

CMPS 6610 – Fall 2018

Graphs

Carola Wenk

Slides courtesy of Charles Leiserson with changes and additions
by Carola Wenk

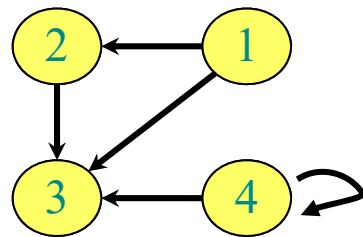
Graphs

Definition. A *directed graph (digraph)* $G = (V, E)$ is an ordered pair consisting of

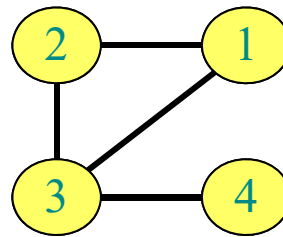
- a set V of *vertices* (singular: *vertex*),
- a set $E \subseteq V \times V$ of *edges*.

In an *undirected graph* $G = (V, E)$, the edge set E consists of *unordered* pairs of vertices.

directed graph



undirected graph



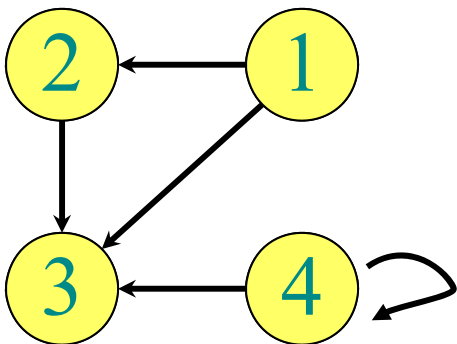
In either case, we have $|E| \in O(|V|^2)$.

Moreover, if G is connected, then $|E| \geq |V| - 1$.

Adjacency-matrix representation

The *adjacency matrix* of a graph $G = (V, E)$, where $V = \{1, 2, \dots, n\}$, is the matrix $A[1 \dots n, 1 \dots n]$ given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E. \end{cases}$$

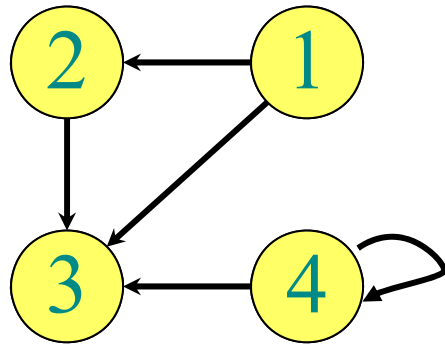


A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	1

$\Theta(|V|^2)$ storage
 \Rightarrow *dense*
representation.

Adjacency-list representation

An *adjacency list* of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to v .



$$Adj[1] = \{2, 3\}$$

$$Adj[2] = \{3\}$$

$$Adj[3] = \{\}$$

$$Adj[4] = \{3, 4\}$$

For undirected graphs, $|Adj[v]| = degree(v)$.

For digraphs, $|Adj[v]| = out-degree(v)$.

Adjacency-list representation

Handshaking Lemma:

Every edge is counted twice

- For undirected graphs:

$$\sum_{v \in V} \text{degree}(v) = 2|E|$$

- For digraphs:

$$\sum_{v \in V} \text{in-degree}(v) = \sum_{v \in V} \text{out-degree}(v) = |E|$$

⇒ adjacency lists use $\Theta(|V| + |E|)$ storage

⇒ a ***sparse*** representation

⇒ We usually use this representation,
unless stated otherwise

Graph Traversal

Let $G=(V,E)$ be a (directed or undirected) graph, given in adjacency list representation.

$$|V| = n , |E| = m$$

A graph traversal visits every vertex:

- Breadth-first search (BFS)
- Depth-first search (DFS)

Breadth-First Search (BFS)

BFS($G=(V,E)$)

Mark all vertices in G as “unvisited” // $\text{time}=0$

Initialize empty queue Q

for each vertex $v \in V$ **do**

if v is unvisited

 visit v // $\text{time}++$

$Q.\text{enqueue}(v)$

 BFS_iter(G)

BFS_iter(G)

while Q is non-empty **do**

$v = Q.\text{dequeue}()$

for each w adjacent to v **do**

if w is unvisited

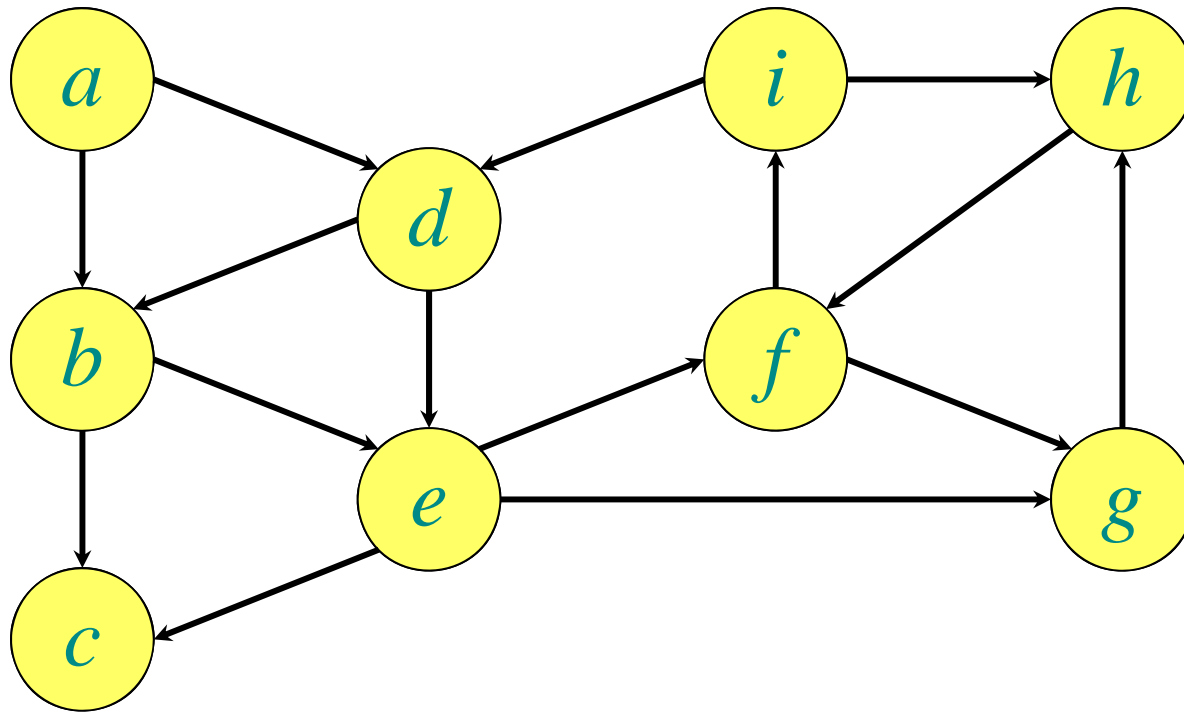
 visit w // $\text{time}++$

 Add edge (v,w) to T

$Q.\text{enqueue}(w)$

Example of breadth-first search

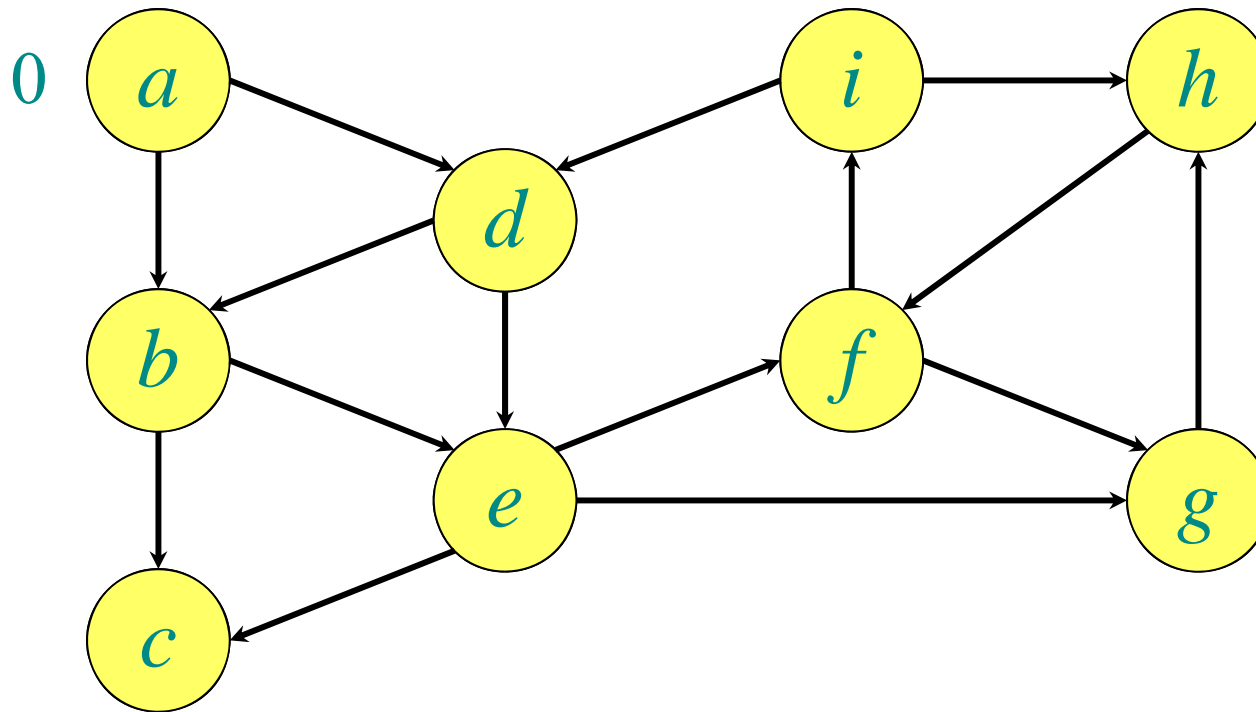
```
while  $Q$  is non-empty do
   $v = Q.dequeue()$ 
  for each  $w$  adjacent to  $v$  do
    if  $w$  is unvisited
      visit  $w$  // time++
      Add edge  $(v,w)$  to  $T$ 
       $Q.enqueue(w)$ 
```



Q :

Example of breadth-first search

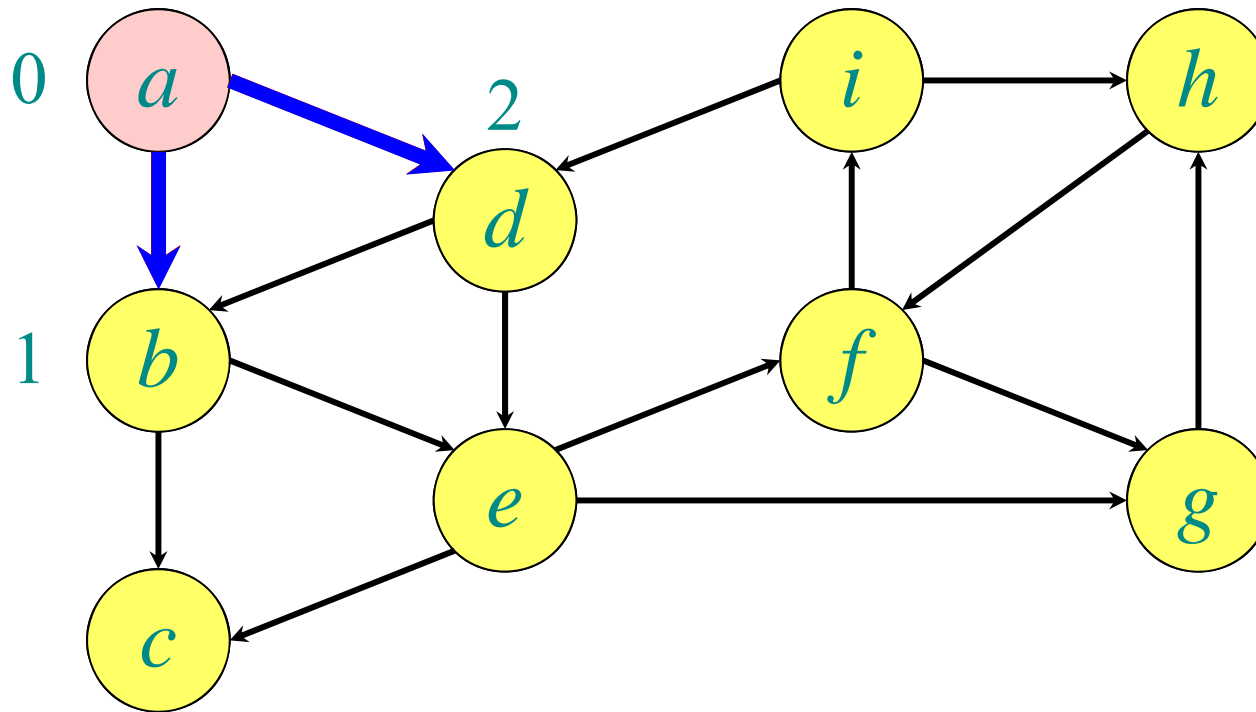
```
while  $Q$  is non-empty do  
   $v = Q.dequeue()$   
  for each  $w$  adjacent to  $v$  do  
    if  $w$  is unvisited  
      visit  $w$  // time++  
      Add edge  $(v,w)$  to  $T$   
       $Q.enqueue(w)$ 
```



$Q: a$

Example of breadth-first search

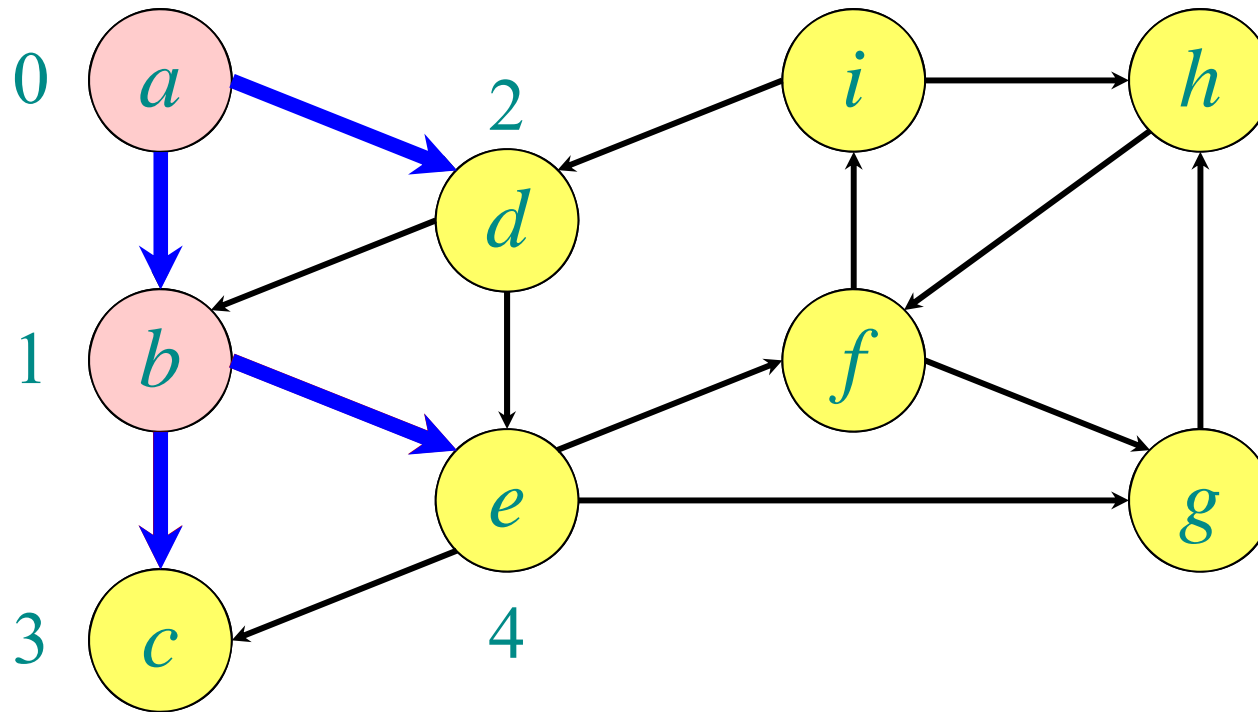
```
while  $Q$  is non-empty do
   $v = Q.dequeue()$ 
  for each  $w$  adjacent to  $v$  do
    if  $w$  is unvisited
      visit  $w$  // time++
      Add edge  $(v,w)$  to  $T$ 
       $Q.enqueue(w)$ 
```



1 2
 $Q: a b d$

Example of breadth-first search

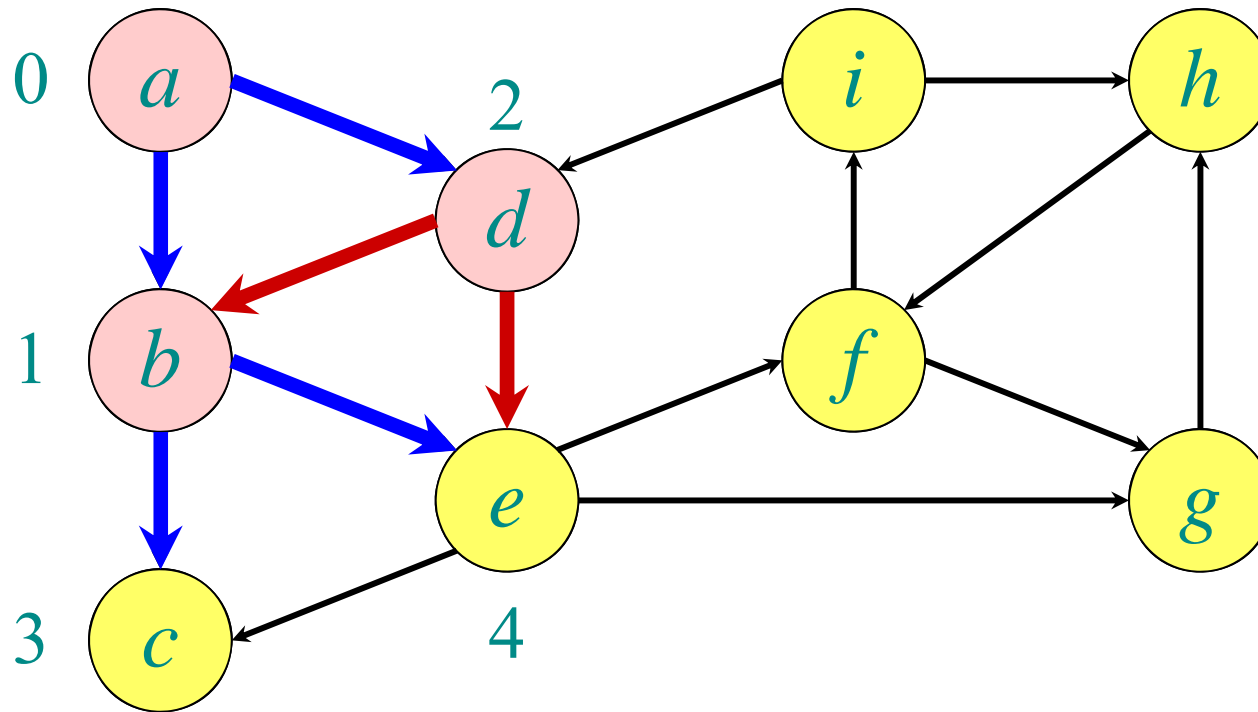
```
while  $Q$  is non-empty do  
   $v = Q.dequeue()$   
  for each  $w$  adjacent to  $v$  do  
    if  $w$  is unvisited  
      visit  $w$  // time++  
      Add edge  $(v,w)$  to  $T$   
       $Q.enqueue(w)$ 
```



2 3 4
 $Q: a, b, d, c, e$

Example of breadth-first search

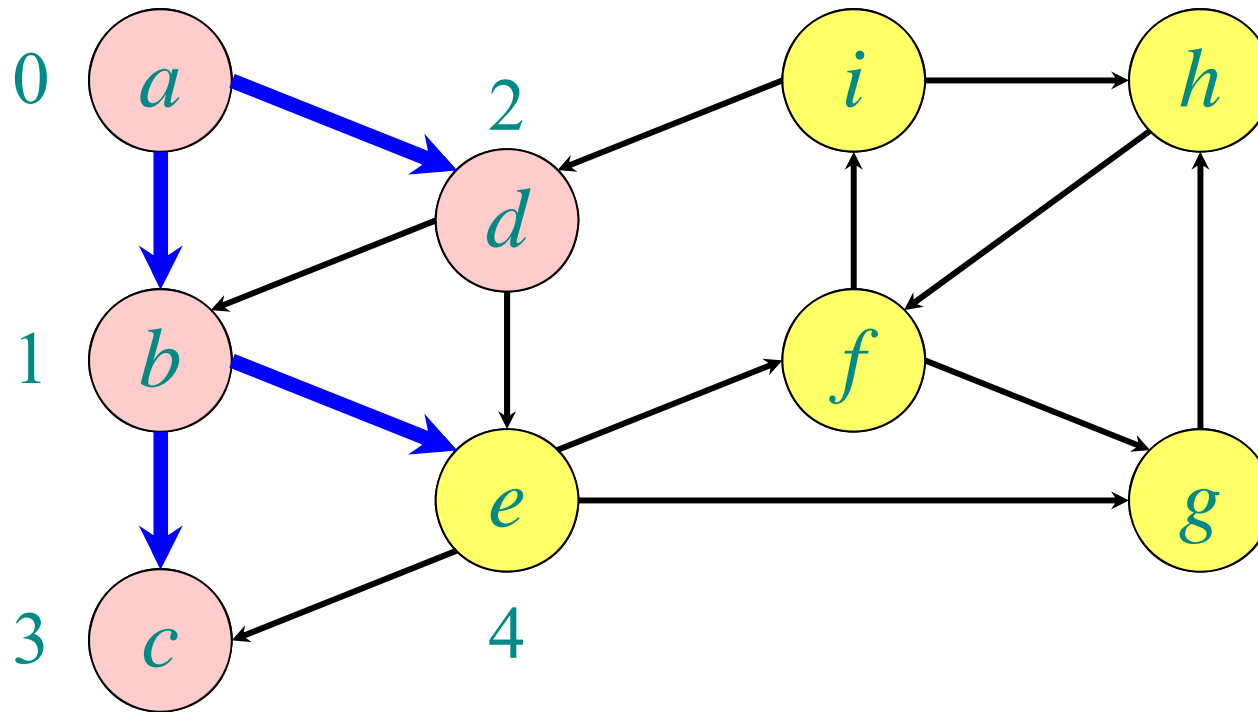
```
while  $Q$  is non-empty do  
   $v = Q.dequeue()$   
  for each  $w$  adjacent to  $v$  do  
    if  $w$  is unvisited  
      visit  $w$  //  $time++$   
      Add edge  $(v,w)$  to  $T$   
       $Q.enqueue(w)$ 
```



3 4
 $Q: a b d c e$

Example of breadth-first search

```
while  $Q$  is non-empty do  
   $v = Q.dequeue()$   
  for each  $w$  adjacent to  $v$  do  
    if  $w$  is unvisited  
      visit  $w$  //  $time++$   
      Add edge  $(v,w)$  to  $T$   
       $Q.enqueue(w)$ 
```

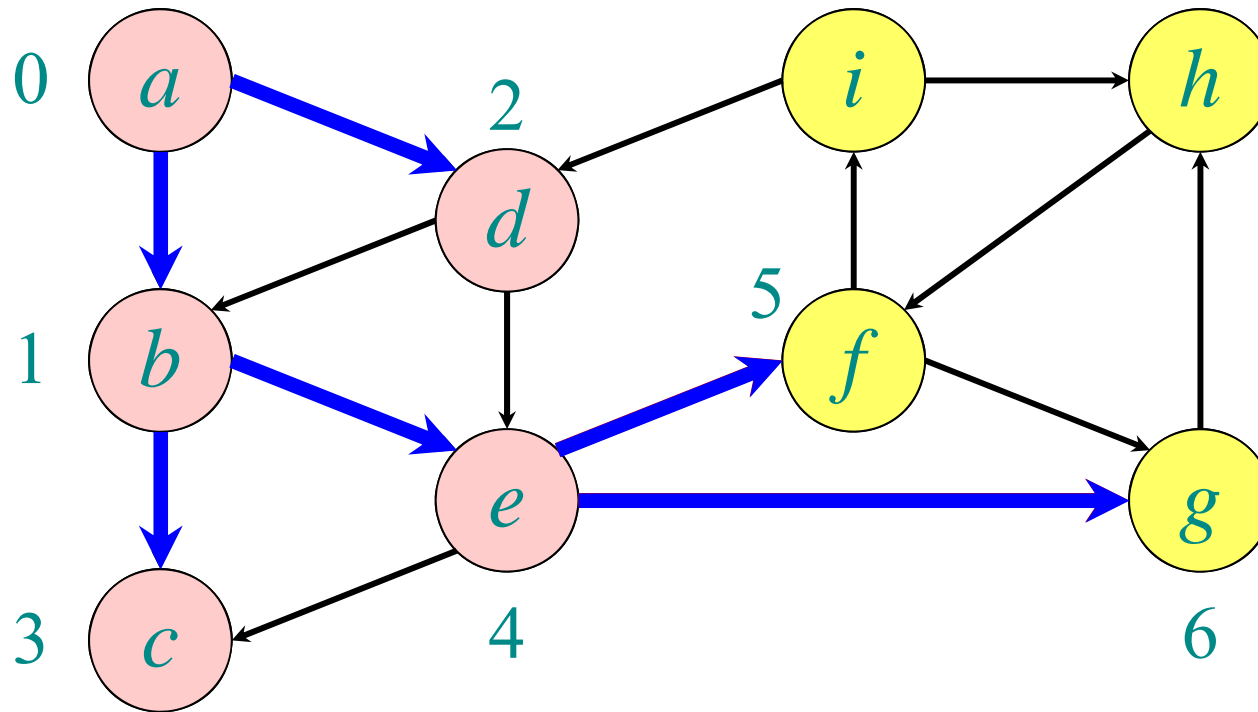


$Q: a b d c e$

Example of breadth-first search

```

while  $Q$  is non-empty do
   $v = Q.dequeue()$ 
  for each  $w$  adjacent to  $v$  do
    if  $w$  is unvisited
      visit  $w$  // time++
      Add edge  $(v,w)$  to  $T$ 
       $Q.enqueue(w)$ 
  
```

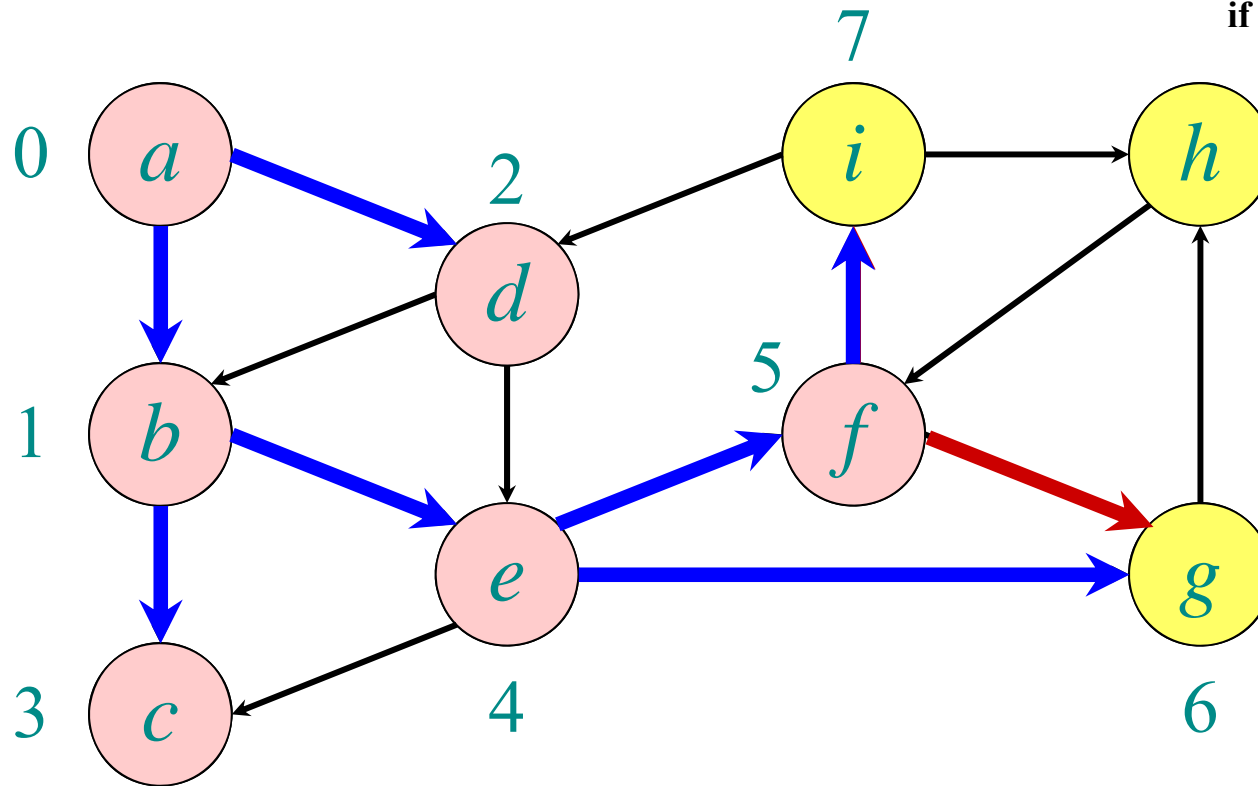


5 6
 $Q: a b d c e f g$

Example of breadth-first search

```

while  $Q$  is non-empty do
   $v = Q.dequeue()$ 
  for each  $w$  adjacent to  $v$  do
    if  $w$  is unvisited
      visit  $w$  // time++
      Add edge  $(v,w)$  to  $T$ 
       $Q.enqueue(w)$ 
  
```

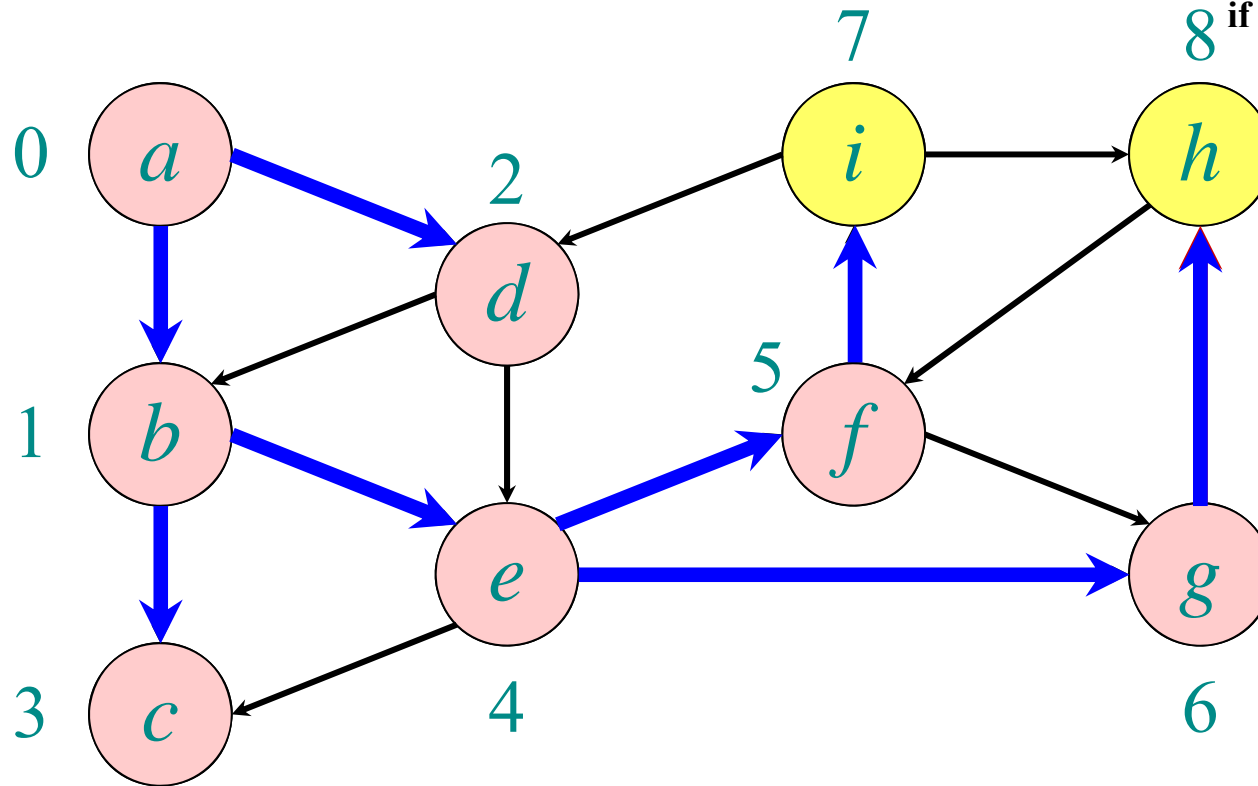


$Q: a b d c e f g i$

Example of breadth-first search

```

while  $Q$  is non-empty do
   $v = Q.dequeue()$ 
  for each  $w$  adjacent to  $v$  do
    8 if  $w$  is unvisited
      visit  $w$  // time++
      Add edge  $(v,w)$  to  $T$ 
       $Q.enqueue(w)$ 
  
```

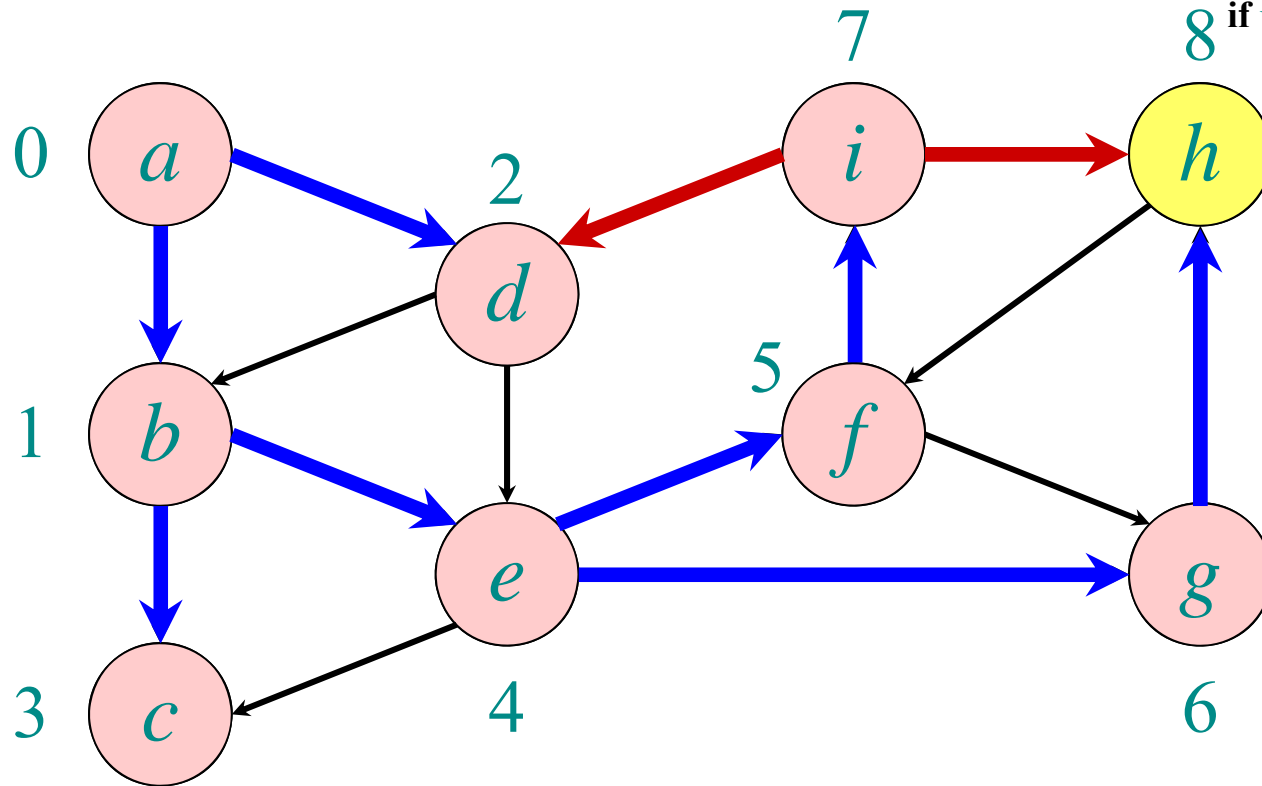


7 8
 $Q: a b d c e f g i h$

Example of breadth-first search

```

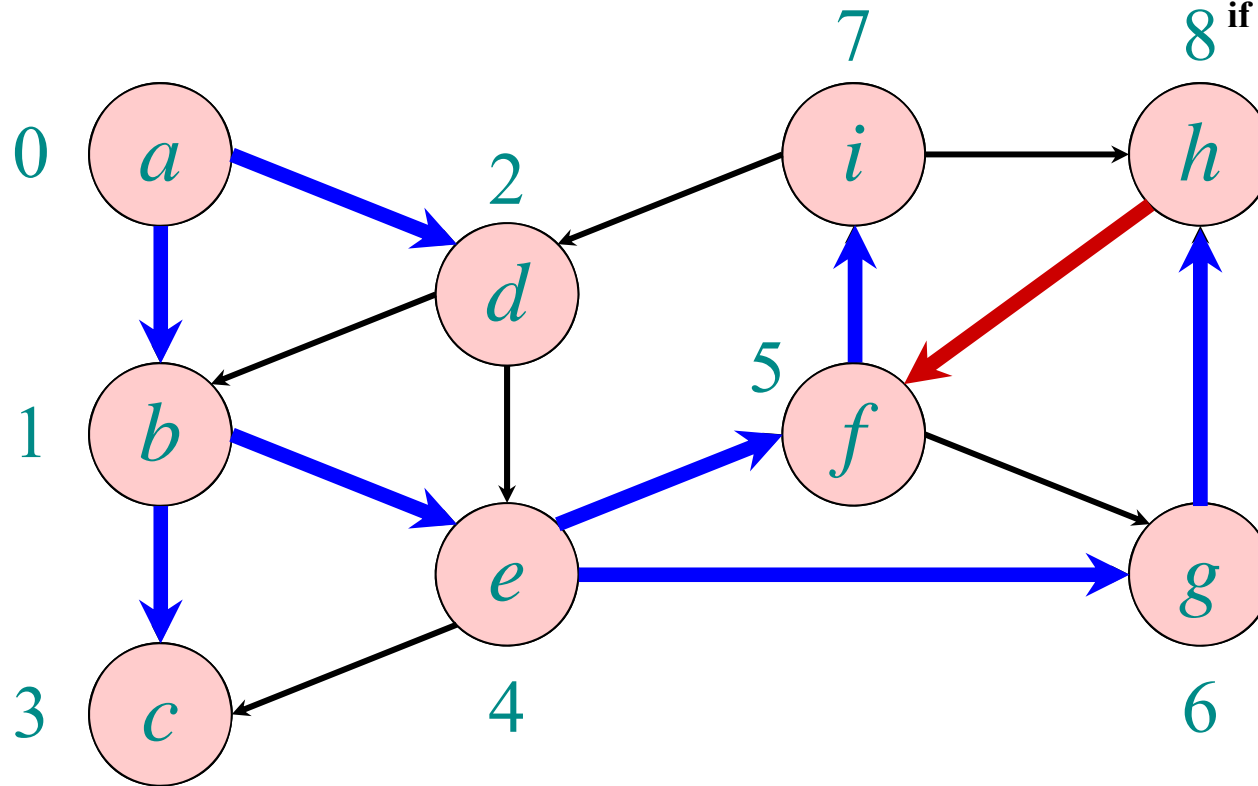
while  $Q$  is non-empty do
   $v = Q.dequeue()$ 
  for each  $w$  adjacent to  $v$  do
    8 if  $w$  is unvisited
      visit  $w$  // time++
      Add edge  $(v,w)$  to  $T$ 
       $Q.enqueue(w)$ 
  
```



$Q: a b d c e f g i h$

Example of breadth-first search

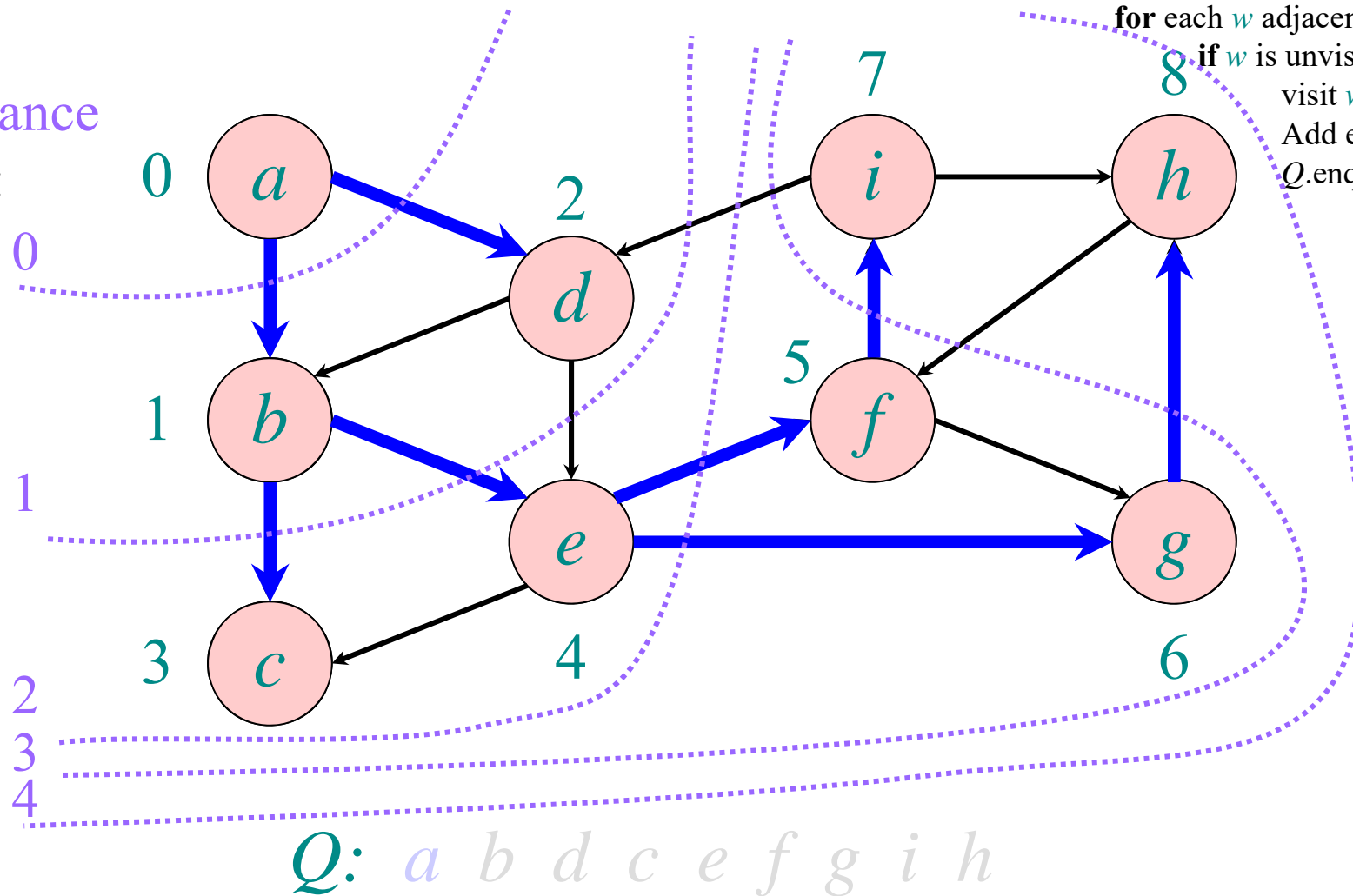
```
while  $Q$  is non-empty do  
   $v = Q.dequeue()$   
  for each  $w$  adjacent to  $v$  do  
    8 if  $w$  is unvisited  
      visit  $w$  // time++  
      Add edge  $(v,w)$  to  $T$   
       $Q.enqueue(w)$ 
```



$Q: a b d c e f g i h$

Example of breadth-first search

Distance
to a :



```

while  $Q$  is non-empty do
   $v = Q.dequeue()$ 
  for each  $w$  adjacent to  $v$  do
    if  $w$  is unvisited
      visit  $w$  // time++
      Add edge  $(v,w)$  to  $T$ 
       $Q.enqueue(w)$ 
  
```

Breadth-First Search (BFS)

BFS($G=(V,E)$)

Mark all vertices in G as “unvisited” // **time=0**

Initialize empty queue Q

for each vertex $v \in V$ **do**

if v is unvisited

visit v // **time++**

$Q.enqueue(v)$

BFS_iter(G)

BFS_iter(G)

while Q is non-empty **do**

$v = Q.dequeue()$

for each w adjacent to v **do**

if w is unvisited

visit w // **time++**

Add edge (v,w) to T

$Q.enqueue(w)$

$O(n)$

$O(1)$

$O(n)$

without
BFS_iter

$O(m)$

$O(deg(v))$

BFS runtime

- Each vertex is marked as unvisited in the beginning $\Rightarrow O(n)$ time
- Each vertex is marked at most once, enqueued at most once, and therefore dequeued at most once
- The time to process a vertex is proportional to the size of its adjacency list (its degree), since the graph is given in adjacency list representation
 $\Rightarrow O(m)$ time
- Total runtime is $O(n+m) = O(|V| + |E|)$

Depth-First Search (DFS)

DFS($G=(V,E)$)

Mark all vertices in G as “unvisited” // $\text{time}=0$

for each vertex $v \in V$ **do**

if v is unvisited

 DFS_rec(G,v)

DFS_rec(G, v)

 mark v as “visited” // $d[v]=++\text{time}$

for each w adjacent to v **do**

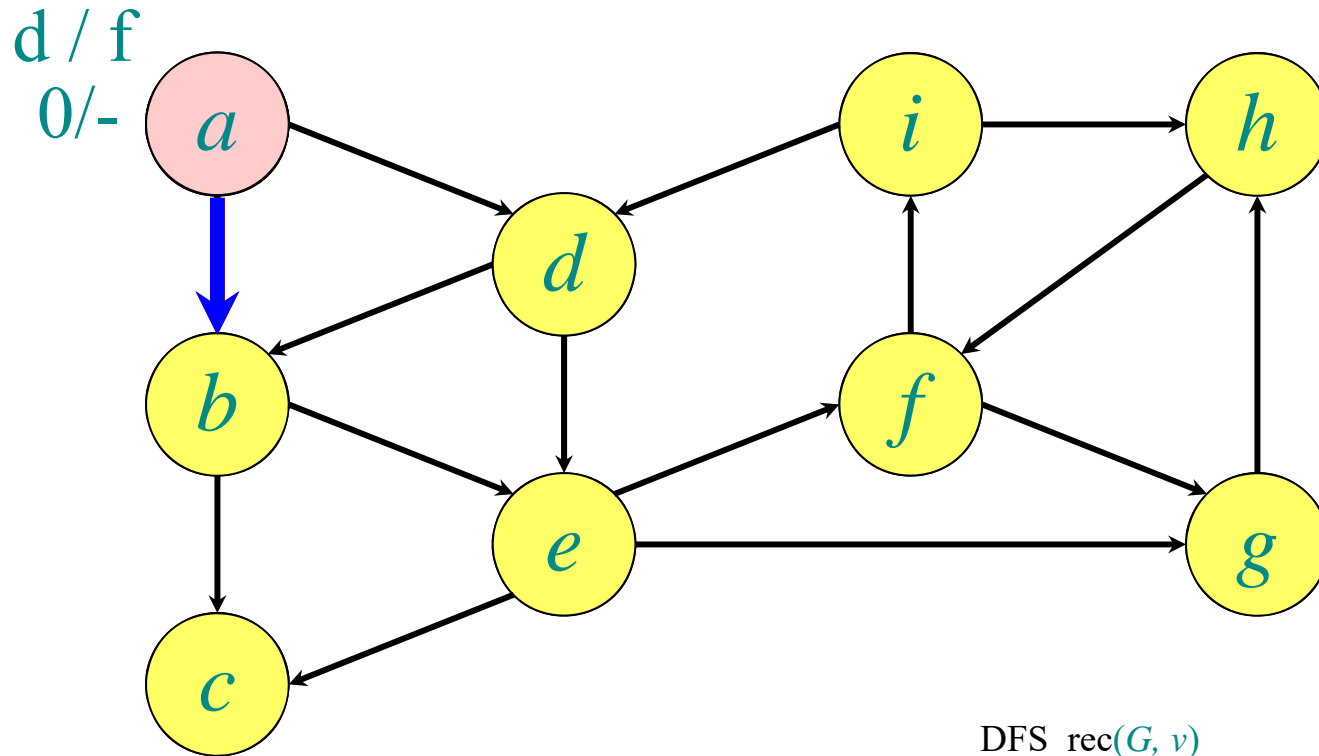
if w is unvisited

 Add edge (v,w) to tree T

 DFS_rec(G,w)

 mark v as “finished” // $f[v]=++\text{time}$

Example of depth-first search



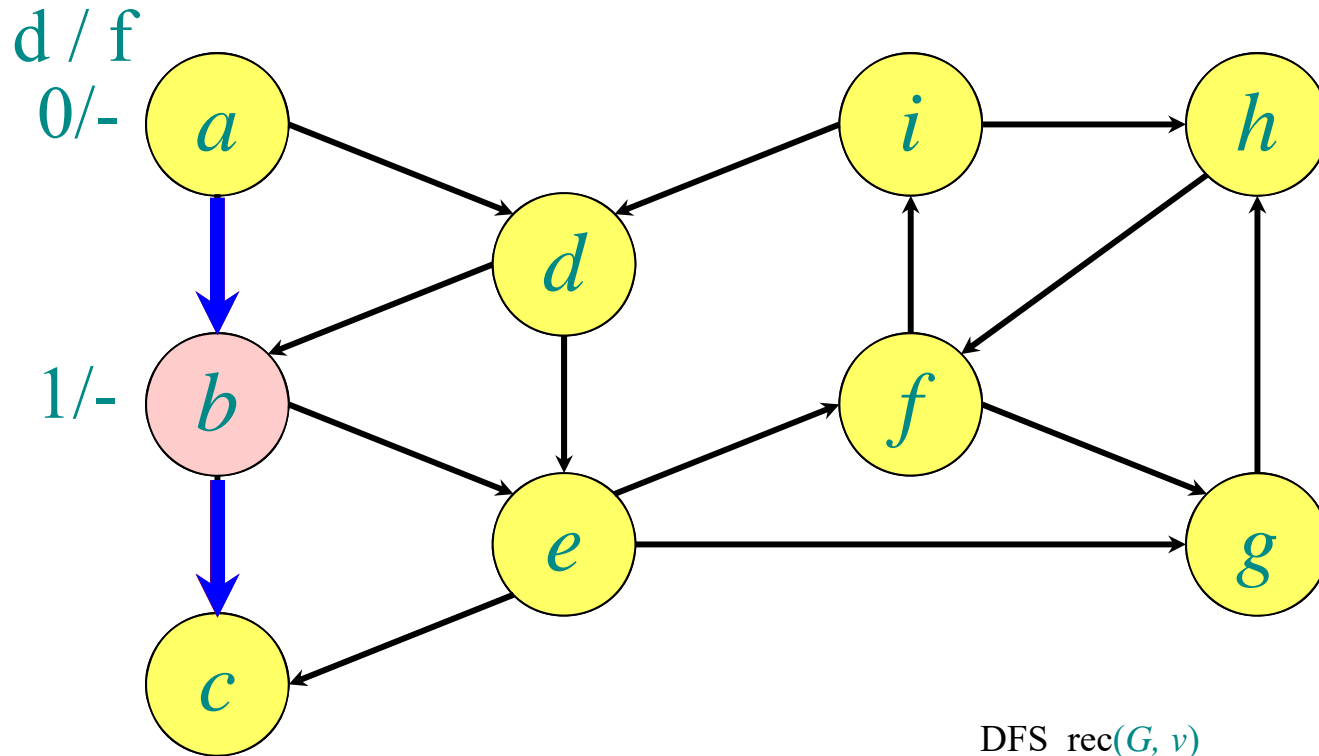
$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 - a

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```

Example of depth-first search



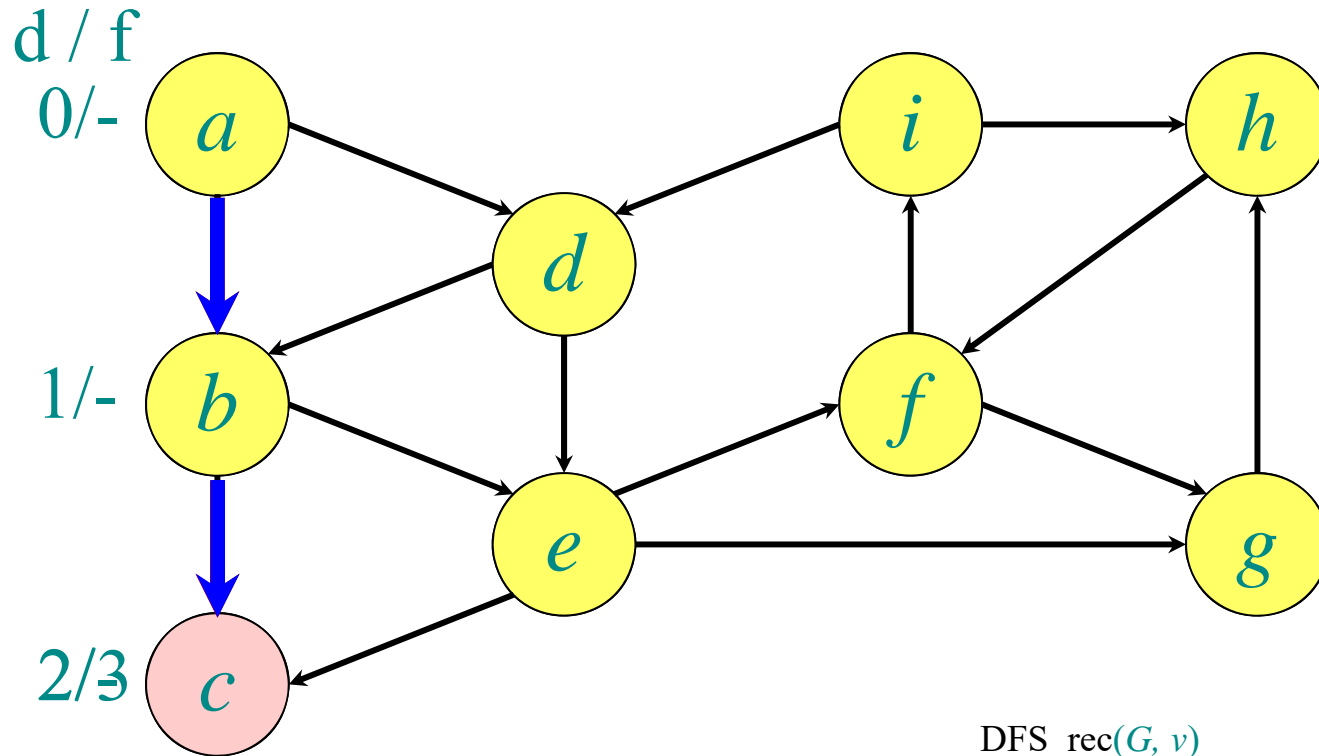
$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 - a b

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```


Example of depth-first search



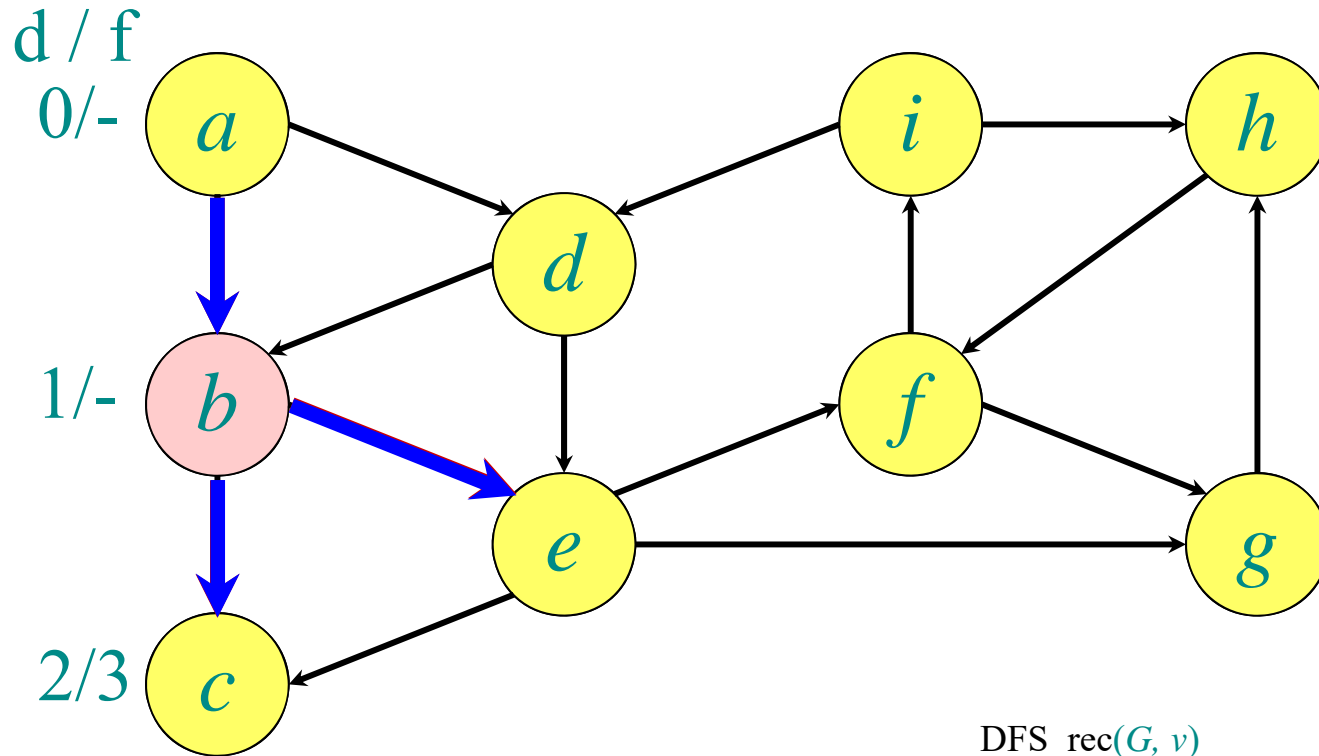
$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 - a b

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```

Example of depth-first search



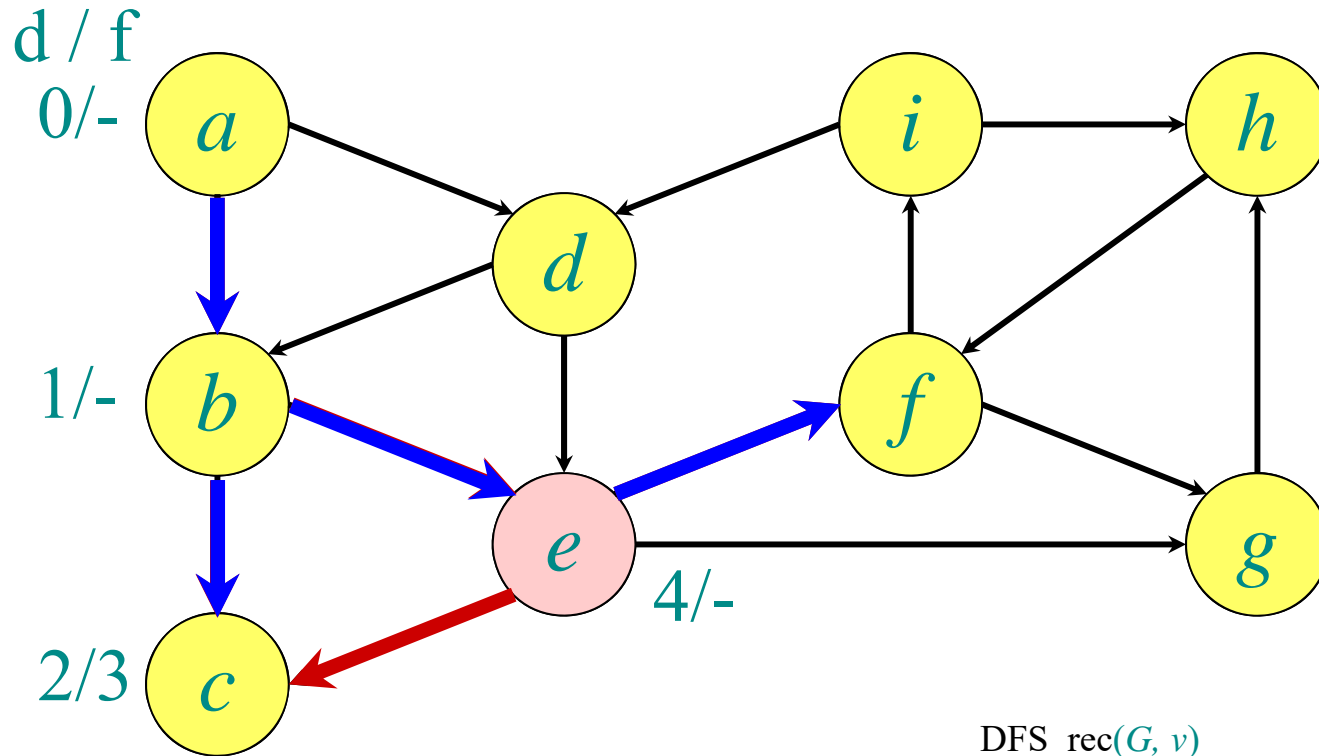
$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 - a b b

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```

Example of depth-first search



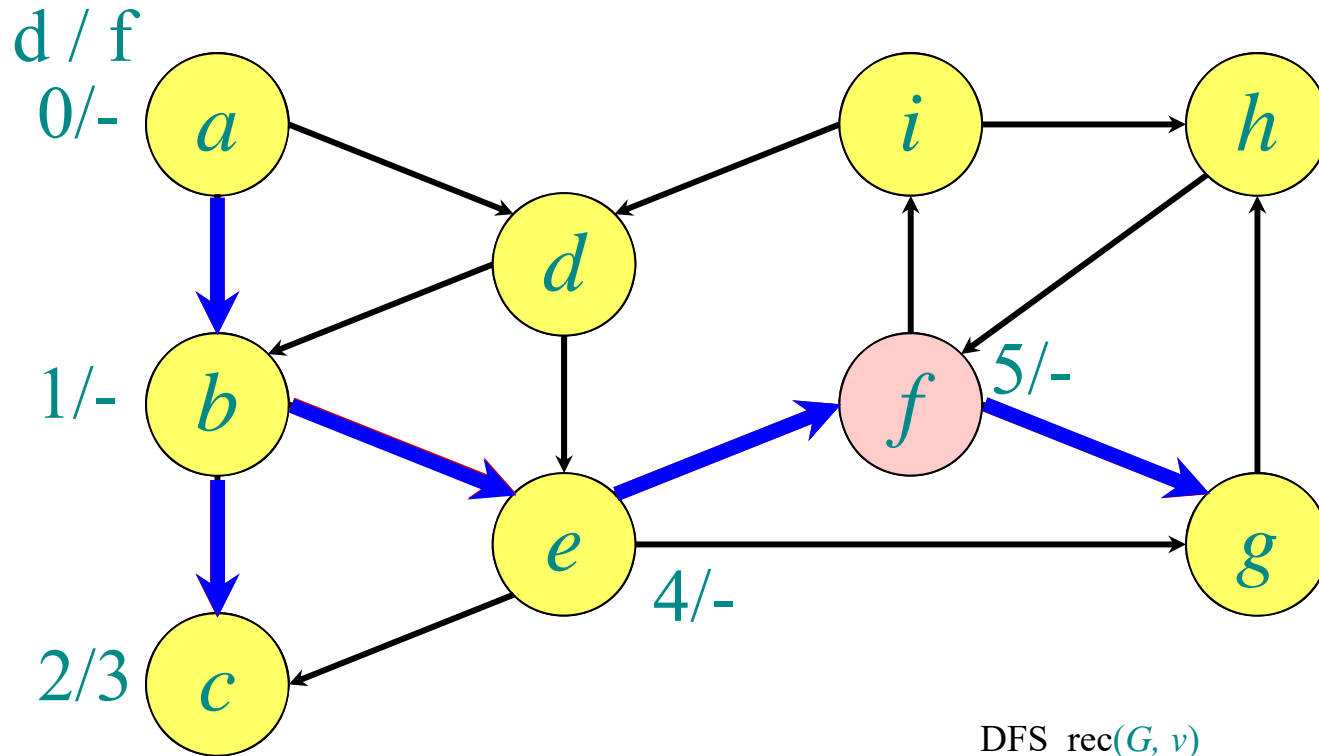
$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 - a b b e

Store edges in predecessor array

```

DFS_rec( $G, v$ )
  mark  $v$  as "visited" //  $d[v]=++time$ 
  for each  $w$  adjacent to  $v$  do
    if  $w$  is unvisited
      Add edge  $(v,w)$  to tree  $T$ 
      DFS_rec( $G,w$ )
  mark  $v$  as "finished" //  $f[v]=++time$ 
    
```

Example of depth-first search



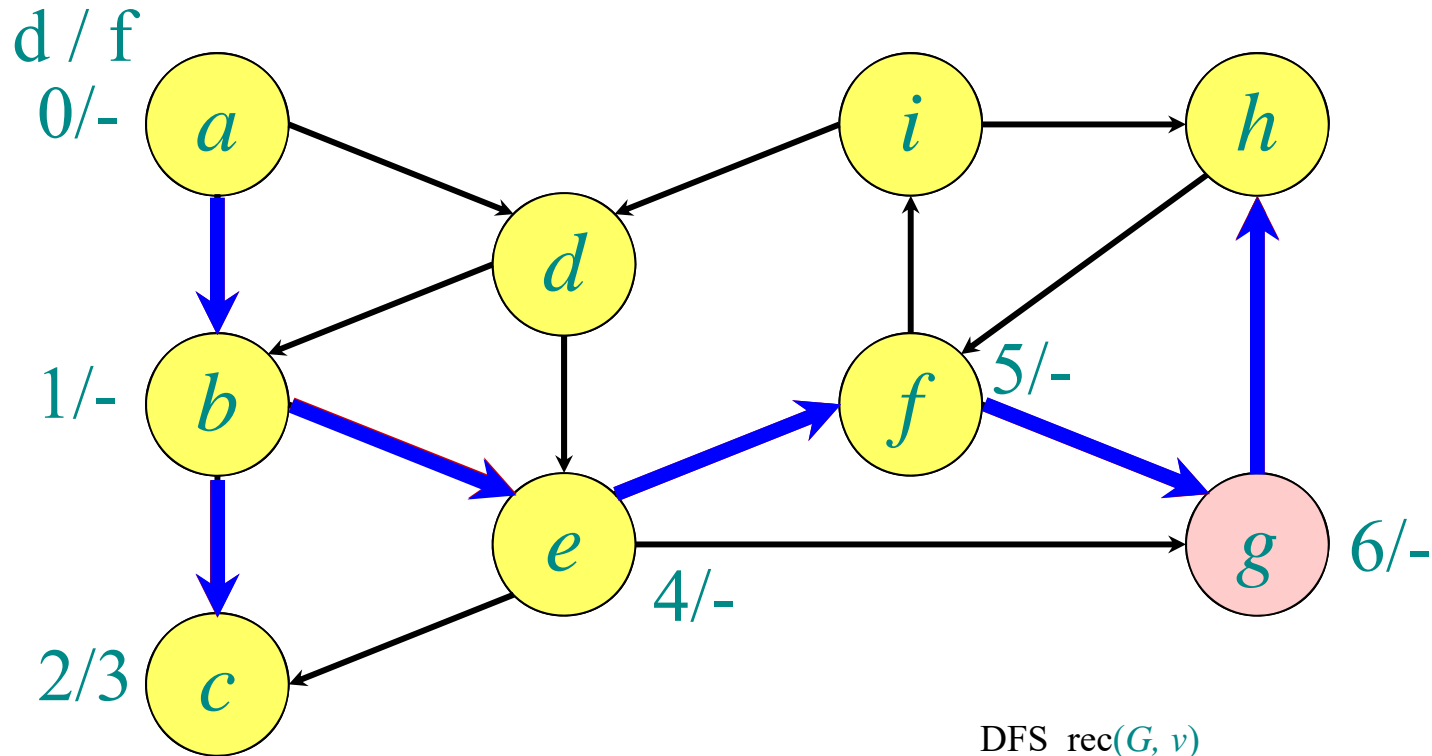
$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 - a b b e f

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```

Example of depth-first search



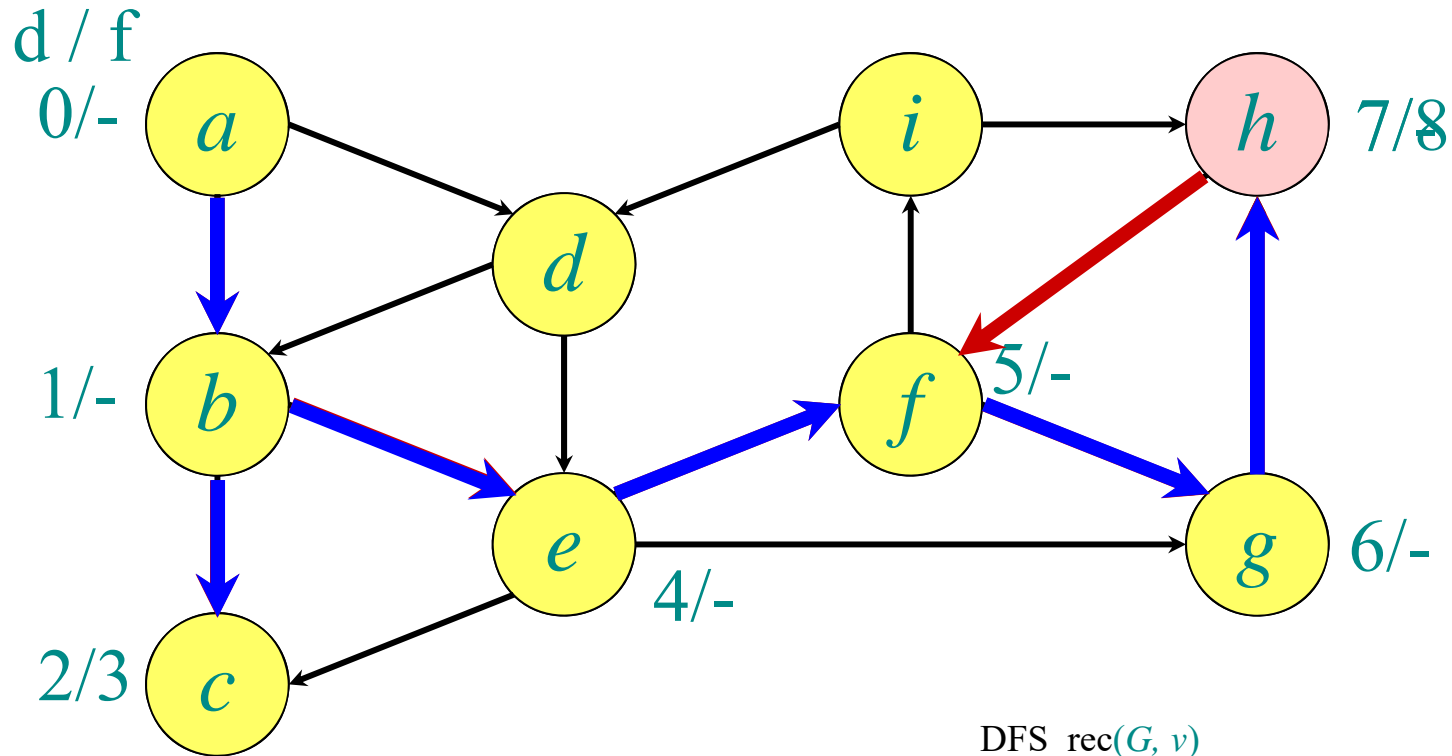
$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 - a b b e f g

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```

Example of depth-first search



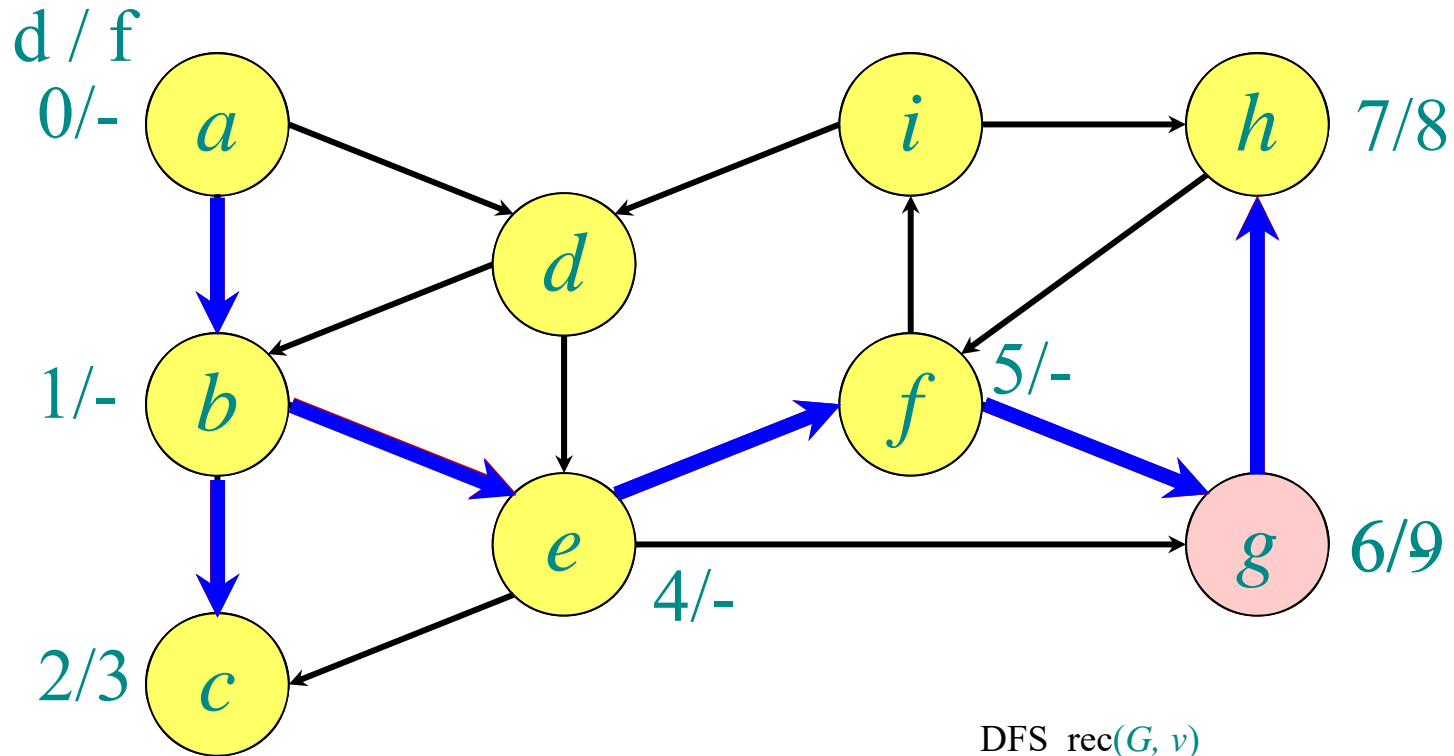
$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 - a b b e f g

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```

Example of depth-first search



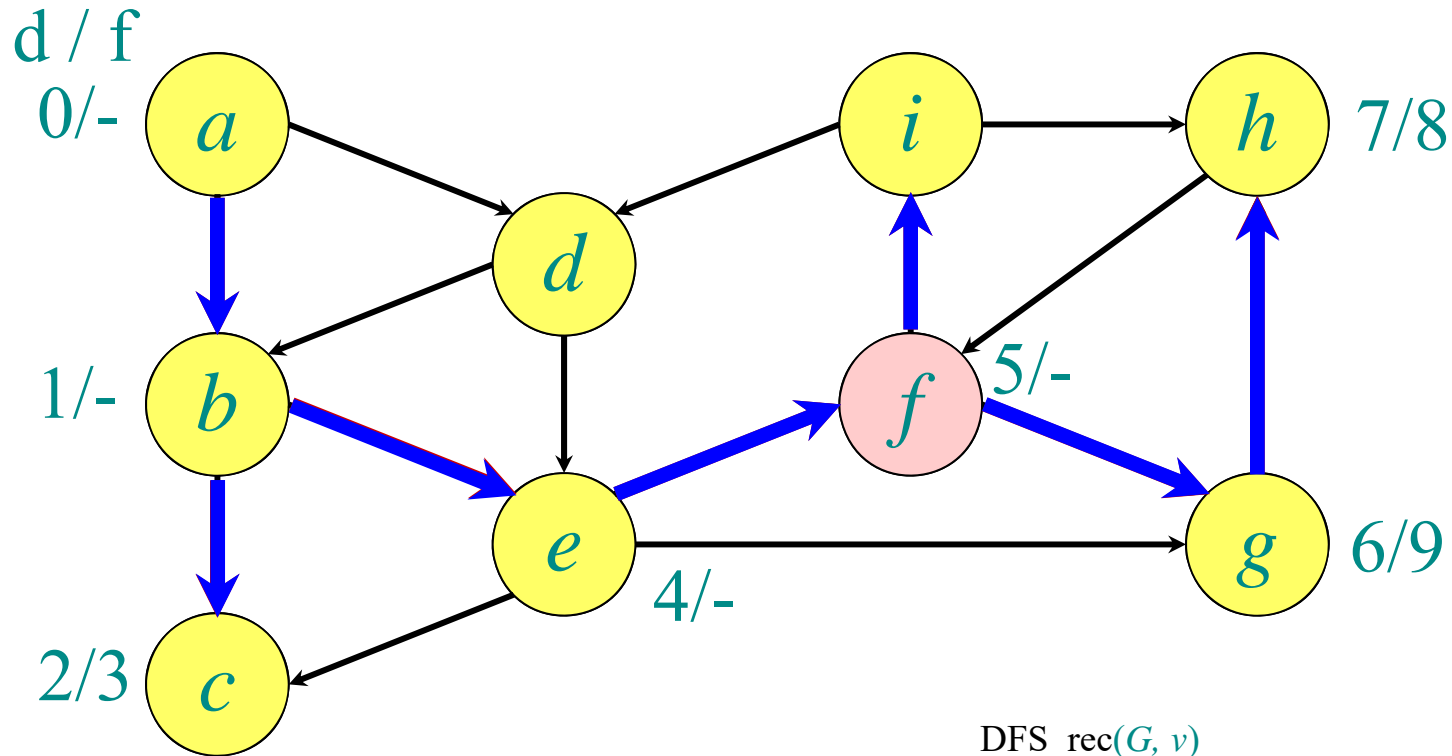
$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 - a b b e f g

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```

Example of depth-first search

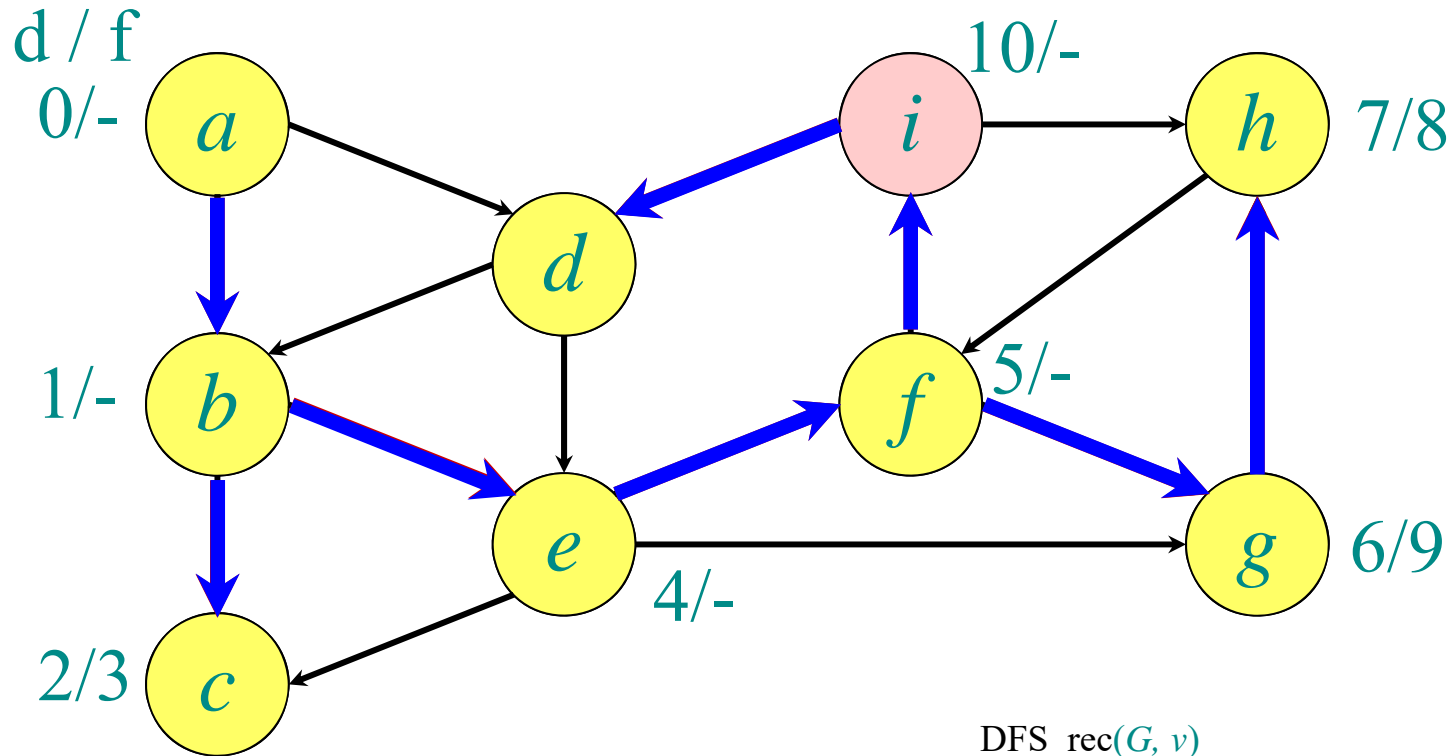


$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 $\quad \quad \underline{- \ a \ b \quad \quad b \ e \ f \ g \ f}$

Store edges in predecessor array

```
DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
```


Example of depth-first search



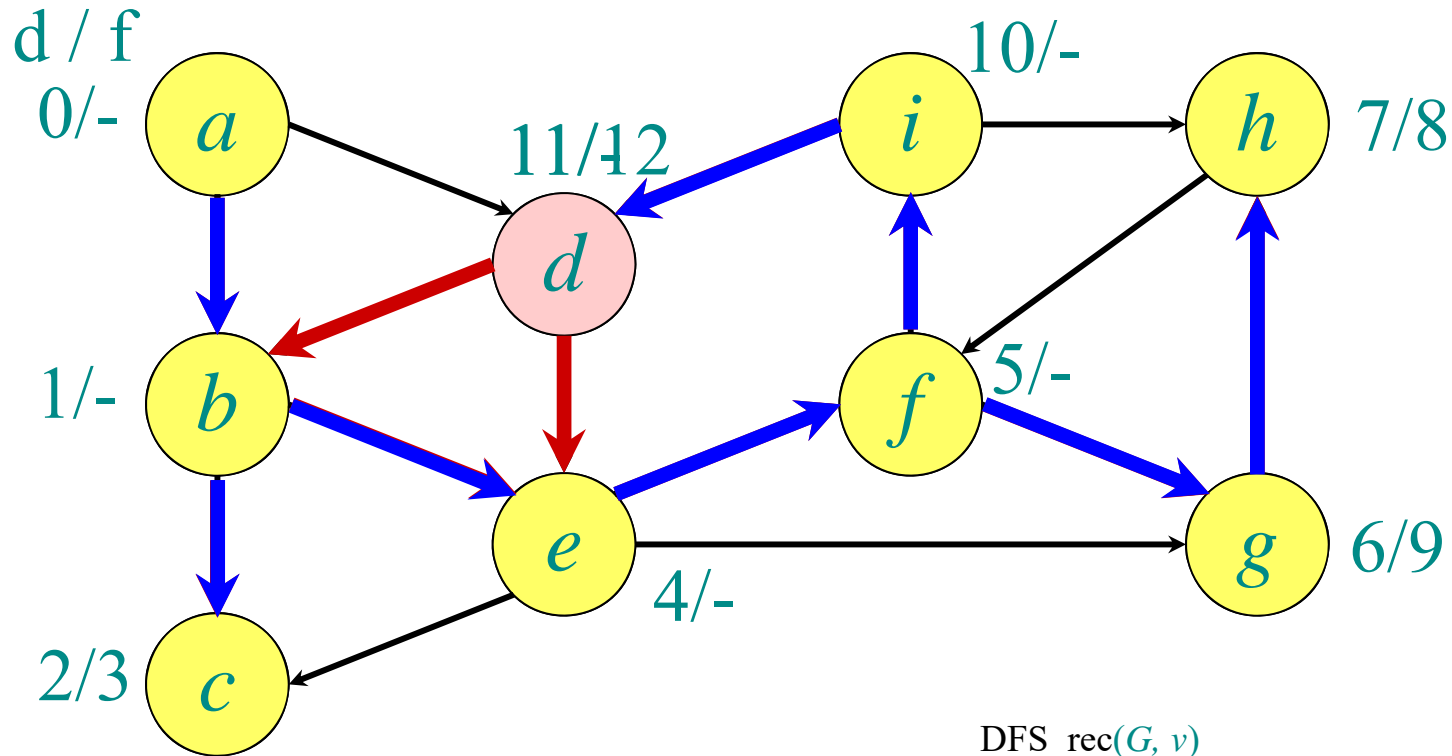
$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 $\quad \underline{- \ a \ b \ i \ b \ e \ f \ g \ f}$

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```

Example of depth-first search



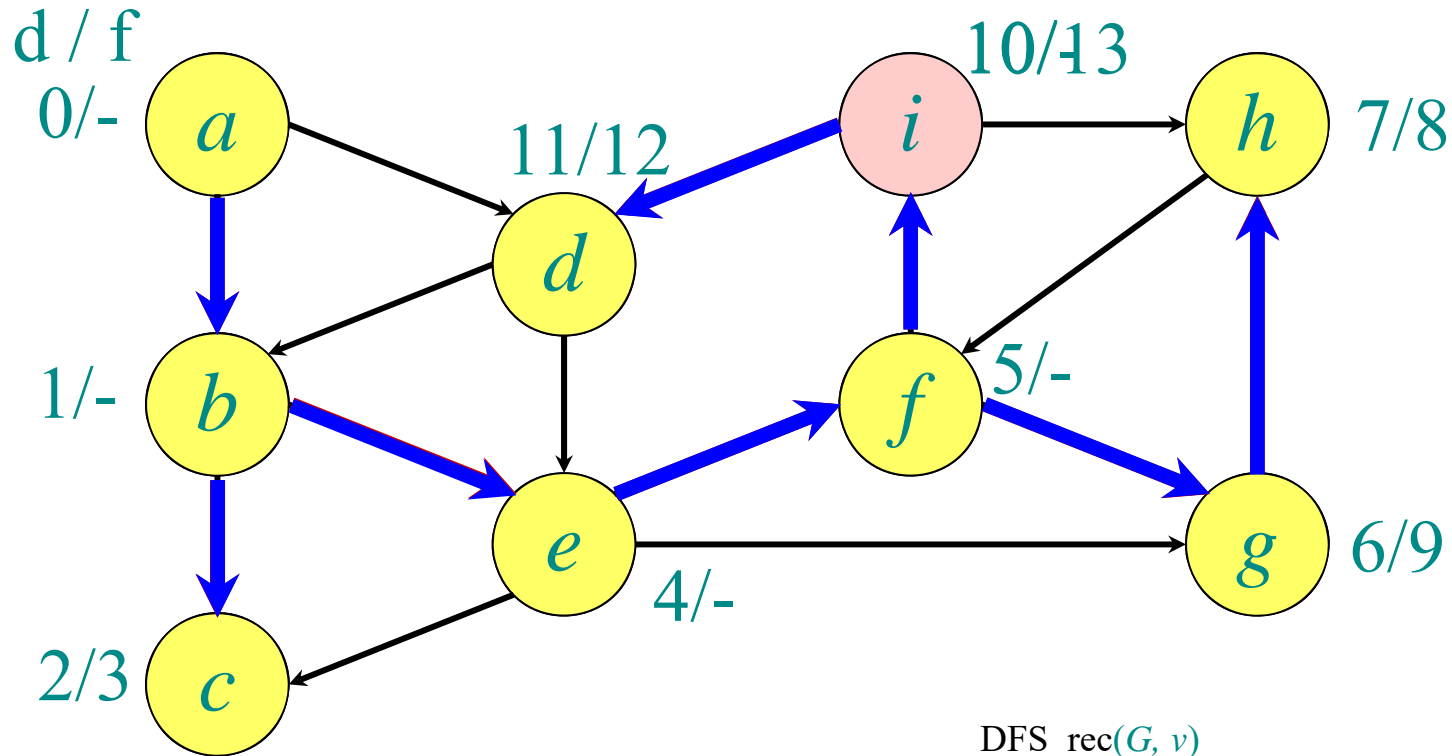
$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 $\quad \quad \quad - \ a \ b \ i \ b \ e \ f \ g \ f$

Store edges in predecessor array

```

DFS_rec( $G, v$ )
  mark  $v$  as "visited" //  $d[v]=++time$ 
  for each  $w$  adjacent to  $v$  do
    if  $w$  is unvisited
      Add edge  $(v,w)$  to tree  $T$ 
      DFS_rec( $G,w$ )
  mark  $v$  as "finished" //  $f[v]=++time$ 
    
```

Example of depth-first search



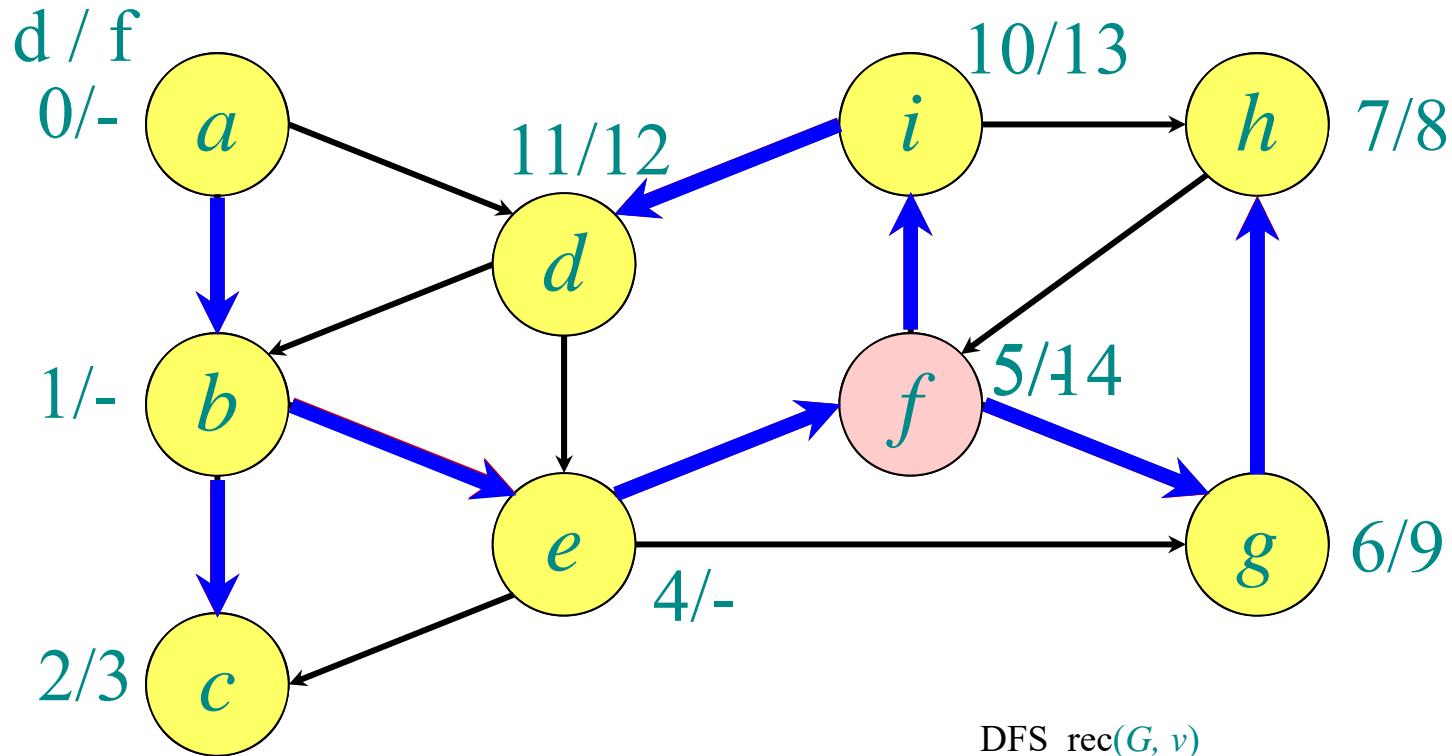
π : a b c d e f g h i
 - a b i b e f g f

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```

Example of depth-first search



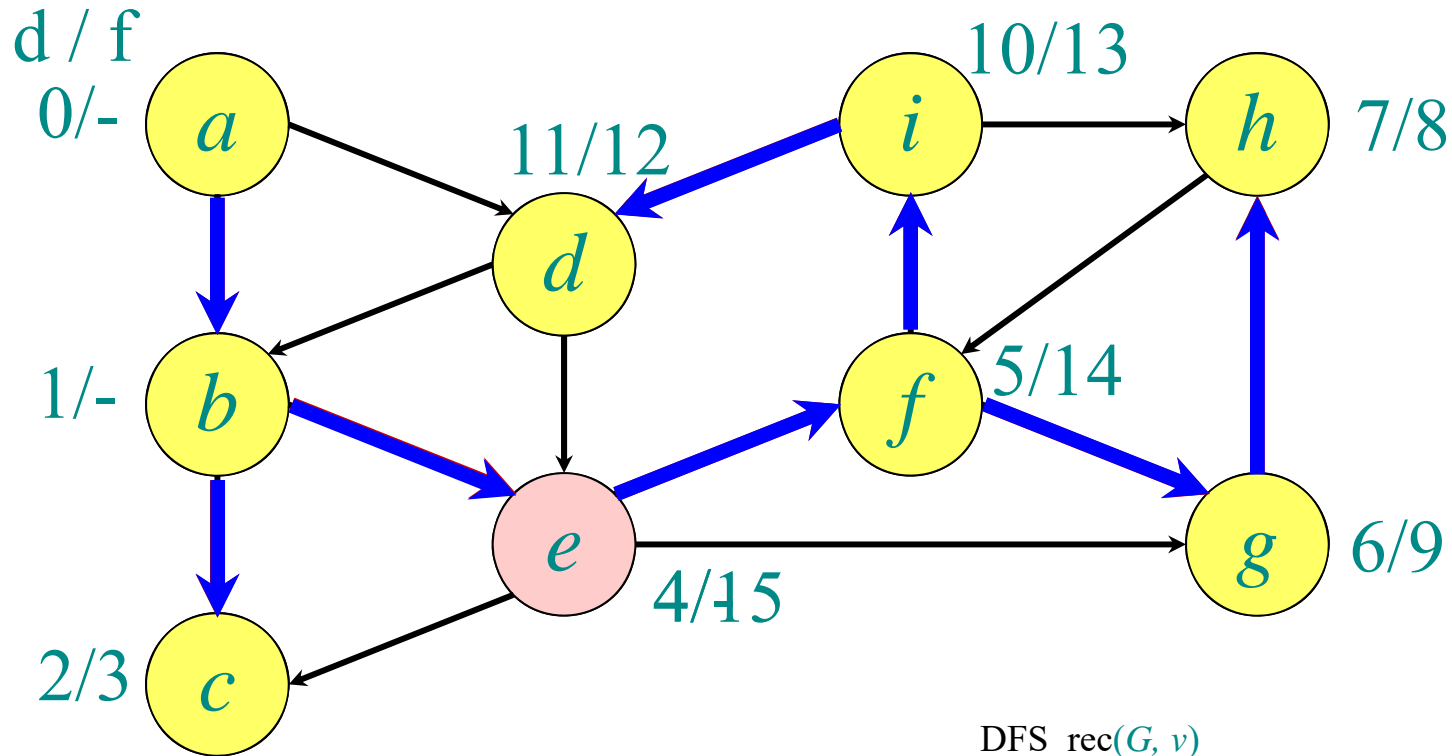
$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 $\quad \underline{- \ a \ b \ i \ b \ e \ f \ g \ f}$

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```

Example of depth-first search



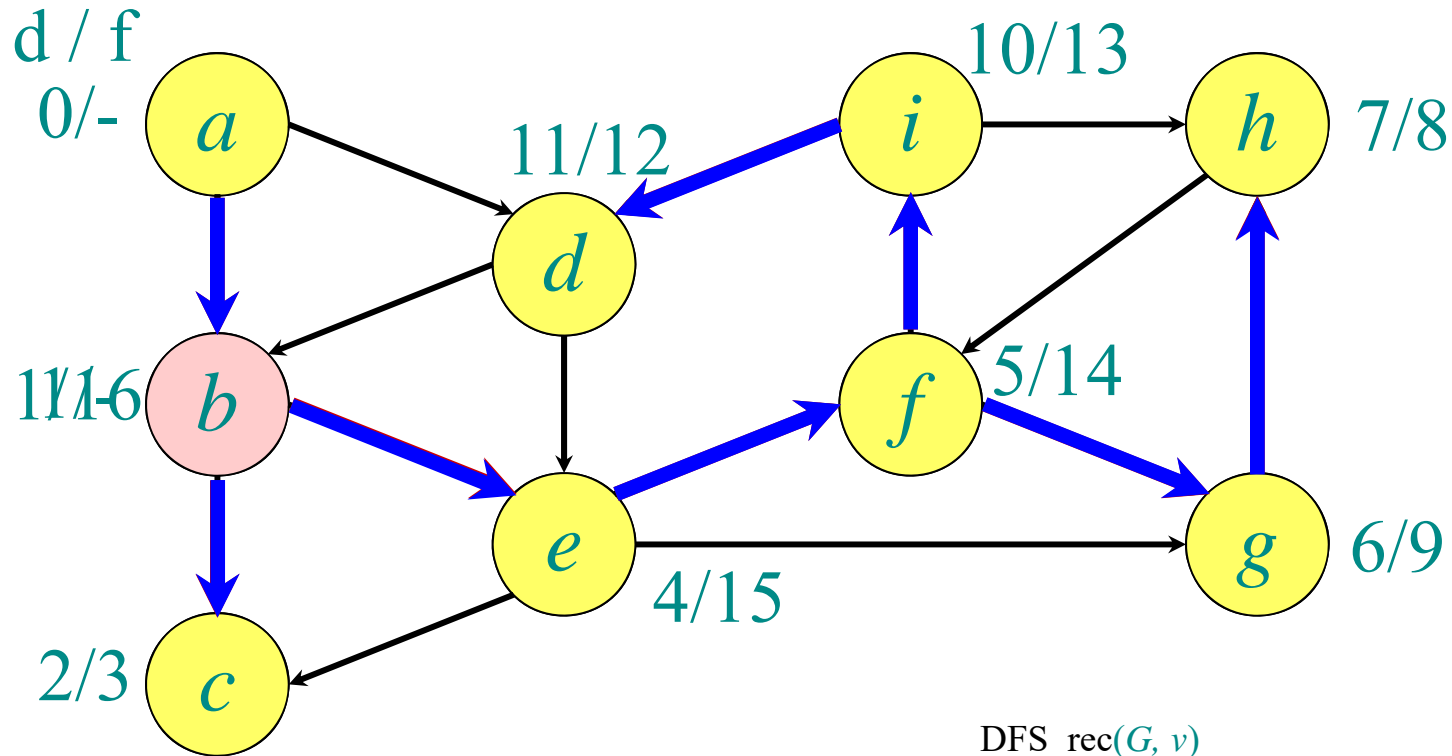
π : a b c d e f g h i
 - a b i b e f g f

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```

Example of depth-first search



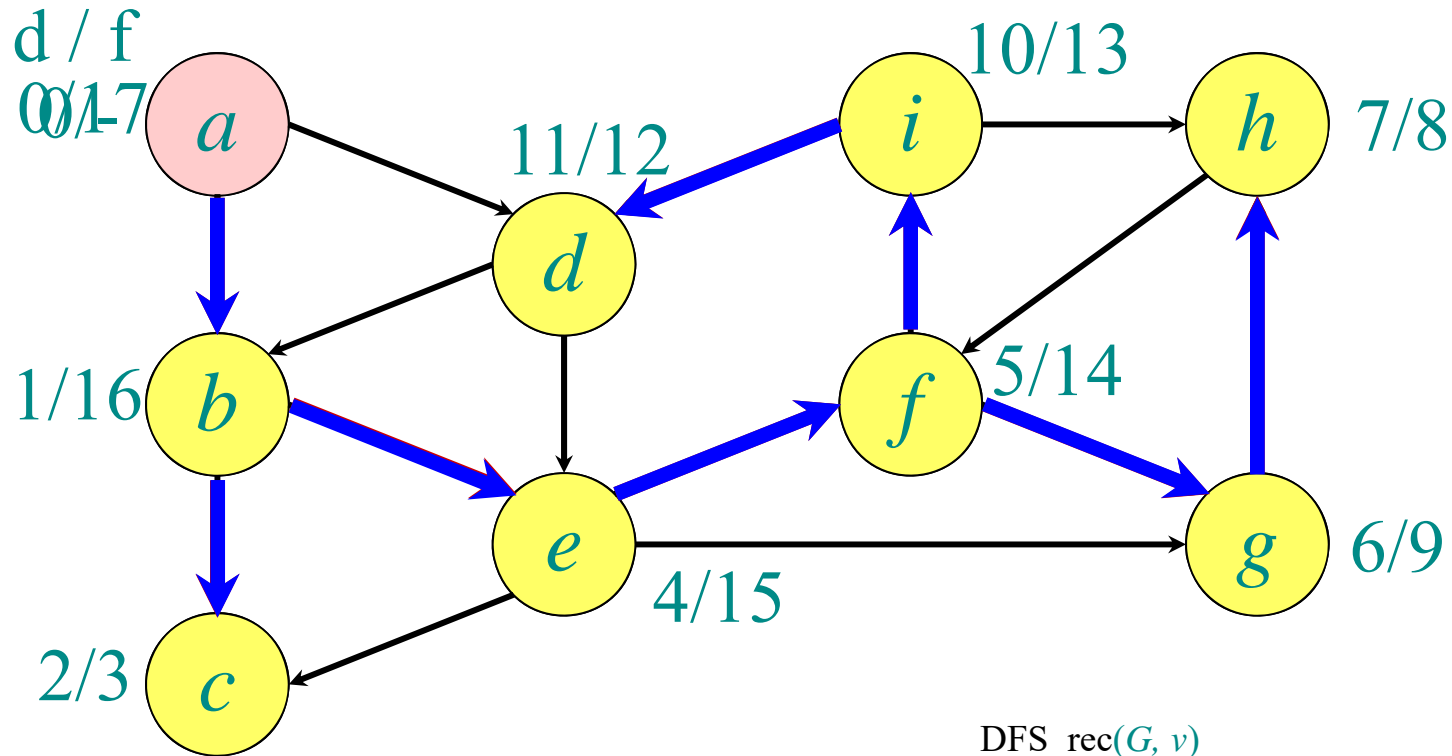
π : a b c d e f g h i
 - a b i b e f g f

Store edges in predecessor array

```

DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
    
```

Example of depth-first search



$\pi: \underline{a \ b \ c \ d \ e \ f \ g \ h \ i}$
 - a b i b e f g f

Store edges in predecessor array

```
DFS_rec(G, v)
  mark v as "visited" // d[v]=++time
  for each w adjacent to v do
    if w is unvisited
      Add edge (v,w) to tree T
      DFS_rec(G,w)
  mark v as "finished" // f[v]=++time
```

Depth-First Search (DFS)

$O(n)$

$O(n)$

without
DFS_rec

DFS($G=(V,E)$)

Mark all vertices in G as “unvisited” // **time=0**

for each vertex $v \in V$ **do**

if v is unvisited

DFS_rec(G,v)

$O(1)$

$O(deg(v))$

without
recursive call

DFS_rec(G, v)

mark v as “visited” // **$d[v]=++time$**

for each w adjacent to v **do**

if w is unvisited

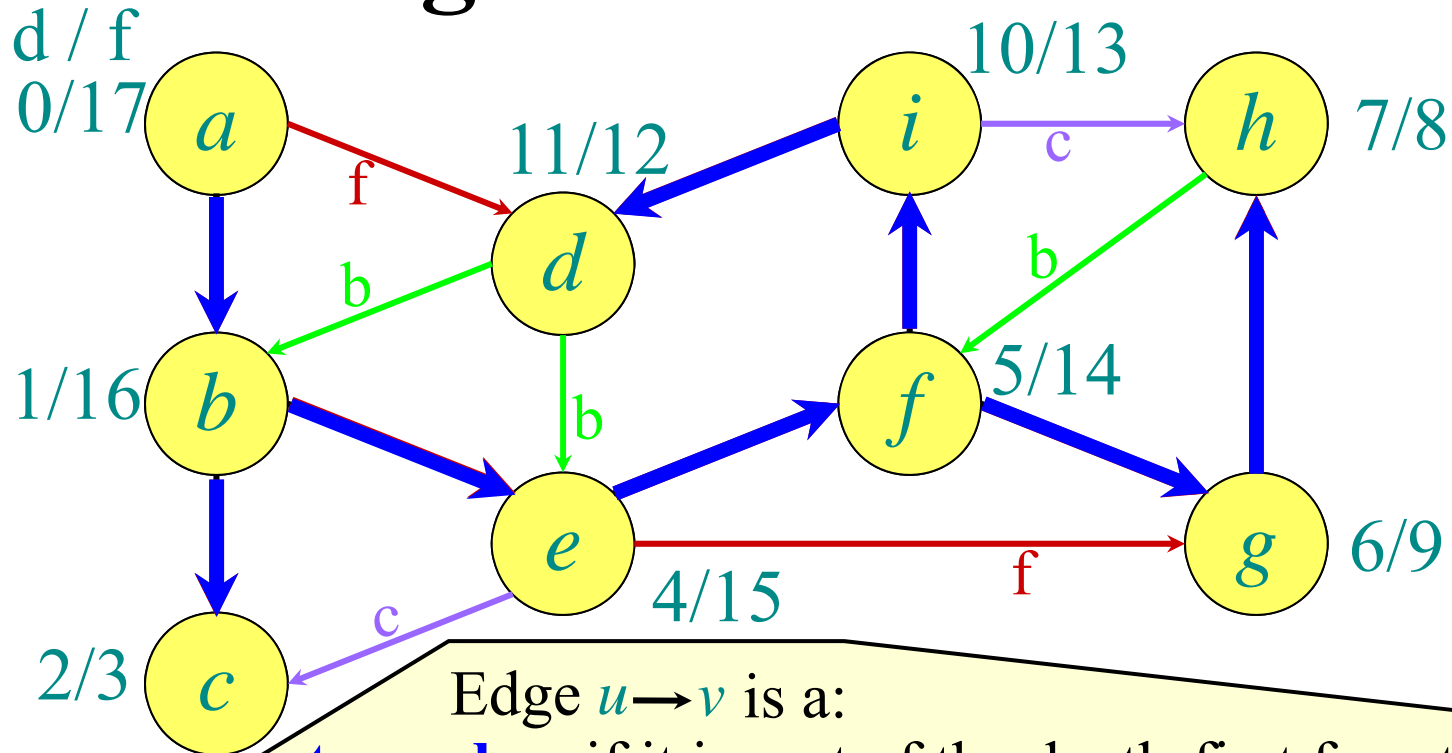
Add edge (v,w) to tree T

DFS_rec(G,w)

mark v as “finished” // **$f[v]=++time$**

\Rightarrow With Handshaking Lemma, all recursive calls are $O(m)$, for a total of $O(n + m)$ runtime

DFS edge classification



Edge $u \rightarrow v$ is a:

- **tree edge**, if it is part of the depth-first forest.
- **back edge**, if u connects to an ancestor v in a depth-first tree. It holds $d(u) > d(v)$ and $f(u) < f(v)$.
- **forward edge**, if it connects u to a descendant v in a depth-first tree. It holds $d(u) < d(v)$.
- **cross edge**, if it is any other edge. It holds $d(u) > d(v)$ and $f(u) > f(v)$.

Paths, Cycles, Connectivity

Let $G=(V,E)$ be a directed (or undirected) graph

- A **path** from v_1 to v_k in G is a sequence of vertices v_1, v_2, \dots, v_k such that $(v_i, v_{i+1}) \in E$ (or $\{v_i, v_{i+1}\} \in E$ if G is undirected) for all $i \in \{1, \dots, k-1\}$.
- A path is **simple** if all vertices in the path are distinct.
- A path v_1, v_2, \dots, v_k forms a **cycle** if $v_1 = v_k$.
- A graph with no cycles is **acyclic**.
 - An undirected acyclic graph is called a **tree**. (Trees do not have to have a root vertex specified.)
 - A directed acyclic graph is a **DAG**. (A DAG can have undirected cycles if the direction of the edges is not considered.)
- An undirected graph is **connected** if every pair of vertices is connected by a path. A directed graph is **strongly connected** if for every pair $u, v \in V$ there is a path from u to v and there is a path from v to u .
- The **(strongly) connected components** of a graph are the equivalence classes of vertices under this reachability relation.

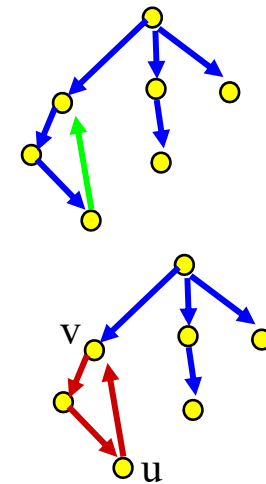
DAG Theorem

Theorem: A directed graph G is acyclic
 \Leftrightarrow a depth-first search of G yields no back edges.

Proof:

“ \Rightarrow ”: Suppose there is a back edge (u,v) . Then by definition of a back edge there would be a cycle.

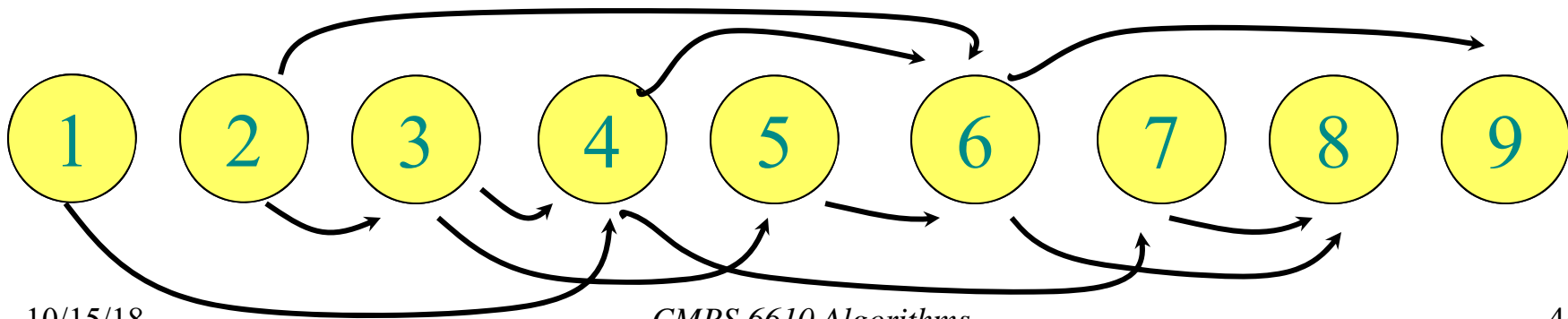
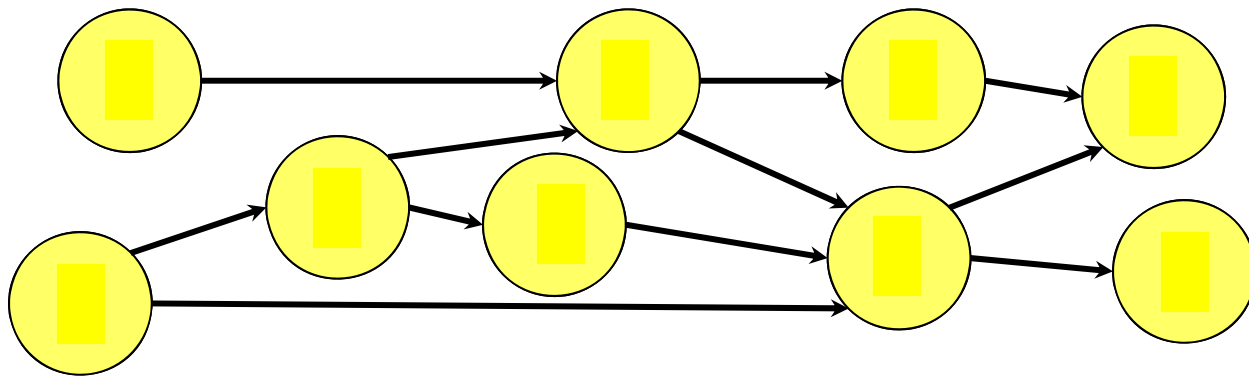
“ \Leftarrow ”: Suppose G contains a cycle c . Let v be the first vertex to be discovered in c , and let u be the preceding vertex in c . v is an ancestor of u in the depth-first forest, hence (u,v) is a back edge.



Topological Sort

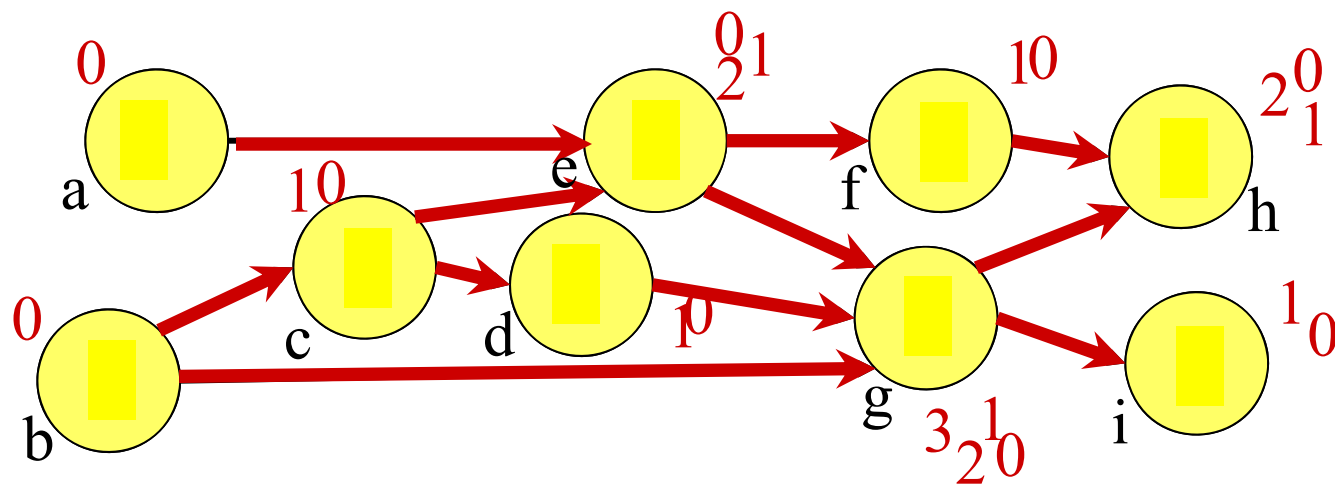
Topologically sort the vertices of a *directed acyclic graph (DAG)*:

- Determine $f: V \rightarrow \{1, 2, \dots, |V|\}$ such that $(u, v) \in E \Rightarrow f(u) < f(v)$.



Topological Sort Algorithm

- Store vertices with **in-degree** 0 in a queue Q.
- While Q is not empty
 - Dequeue vertex v, and give it the next number
 - Decrease **in-degree** of all adjacent vertices by 1
 - Enqueue all vertices with **in-degree** 0



Q: a , b , c, e, d, f, g, i, h

Topological Sort Runtime

Runtime:

- $O(|V|+|E|)$ because every edge is touched once, and every vertex is enqueued and dequeued exactly once

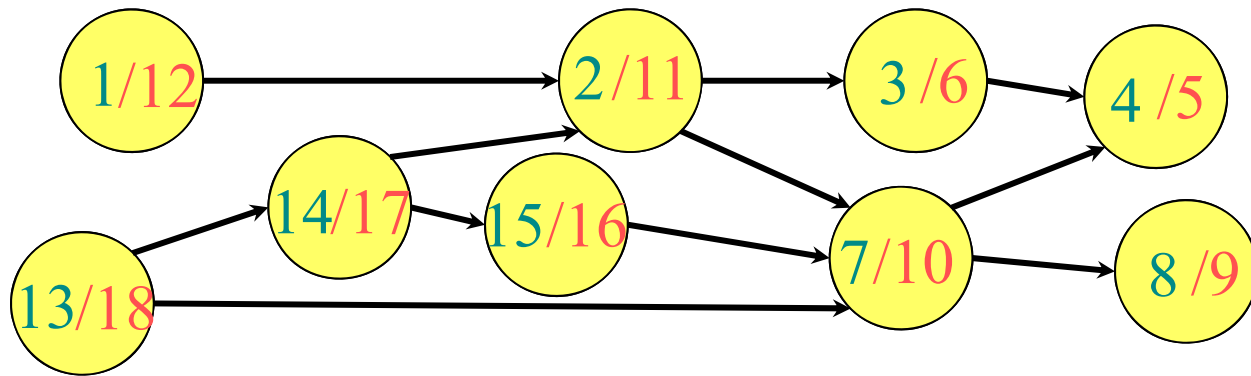
DFS-Based Topological Sort Algorithm

- Call DFS on the directed acyclic graph $G=(V,E)$
 - \Rightarrow Finish time for every vertex
- Reverse the finish times (highest finish time becomes the lowest finish time,...)
 - \Rightarrow Valid function $f': V \rightarrow \{1, 2, \dots, |V|\}$ such that $(u, v) \in E \Rightarrow f'(u) < f'(v)$

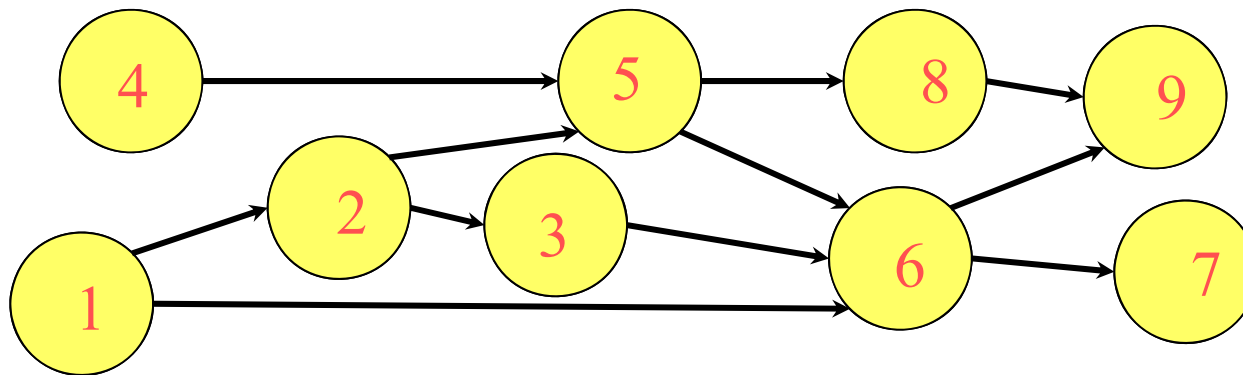
Runtime: $O(|V|+|E|)$

DFS-Based Topological Sort

- Run DFS:



- Reverse finish times:



DFS-Based Top. Sort Correctness

- Need to show that for any $(u, v) \in E$ holds $f(v) < f(u)$.
(since we consider reversed finish times)
- Consider exploring edge (u, v) in DFS:
 - v cannot be visited and unfinished (and hence an ancestor in the depth first tree), since then (u, v) would be a back edge (which by the DAG lemma cannot happen).
 - If v has not been visited yet, it becomes a descendant of u , and hence $f(v) < f(u)$. (tree edge)
 - If v has been finished, $f(v)$ has been set, and u is still being explored, hence $f(u) > f(v)$ (forward edge, cross edge) .

Topological Sort Runtime

Runtime:

- $O(|V|+|E|)$ because every edge is touched once, and every vertex is enqueued and dequeued exactly once
- DFS-based algorithm: $O(|V| + |E|)$