CMPS 6610/4610 – Fall 2016

Knapsack Problem

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Slides courtesy of Charles Leiserson with changes and additions by Carola Wenk

Knapsack Problem

- Given a knapsack with weight capacity $W > 0$, and given n items of positive integer weights $w_1, \dots, w_n$ and positive integer values $v_1, \dots, v_n$.
(So, item i has value v_i and weight w_i.)

- **0-1 Knapsack Problem:** Compute a subset of items that maximize the total value (sum), and they all fit into the knapsack (total weight at most W).

- **Fractional Knapsack Problem:** Same as before but we are allowed to take fractions of items ($\rightarrow$ gold dust).

Greedy Knapsack

- Greedy Strategy:
  - Compute $\frac{v_i}{w_i}$ for each i
  - Greedily take as much as possible of the item with the highest value/weight. Then repeat/recurse.

  $\Rightarrow$ Sort items by value/weight

  $\Rightarrow$ $O(n \log n)$ runtime

Knapsack Example

item	1	2	3	
value	12	15	4	W=4
weight	2	3	1	
value/weight	6	5	4	

- **Greedy fractional:** Take item 1 and 2/3 of item 2

$\Rightarrow$ weight=4, value=12+2/3·15 = 12+10 = 22

- **Greedy 0-1:** Take item 1 and then item 3

$\Rightarrow$ weight = 1+2=3, value=12+4=16

- **Optimal 0-1:** Take items 2 and 3, value =19

Optimal Substructure

- Let $s_1, \ldots, s_n$ be an optimal solution, where s_i = amount of item i that is taken; $0 \leq s_i \leq 1$
- Suppose we remove one item. $\rightarrow n - 1$ items left
- Is the remaining "solution" still an optimal solution for $n - 1$ items?
- Yes; cut-and-paste.

Correctness Proof for Greedy

• Suppose items $1, \ldots, n$ are numbered in decreasing order by value/weight.

• Greedy solution G: Takes all elements $1, \ldots, j, \ldots, i^*$-1 and a fraction of i^*.

• Assume optimal solution S is different from G. Assume S takes only a fraction $\frac{1}{a}$ of item j, for $j \leq i^*$-1.

• Create new solution S' from S by taking $w_j - 1/a$ weight away from items $> j$, and add $w_j - 1/a$ of item j back in. Hence, all of item j is taken.

$\Rightarrow$ New solution S' has the same weight but increased value. This contradicts the assumption that S was optimal.
$\Rightarrow$ S=G. □

General Solution: DP

- $D[i, w]$= max value possible for taking a subset of items $1, \ldots, i$ with knapsack constraint w.
- $D[0, w] = D[i, 0] = 0$ for all $0 \leq i \leq n$ and $0 \leq w \leq W$

  $D[i, w] = 0$ for $w < 0$

  $$D[i, w] = \max(\underbrace{D[i-1, w]}_{\text{don't take item i}}, \underbrace{v_i + D[i-1, w - w_i]}_{\text{take item i}})$$
- Compute $D[n, W]$ by filling an $n \times W$ DP-table.

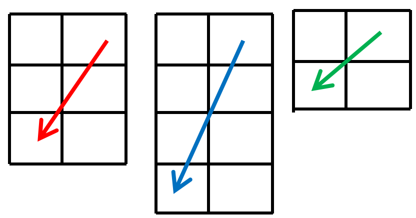
  $\Rightarrow$ Two nested for-loops, runtime and space $\Theta(nW)$
- Trace back from $D[n, W]$ by redoing computation or following arrows. $\Rightarrow$ $\Theta(n + W)$ runtime

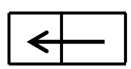
DP Example

Solution:
Take items 3 and 2

W=4

item	1	2	3
value	12	15	4
weight	2	3	1
value/weight	6	5	4

Take item i: 

Don’t take item i: 

w ↑				
W= 4	0	12	15	19
3	0	12	15	16
2	0	12	12	12
1	0	0	0	4
0	0 ←	0	0	0
	0	1	2	3 n → i

$$D[i,w] = \max(\underbrace{D[i-1,w]}_{\text{don't take item i}}, \underbrace{v_i + D[i-1,w-w_i]}_{\text{take item i}})$$