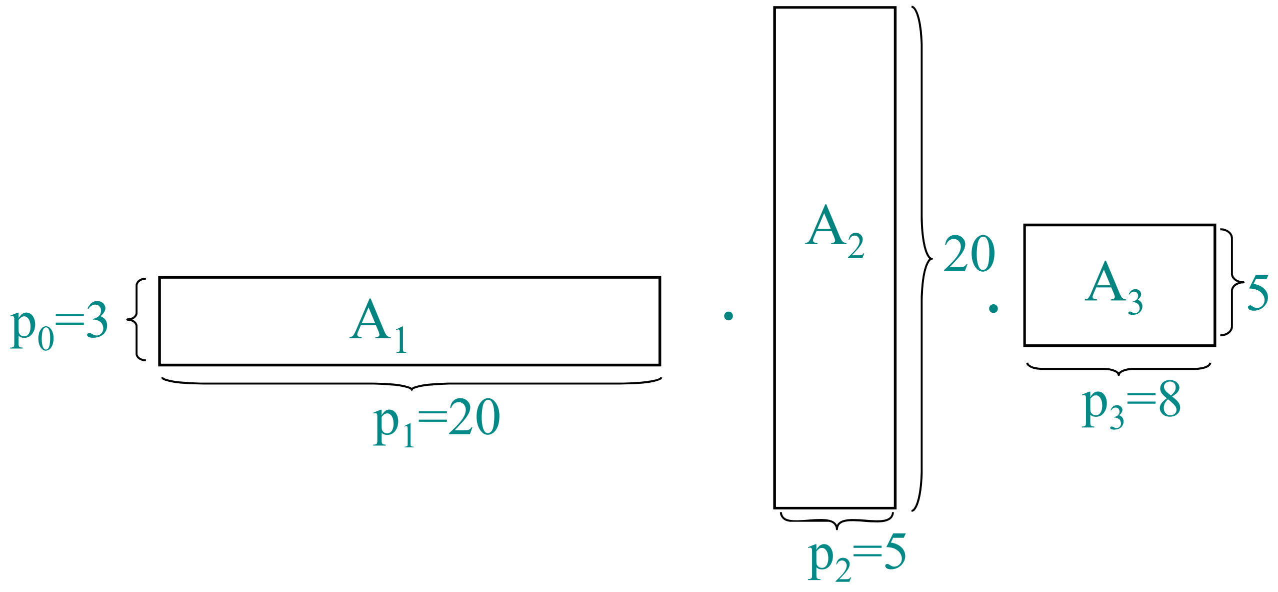
Matrix-chain multiplication

Given: A sequence/chain of n matrices $A_1, A_2, \ldots, A_n$, where A_i is a $p_{i-1} \times p_i$ matrix

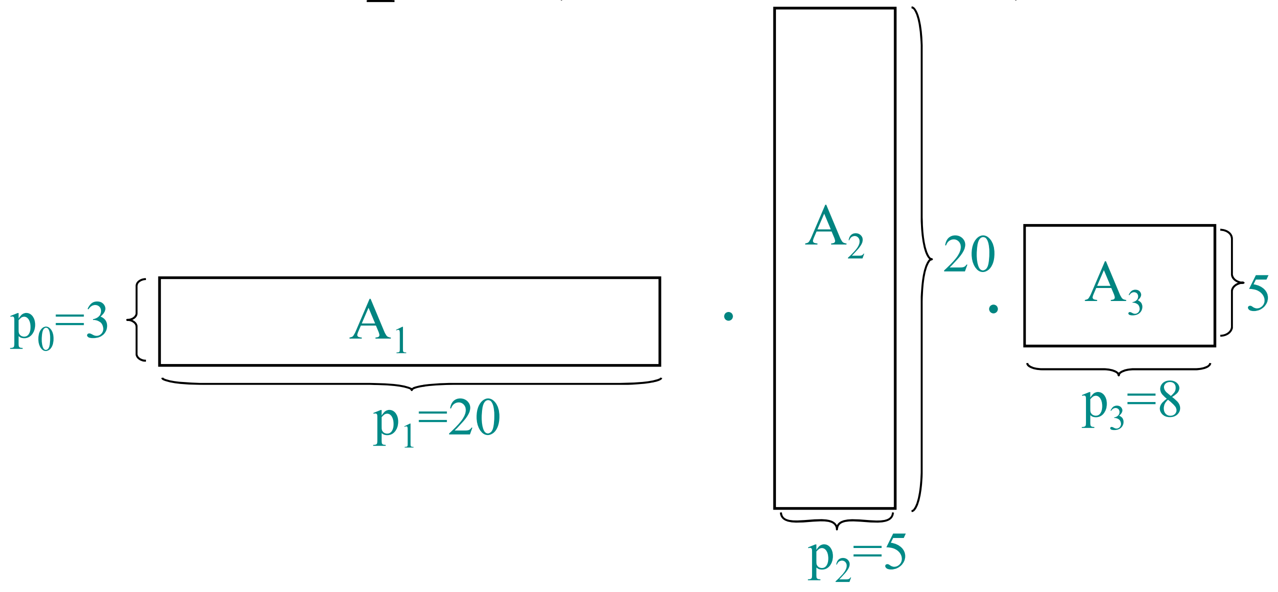
Task: Compute their product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$ using the minimum number of scalar multiplications.

Matrix-chain multiplication example

Example: n=3, p_0=3, p_1=20, p_2=5, p_3=8. A_1 is a 3×20 matrix, A_2 is a 20×5 matrix, A_3 is a 5×2 matrix. Compute $A_1 \cdot A_2 \cdot A_3$.

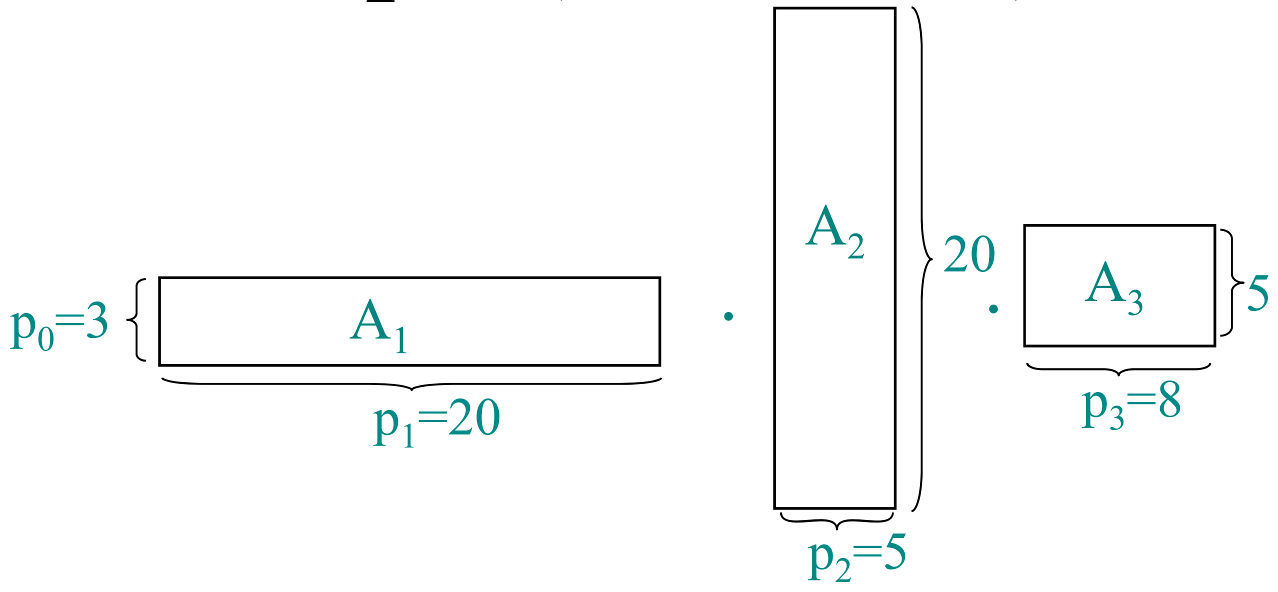


Matrix-chain multiplication example (continued)



- Computing $A_1 \cdot A_2$ takes $3 \cdot 20 \cdot 5$ multiplications and results in a 3×5 matrix.
- Computing $A_i \cdot A_{i+1}$ takes $p_{i-1} \cdot p_i \cdot p_{i+1}$ multiplications and results in a $p_{i-1} \times p_{i+1}$ matrix.

Matrix-chain multiplication example (continued)



- Computing $(A_1 \cdot A_2) \cdot A_3$ takes $3\cdot20\cdot5+3\cdot5\cdot8 = 300+120 = 420$ multiplications
- Computing $A_1 \cdot (A_2 \cdot A_3)$ takes $20\cdot5\cdot8+3\cdot20\cdot8 = 800+480 = 1{,}280$ multiplications

Matrix-chain multiplication

Given: A sequence/chain of n matrices $A_1, A_2, \ldots, A_n$, where A_i is a $p_{i-1} \times p_i$ matrix

Task: Compute their product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$ using the minimum number of scalar multiplications.

$\Rightarrow$ Find a parenthesization that minimizes the number of multiplications

Would greedy work?

- Parenthesizing like this $(\ldots((A_1 \cdot A_2) \cdot A_3) \ldots \cdot A_n)$ does not work (e.g., reverse our running example).

$\Rightarrow$ Try dynamic programming

1) Optimal substructure

Let $A_{i,j} = A_i \cdot \ldots \cdot A_j$ for $i \leq j$

- Consider an optimal parenthesization for $A_{i,j}$. Assume it splits it at k, so
$$A_{i,j}=(A_i \cdot \ldots \cdot A_k)\cdot(A_{k+1} \ldots \cdot A_j)$$
- Then, the par. of the prefix $A_i \cdot \ldots \cdot A_k$ within the optimal par. of $A_{i,j}$ must be an optimal par. of $A_{i,k}$. (Assume it is not optimal, then there exists a better par. for $A_{i,k}$. **Cut and paste** this par. into the par. for $A_{i,j}$. This yields a better par. for $A_{i,j}$. Contradiction.)

2) Recursive solution

a) First compute the minimum number of multiplications

b) Then compute the actual parenthesization

We will concentrate on solving a) now.

2) Recursive solution (cont.)

$m[i,j]$ = minimum number of scalar multiplications to compute A_{ij}

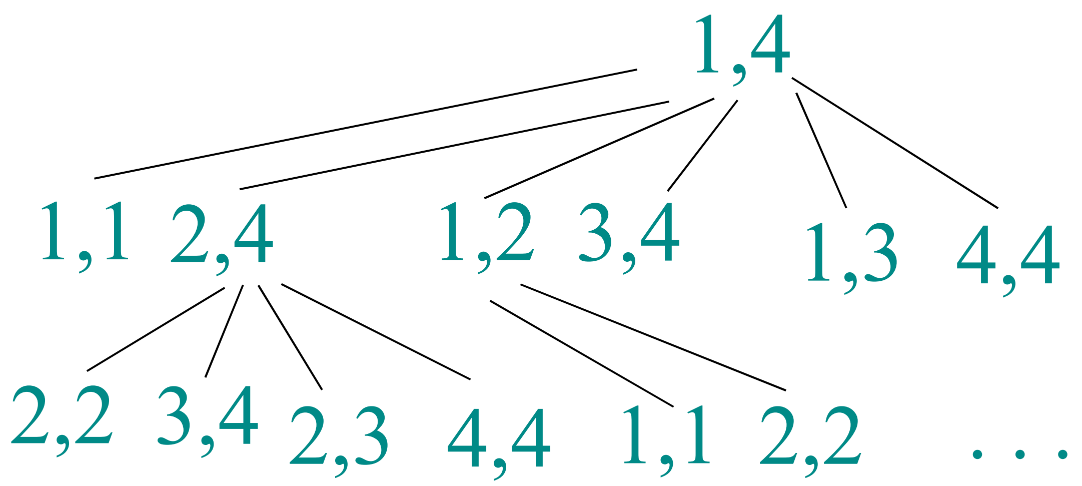
Goal: Compute $m[1,n]$

$$A_{i,j} = \underbrace{(A_i \cdot \ldots \cdot A_k)}_{p_{i-1} \times p_k} \cdot \underbrace{(A_{k+1} \ldots \cdot A_j)}_{p_k \times p_j}$$

Recurrence:

- $m[i,i] = 0$ for $i=1,2,\ldots,n$
- $m[i,j] = \min\limits_{i \le k < j} (m[i,k]+m[k+1,j] + p_{i-1}\, p_k\, p_j)$

Recursion tree



- The runtime of the straight-forward recursive algorithm is $\Omega(2^n)$

- But only $\Theta(n^2)$ different subproblems !

Dynamic programming

```
MATRIX_CHAIN_DP($p$, $n$):
  for $i$:=1 to $n$ do $m[i,i]=0$
  for $l$:=2 to $n$ do // $l$ is length of chain
    for $i$:=1 to $n-l+1$ do
      $j:=i+l-1$
      $m[i,j]=\infty$
      for $k$:=$i$ to $j-1$ do
        $q:=m[i,k]+m[k+1,j]+p_{i-1}*p_k*p_j$
        if $q<m[i,j]$ then
          $m[i,j]=q$
          $s[i,j]:=k$ //index that optimizes m[i,j]
  return $m$ and $s$;
```

Dynamic programming

- Use dynamic programming to fill the 2-dimensional $m[i,j]$-table
- Bottom-up: Diagonal by diagonal
- For the construction of the optimal parenthesization, use an additional array $s[i,j]$ that records that value of k for which the minimum is attained and stored in $m[i,j]$
- $O(n^3)$ runtime ($n \times n$ table, $O(n)$ min-computation per entry), $O(n^2)$ space
- $m[1,n]$ is the desired value

Construction of an optimal parenthesization

```
PRINT_PARENS($s,i,j$) // initial call: print_parens(s,1,n)
  if i=j then print "A"i
  else print "("
        PRINT_PARENS($s,i,s[i,j]$)
        print ")·("
        PRINT_PARENS($s,s[i,j]+1,j$)
        print ")"
```

Runtime: Recursion tree = binary tree with n leaves. Spend O(1) per node. O(n) total runtime.