# CMPS 6610 Algorithms – Fall 2018

# *Greedy Algorithms: Knapsack Problem*

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# Greedy Strategy

1. Repeatedly identify a decision to be made ($\rightarrow$ recursion)
2. Make a **locally optimal** choice for each decision

In order to reach a globally optimal solution, the problem must have appropriate recursive substructure:

optimal solution = locally optimal choice
+ optimal solution for the remainder of the problem

# Knapsack Problem

- Given a knapsack with weight capacity $W > 0$, and given $n$ items of positive integer weights $w_1, \ldots, w_n$ and positive integer values $v_1, \ldots, v_n$.
(So, item $i$ has value $v_i$ and weight $w_i$.)

- **0-1 Knapsack Problem:** Compute a subset of items that maximize the total value (sum), and they all fit into the knapsack (total weight at most W).

- **Fractional Knapsack Problem:** Same as before but we are allowed to take fractions of items (→ gold dust).

# Greedy Knapsack

- Greedy Strategy:
  - Compute $\frac{v_i}{w_i}$ for each $i$
  - Greedily take as much as possible of the item with the highest value/weight. Then repeat/recurse.

$\Rightarrow$ Sort items by value/weight

$\Rightarrow$ $O(n \log n)$ runtime

# Knapsack Example

| | | | |
|---|---|---|---|
| item | 1 | 2 | 3 |
| value | 12 | 15 | 4 |
| weight | 2 | 3 | 1 |
| value/weight | 6 | 5 | 4 |

W=4

- **Greedy fractional:** Take item 1 and 2/3 of item 2

$\Rightarrow$ weight=4, value=$12+2/3\cdot 15 = 12+10 = 22$

- **Greedy 0-1:** Take item 1 and then item 3

$\Rightarrow$ weight = $1+2=3$, value=$12+4=16$

greedy 0-1
$\neq$ optimal 0-1

- **Optimal 0-1:** Take items 2 and 3, value =19

# Optimal Substructure

- Let $s_1, \ldots, s_n$ be an optimal solution, where $s_i =$ amount of item $i$ that is taken; $0 \leq s_i \leq 1$
- Suppose we remove one item. → $n-1$ items left
- Is the remaining "solution" still an optimal solution for $n-1$ items?
- Yes; cut-and-paste.

# Correctness Proof for Greedy

- Suppose items $1, \ldots, n$ are numbered in decreasing order by value/weight.
- Greedy solution G: Takes all elements $1, \ldots, j, \ldots, i^*$-1 and a fraction of $i^*$.
- Assume optimal solution S is different from G. Assume S takes only a fraction $\frac{1}{a}$ of item $j$, for $j \leq i^*$-1.
- Create new solution S' from S by taking $w_j - 1/a$ weight away from items $> j$, and add $w_j - 1/a$ of item $j$ back in. Hence, all of item $j$ is taken.

$\Rightarrow$ New solution S' has the same weight but increased value. This contradicts the assumption that S was optimal.
$\Rightarrow$ S=G. □

# General Solution: DP

- $D[i, w]$ = max value possible for taking a subset of items $1, \ldots, i$ with knapsack constraint $w$.

- $D[0, w] = D[i, 0] = 0$ for all $0 \leq i \leq n$ and $0 \leq w \leq W$
  $D[i, w] = -\infty$ for $w < 0$
  $$D[i, w] = \max(\underbrace{D[i-1, w]}_{\text{don't take item } i}, \underbrace{v_i + D[i-1, w - w_i]}_{\text{take item } i})$$

- Compute $D[n, W]$ by filling an $n \times W$ DP-table.
  $\Rightarrow$ Two nested for-loops, runtime and space $\Theta(nW)$

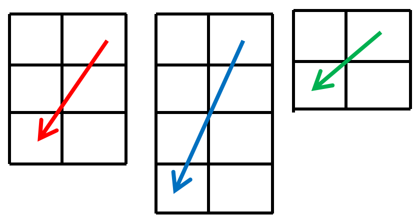
- Trace back from $D[n, W]$ by redoing computation or following arrows. $\Rightarrow$ $\Theta(n + W)$ runtime

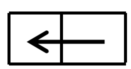
# DP Example

Solution:
Take items 3 and 2

W=4

| | | | |
|---|---|---|---|
| item | 1 | 2 | 3 |
| value | 12 | 15 | 4 |
| weight | 2 | 3 | 1 |
| value/weight | 6 | 5 | 4 |

Take item i: 

Don't take item i: 

w ↑

| w \ i | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| W= 4 | 0 | 12 | 15 | 19 |
| 3 | 0 | 12 | 15 | 16 |
| 2 | 0 | 12 | 12 | 12 |
| 1 | 0 | 0 | 0 | 4 |
| 0 | 0 ← | 0 | 0 | 0 |

0 1 2 3 → i
n

$$D[i,w] = \max(\underbrace{D[i-1,w]}_{\text{don't take item } i}, \underbrace{v_i + D[i-1,w-w_i]}_{\text{take item } i})$$