CMPS 6610 Algorithms – Fall 18

10/22/18

7. Homework

Due **10/29/18** at the beginning of class

Justify all your answers.

1. **Union-Find (6 points)**
   (a) (3 points) Consider a disjoint-set forest data structure with path compression. Assume a sequence of MAKE-SET, UNION, and FIND-SET operations have already been performed, and the data structure contains n elements. Prove that then any consecutive sequence of FIND-SET operations takes $O(n)$ time.
   (b) (3 points) Now consider a disjoint-set forest data structure with path compression and union-by-weight. Give a sequence of MAKE-SET, FIND-SET, and UNION operations which causes a taller tree to be linked underneath a shorter tree.

2. **MST (9 points)**
   Let $G = (V, E, w)$ be a connected undirected weighted graph.
   (a) (2 points) What are the runtimes of Prim's, Kruskal's, and Boruvka's algorithms if G is given in an adjacency matrix? How can you speed up these runtimes?
   (b) (2 points) Assume a list of all edges E is given in non-decreasing order of edge weights. In this case, what is the fastest runtime to compute an MST for G?
   (c) (5 points) Assume for each vertex $v \in V$, the adjacency list $Adj(v)$ stores adjacent vertices in non-decreasing order of edge weights. Show how to modify Boruvka's algorithm to run in $O(|E| + |V| \log |V|)$ time in this case.

3. **Amortized Runtime for Binary Heaps (8 points)**
   Consider a regular binary min-heap data structure that supports the operations INSERT and EXTRACT-MIN in $O(\log n)$ worst-case time, where n is the number of elements in the heap. Give a potential function Φ such that the amortized cost of INSERT is $O(\log n)$ and the amortized cost of EXTRACT-MIN is $O(1)$, and show that it works.
   (Hint 1: One possible potential function is based on $n_i \log n_i$, where n_i is the number of elements in the heap after i operations.
   Hint 2: A fact that might be useful: $n \log \frac{n}{n-1} \leq \frac{2}{\ln 2}$)