CMPS 6610/4610 – Fall 2016

Graphs

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Slides courtesy of Charles Leiserson with changes and additions by Carola Wenk

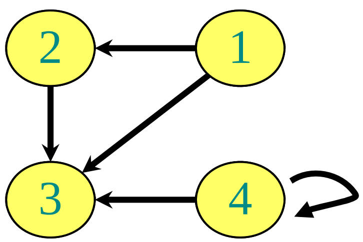
Graphs

Definition. A ***directed graph (digraph)*** $G = (V, E)$ is an ordered pair consisting of

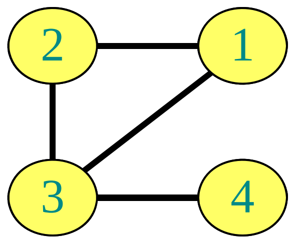
- a set V of ***vertices*** (singular: ***vertex***),
- a set $E \subseteq V \times V$ of ***edges***.

In an ***undirected graph*** $G = (V, E)$, the edge set E consists of *unordered* pairs of vertices.

directed graph



undirected graph

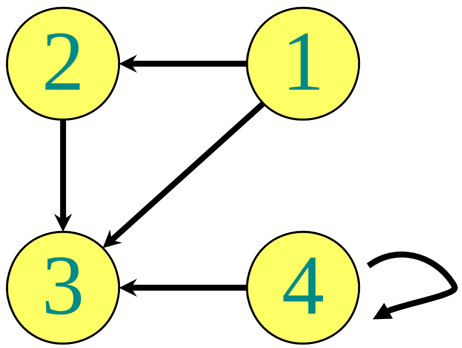


In either case, we have $|E| = O(|V|^2)$.
Moreover, if G is connected, then $|E| \geq |V| - 1$.

Adjacency-matrix representation

The ***adjacency matrix*** of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1 \,.\,.\, n, 1 \,.\,.\, n]$ given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E. \end{cases}$$

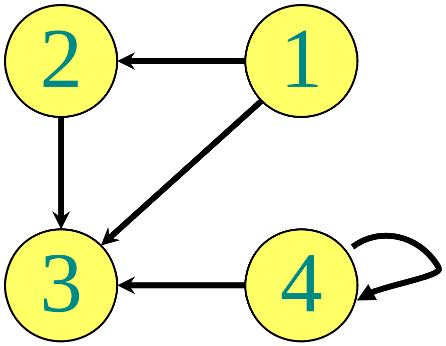


A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	1

$\Theta(|V|^2)$ storage $\Rightarrow$ ***dense*** representation.

Adjacency-list representation

An ***adjacency list*** of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to v.



$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3, 4\}$

For undirected graphs, $|Adj[v]| = degree(v)$.

For digraphs, $|Adj[v]| = out\text{-}degree(v)$.

Adjacency-list representation

Handshaking Lemma:

Every edge is counted twice

- For undirected graphs:
  $\sum_{v \in V} degree(v) = 2|E|$
- For digraphs:
  $\sum_{v \in V} in\text{-}degree(v) = \sum_{v \in V} out\text{-}degree(v) = |E|$

$\Rightarrow$ adjacency lists use $\Theta(|V| + |E|)$ storage
$\Rightarrow$ a ***sparse*** representation
$\Rightarrow$ We usually use this representation, unless stated otherwise

Graph Traversal

Let $G=(V,E)$ be a (directed or undirected) graph, given in adjacency list representation.

$|V| = n$, $|E| = m$

A graph traversal visits every vertex:

- Breadth-first search (BFS)
- Depth-first search (DFS)

Breadth-First Search (BFS)

```
BFS(G=(V,E))
    Mark all vertices in G as "unvisited" // time=0
    Initialize empty queue Q
    for each vertex v ∈ V do
        if v is unvisited
            visit v // time++
            Q.enqueue(v)
            BFS_iter(G)
```

```
BFS_iter(G)
    while Q is non-empty do
        v = Q.dequeue()
        for each w adjacent to v do
            if w is unvisited
                visit w   // time++
                Add edge (v,w) to T
                Q.enqueue(w)
```

Example of breadth-first search

while *Q* is non-empty **do**
    v = *Q*.dequeue()
    for each *w* adjacent to *v* **do**
        if *w* is unvisited
            visit *w*   // time++
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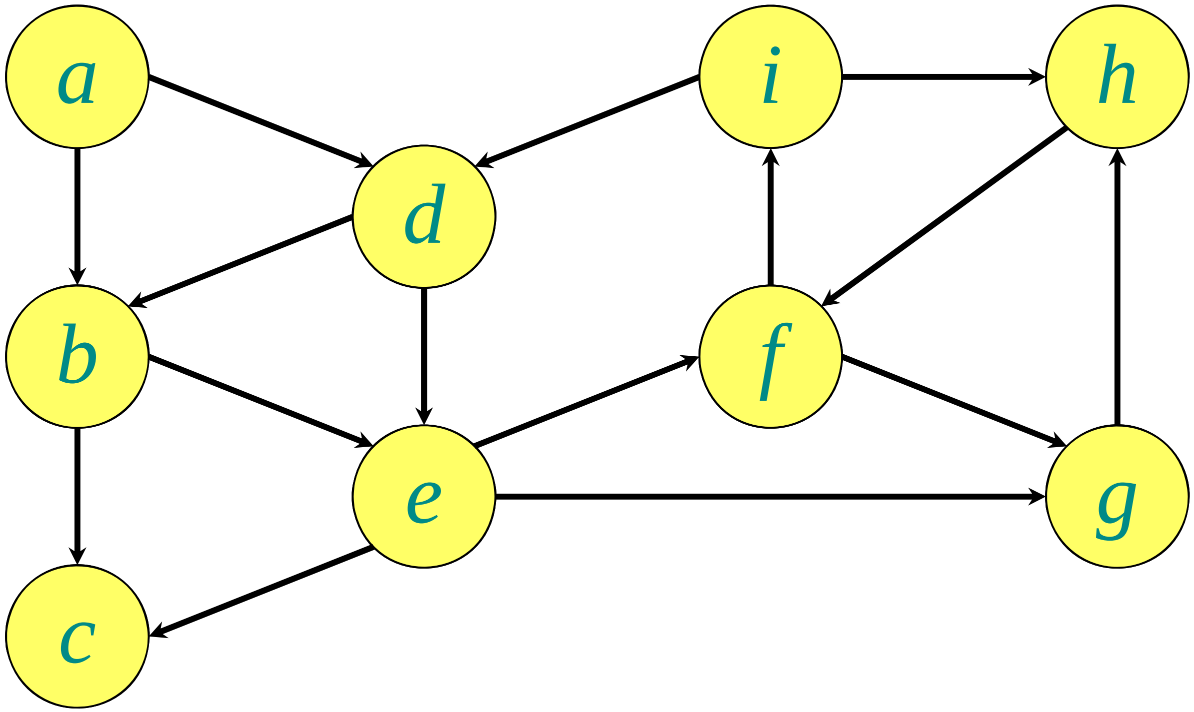

Q:

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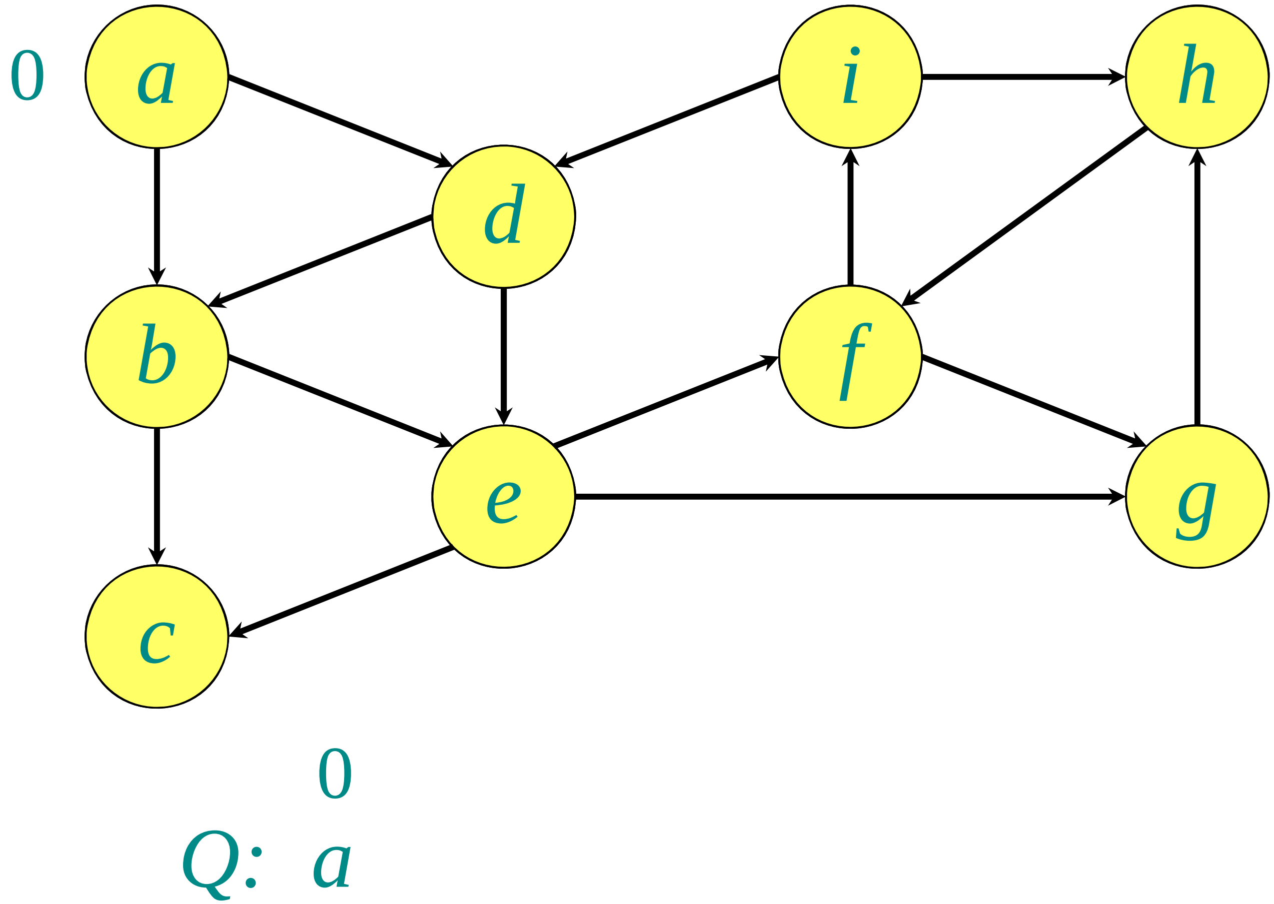

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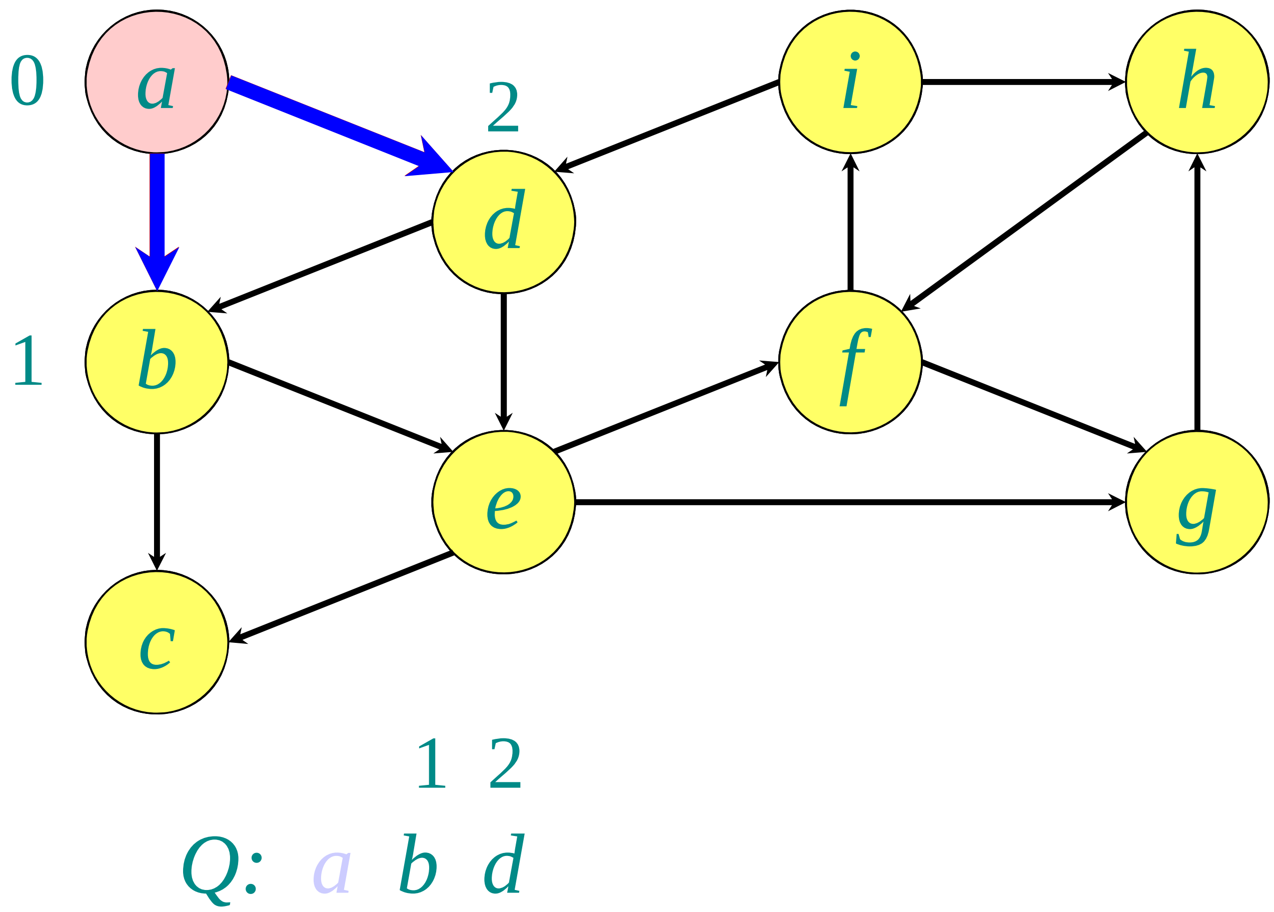

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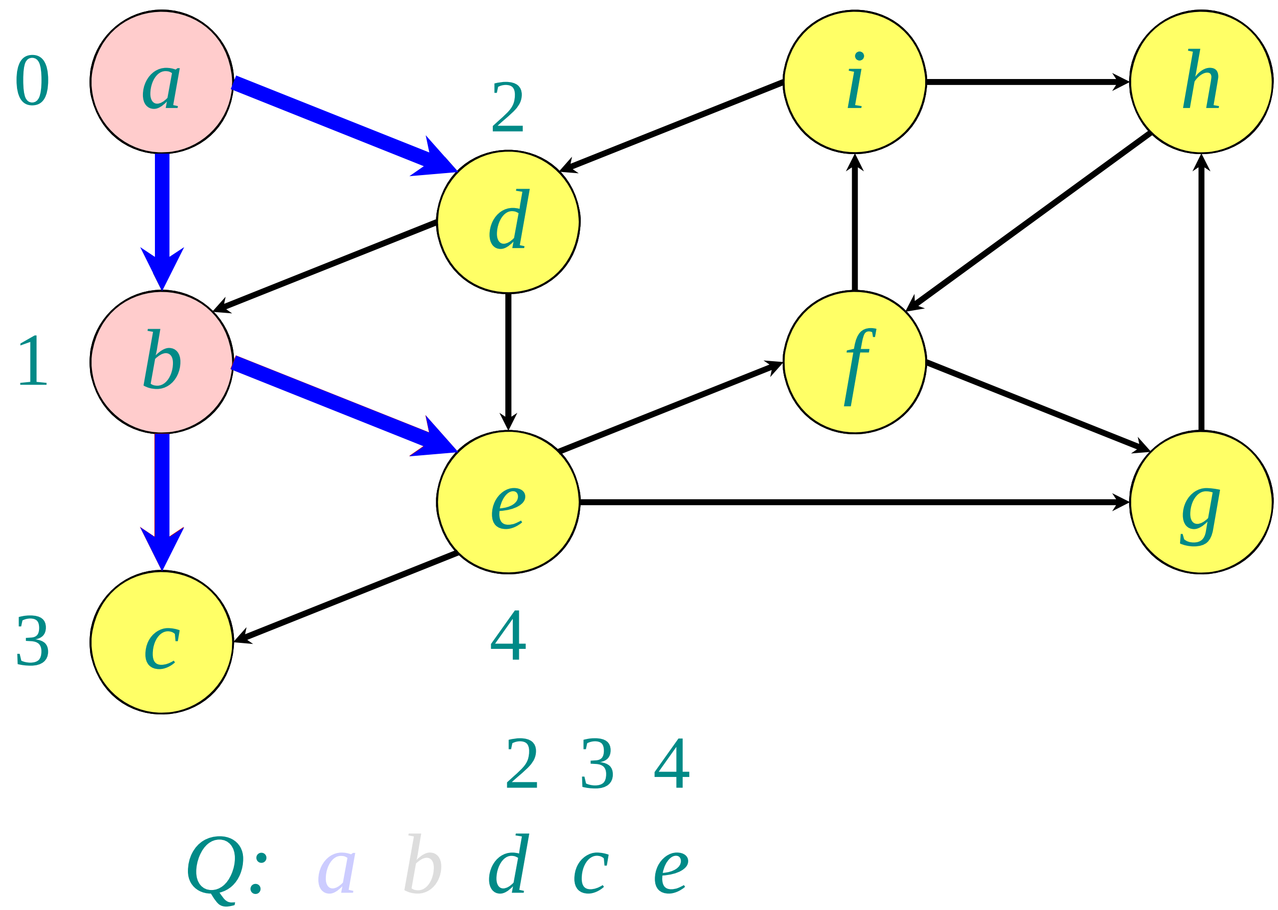

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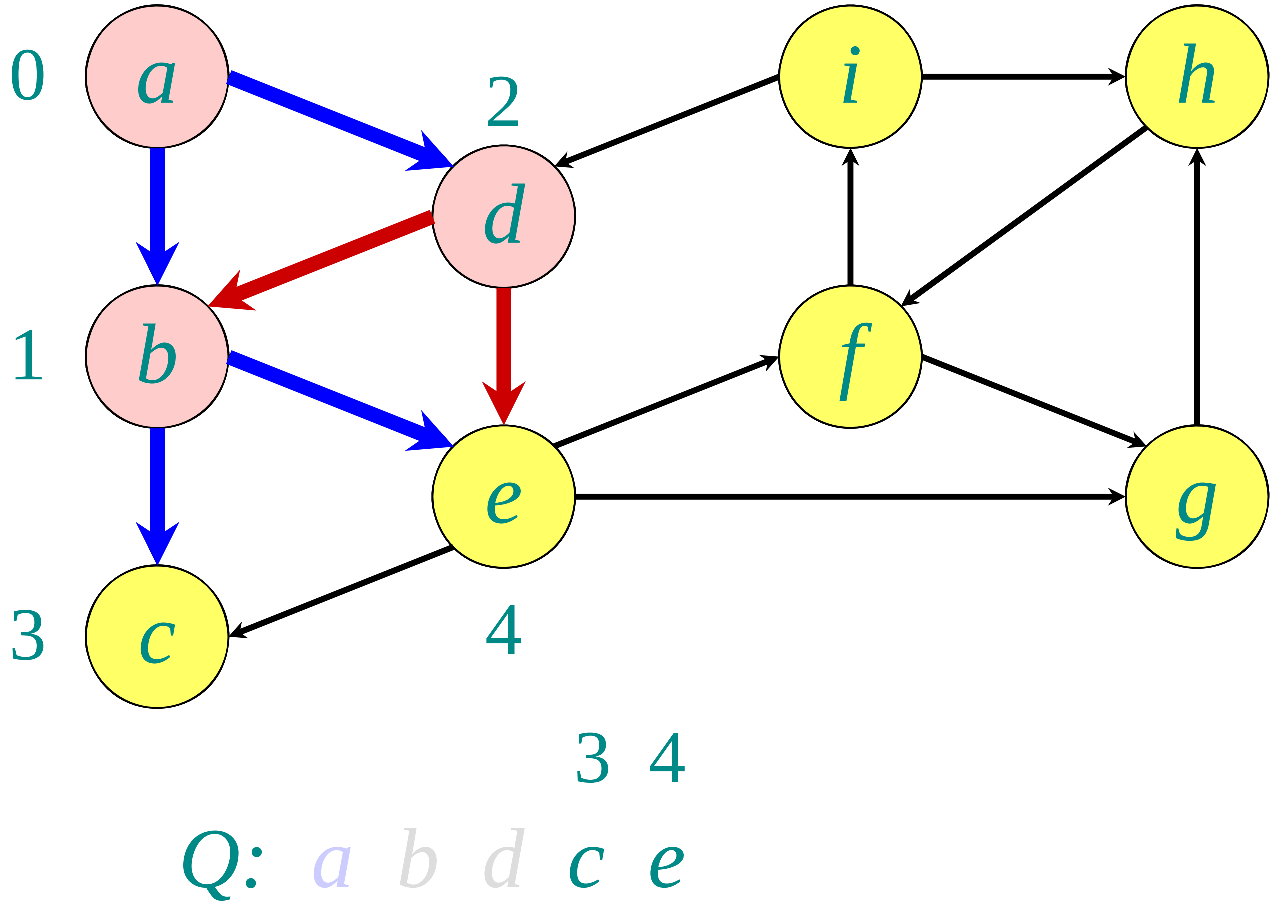

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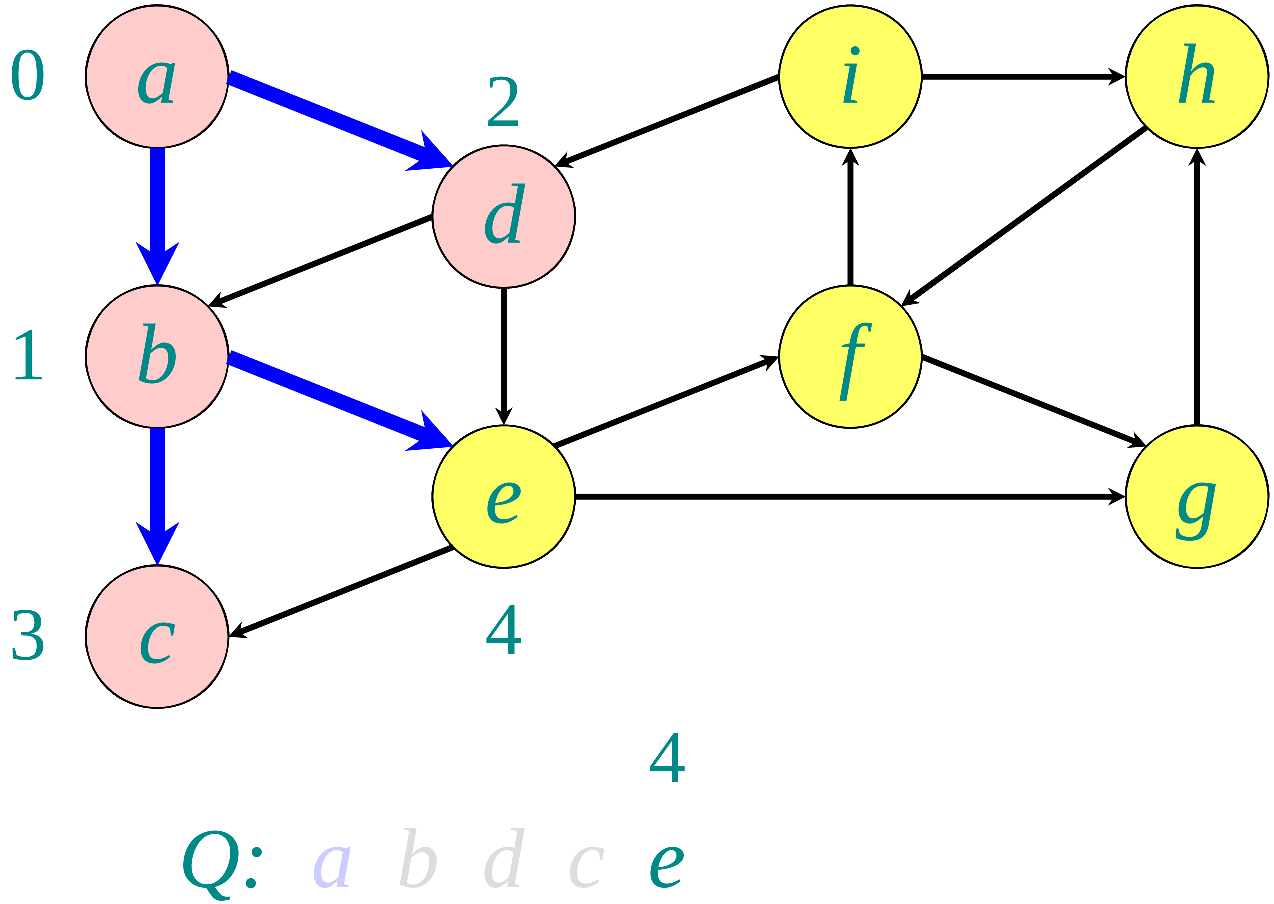

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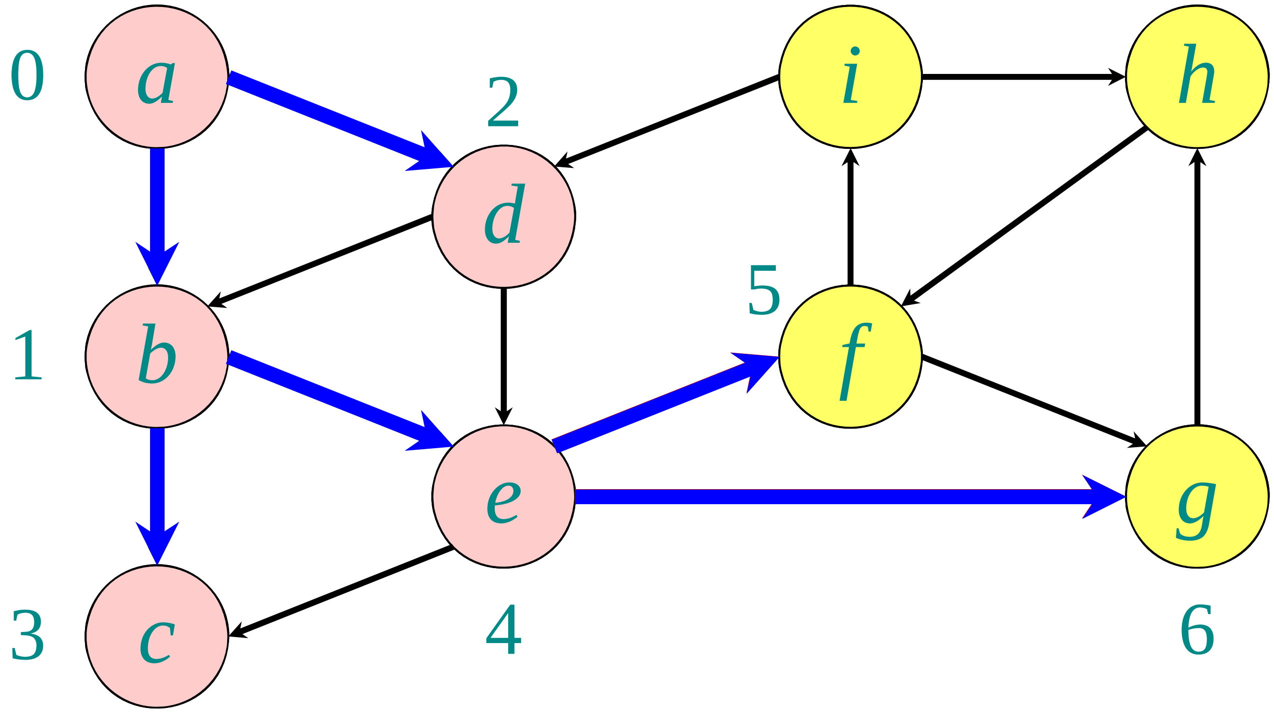

5 6

Q: *a* *b* *d* *c* *e* *f* *g*

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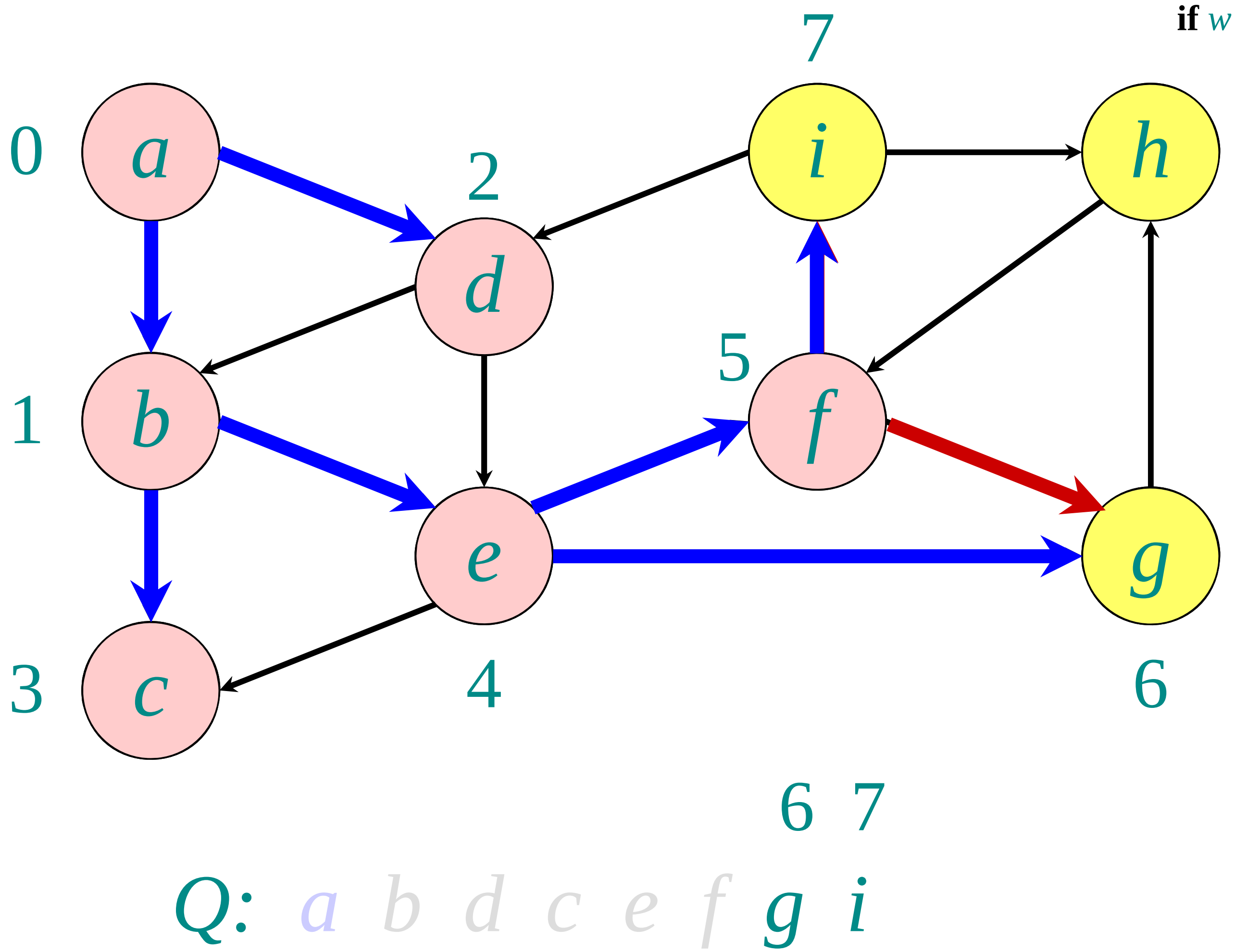

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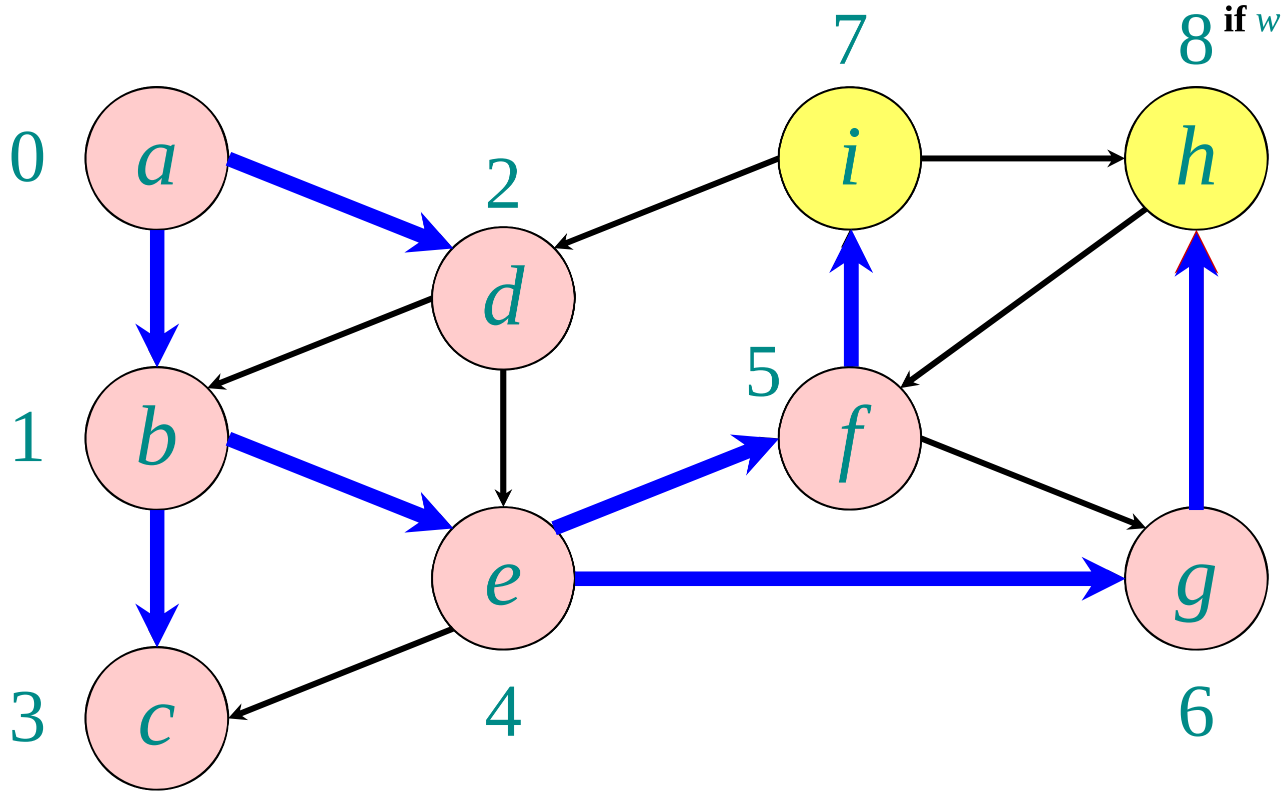

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Q: *a b d c e f g i h*

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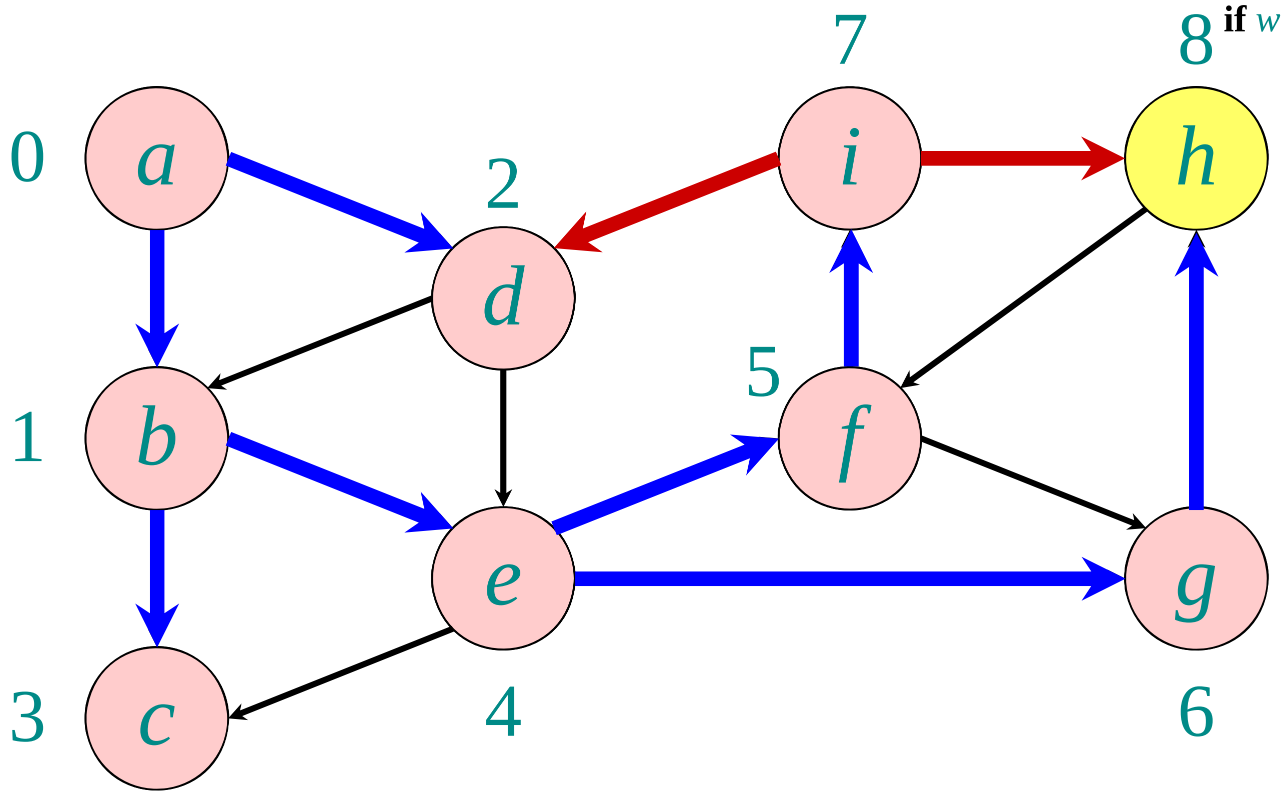

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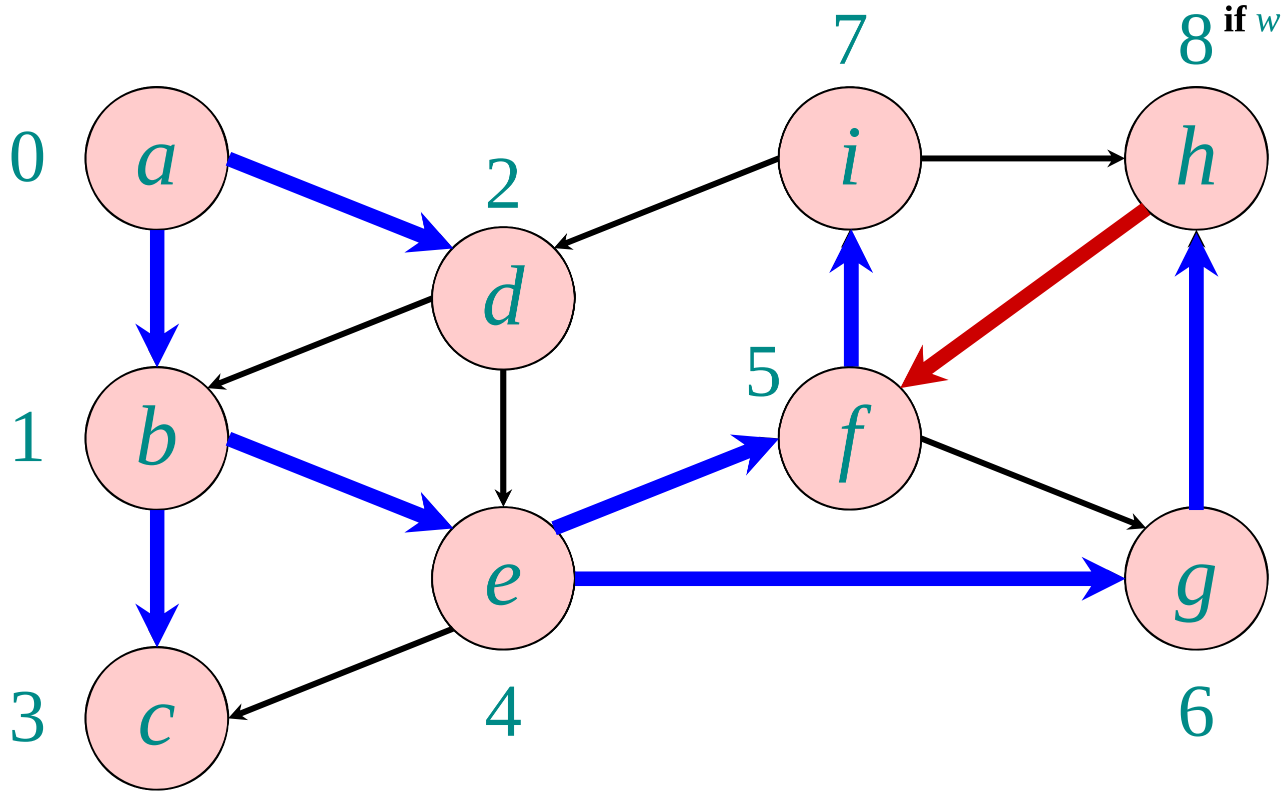

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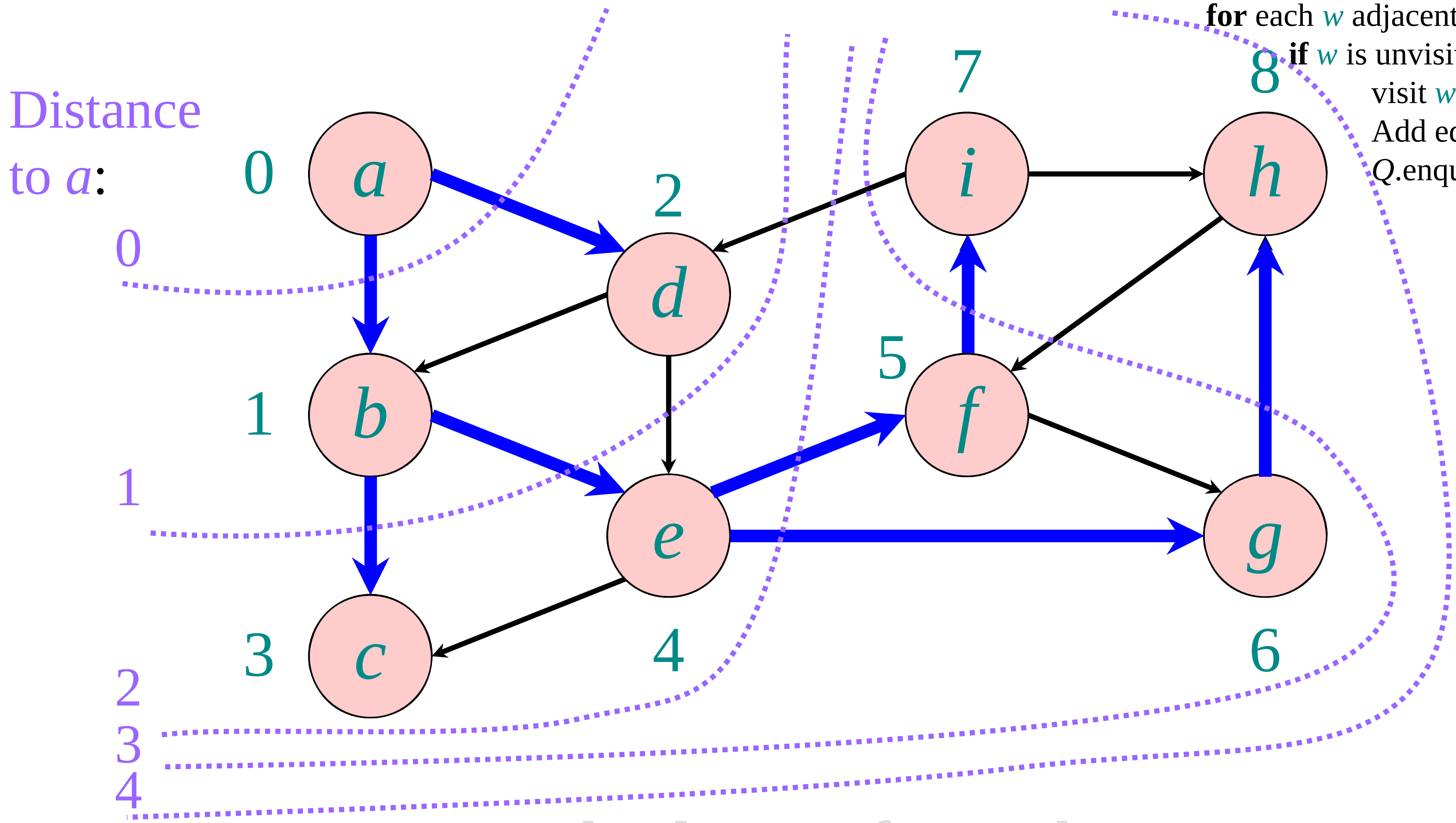

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Breadth-First Search (BFS)

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BFS(G=(V,E))
    Mark all vertices in G as "unvisited" // time=0
    Initialize empty queue Q
    for each vertex v ∈ V do
        if v is unvisited
            visit v // time++
            Q.enqueue(v)
            BFS_iter(G)
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```
BFS_iter(G)
    while Q is non-empty do
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                visit w  // time++
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```

O(n) — Mark all vertices

O(1) — Initialize empty queue

O(n) without BFS_iter — for-each-vertex loop

O($deg(v)$) — for each w adjacent to v

O(m) — while loop

BFS runtime

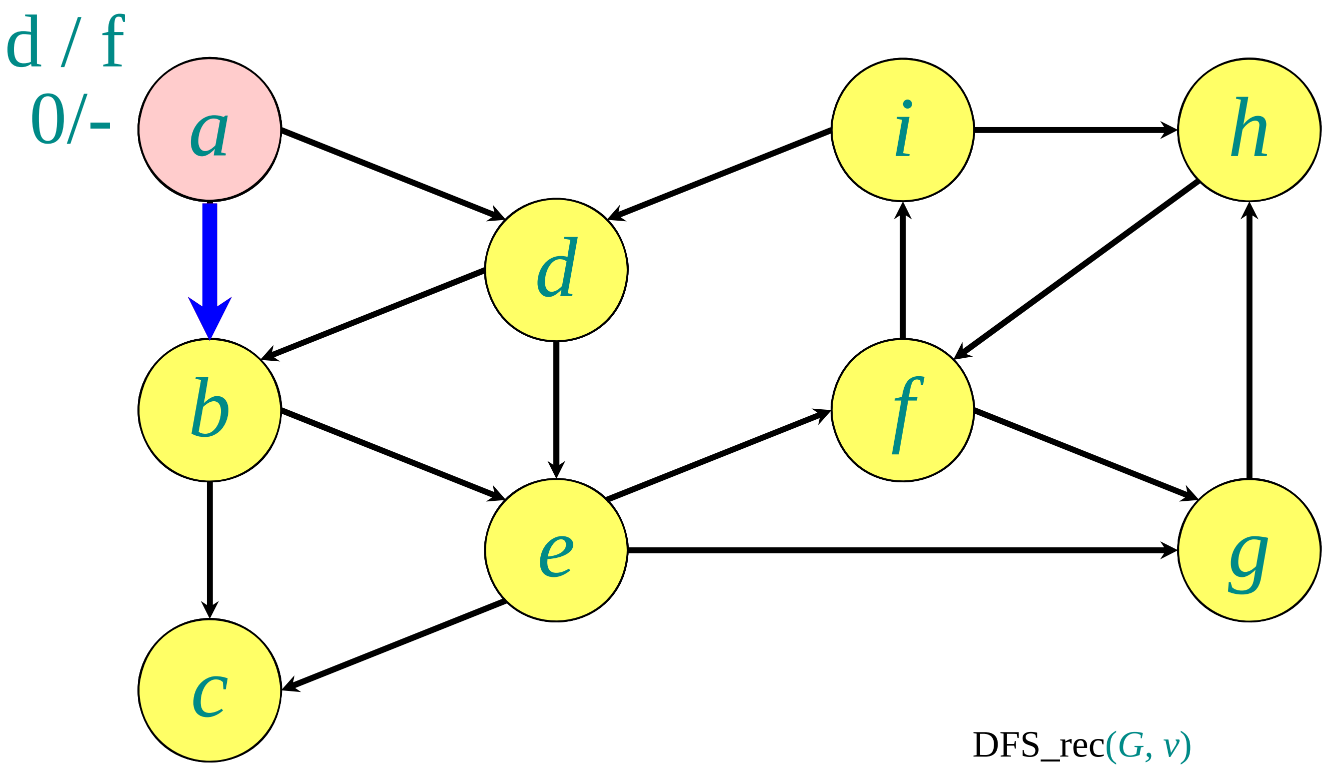
- Each vertex is marked as unvisited in the beginning $\Rightarrow$ O(n) time
- Each vertex is marked at most once, enqueued at most once, and therefore dequeued at most once
- The time to process a vertex is proportional to the size of its adjacency list (its degree), since the graph is given in adjacency list representation
 $\Rightarrow$ O(m) time
- Total runtime is O(n+m) = O(|V| + |E|)

Depth-First Search (DFS)

```
DFS(G=(V,E))
    Mark all vertices in G as "unvisited"  // time=0
    for each vertex v ∈ V do
        if v is unvisited
            DFS_rec(G,v)
```

```
DFS_rec(G, v)
    mark v as "visited" // d[v]=++time
    for each w adjacent to v do
        if w is unvisited
            Add edge (v,w) to tree T
            DFS_rec(G,w)
    mark v as "finished" // f[v]=++time
```

Example of depth-first search



π: a b c d e f g h i

- a

Store edges in predecessor array

DFS_rec(*G*, *v*)
mark *v* as "visited" // *d*[*v*]=++time
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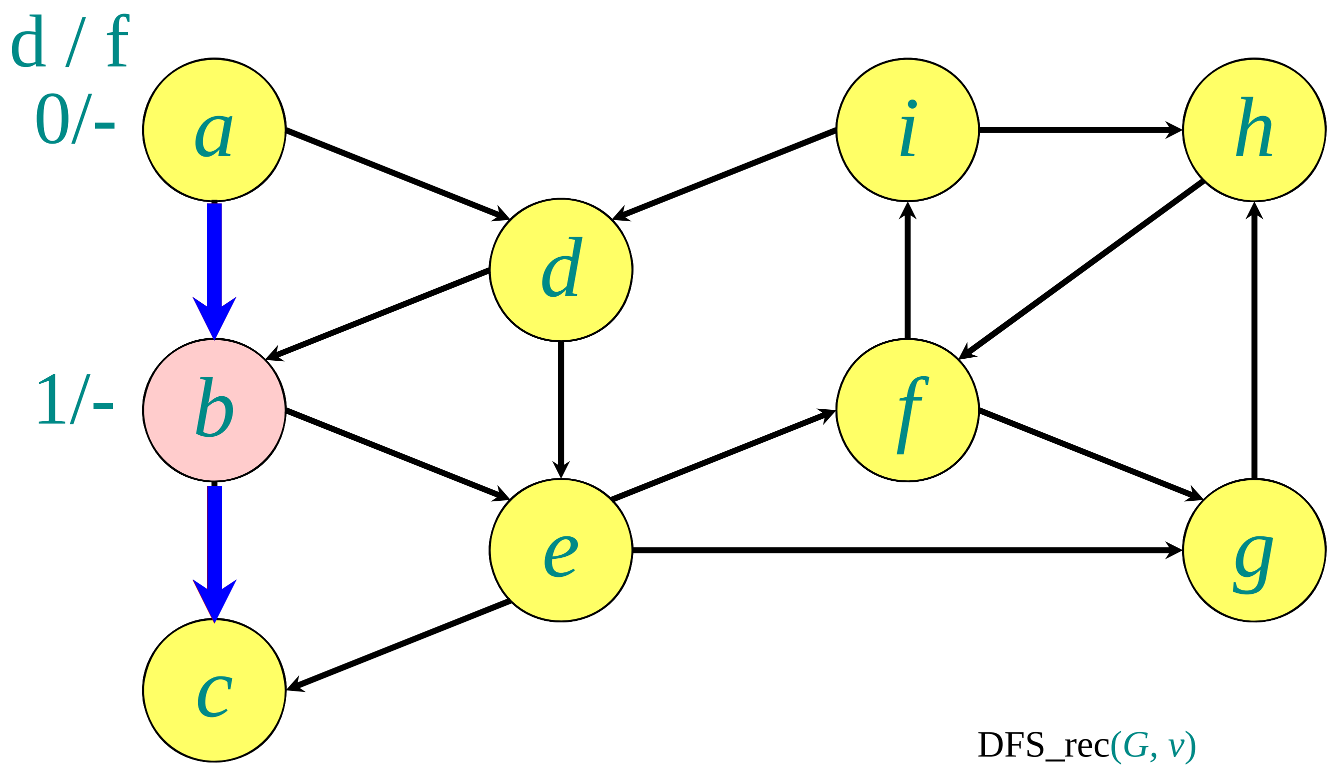

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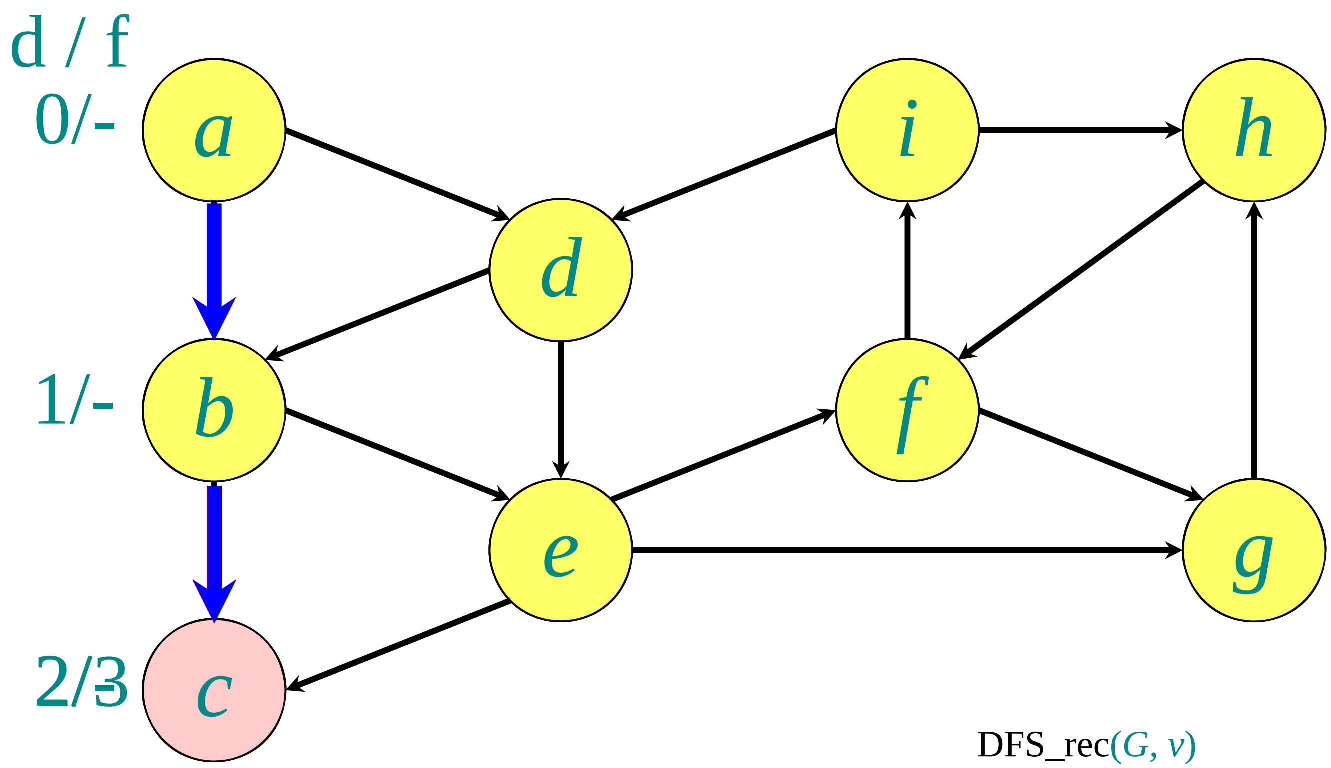

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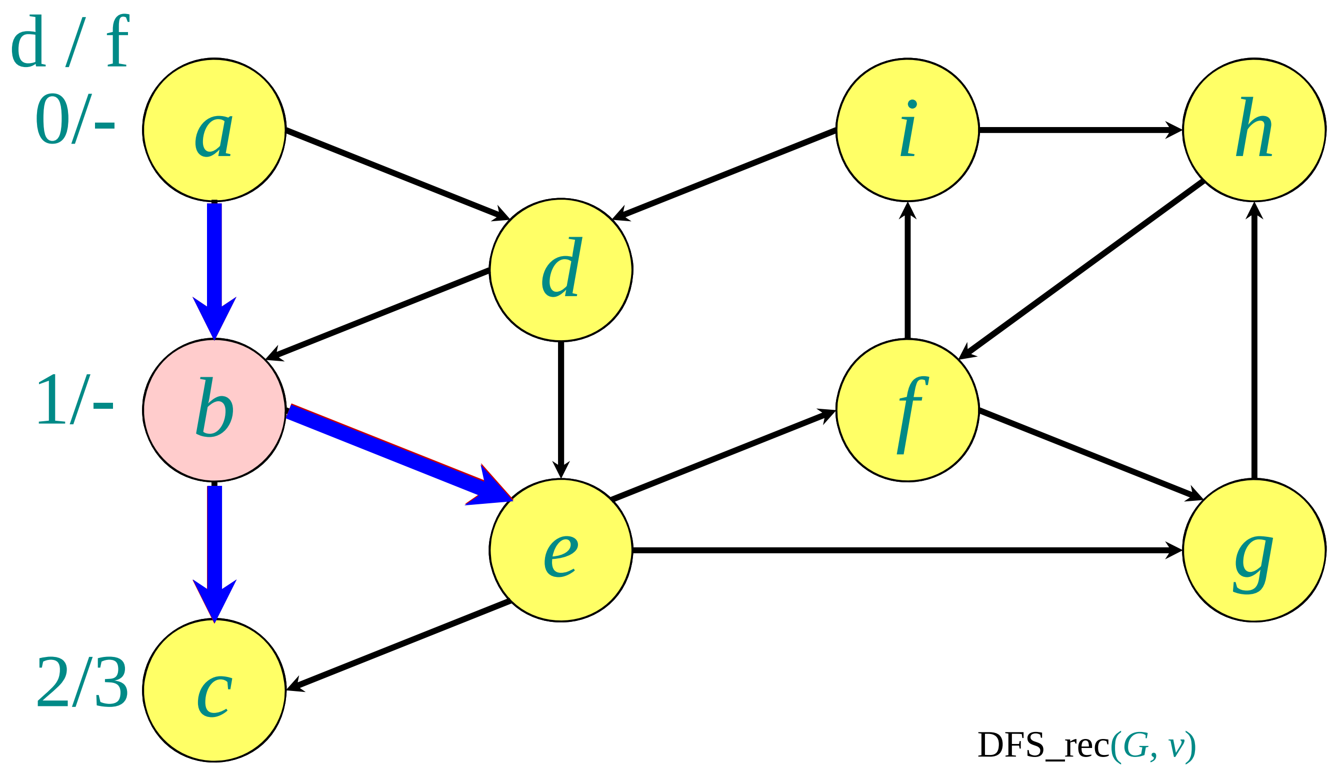

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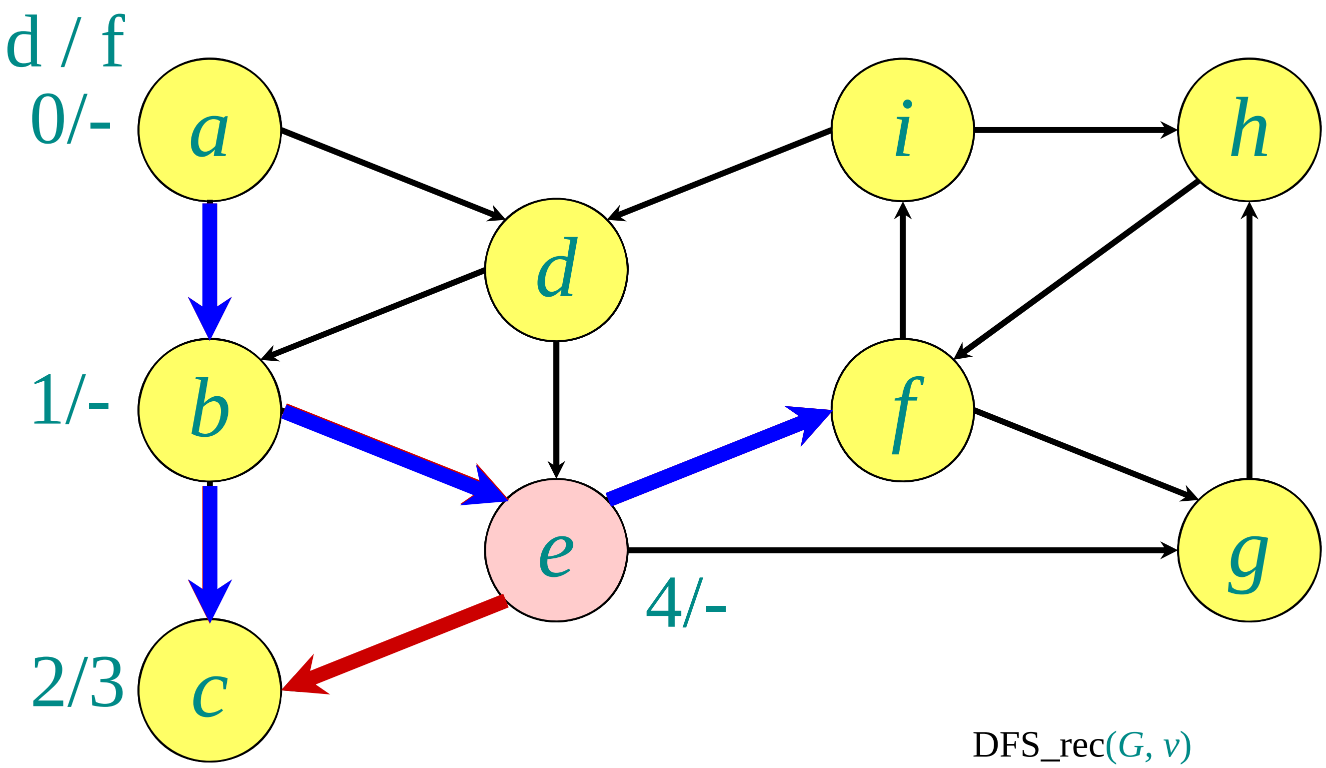

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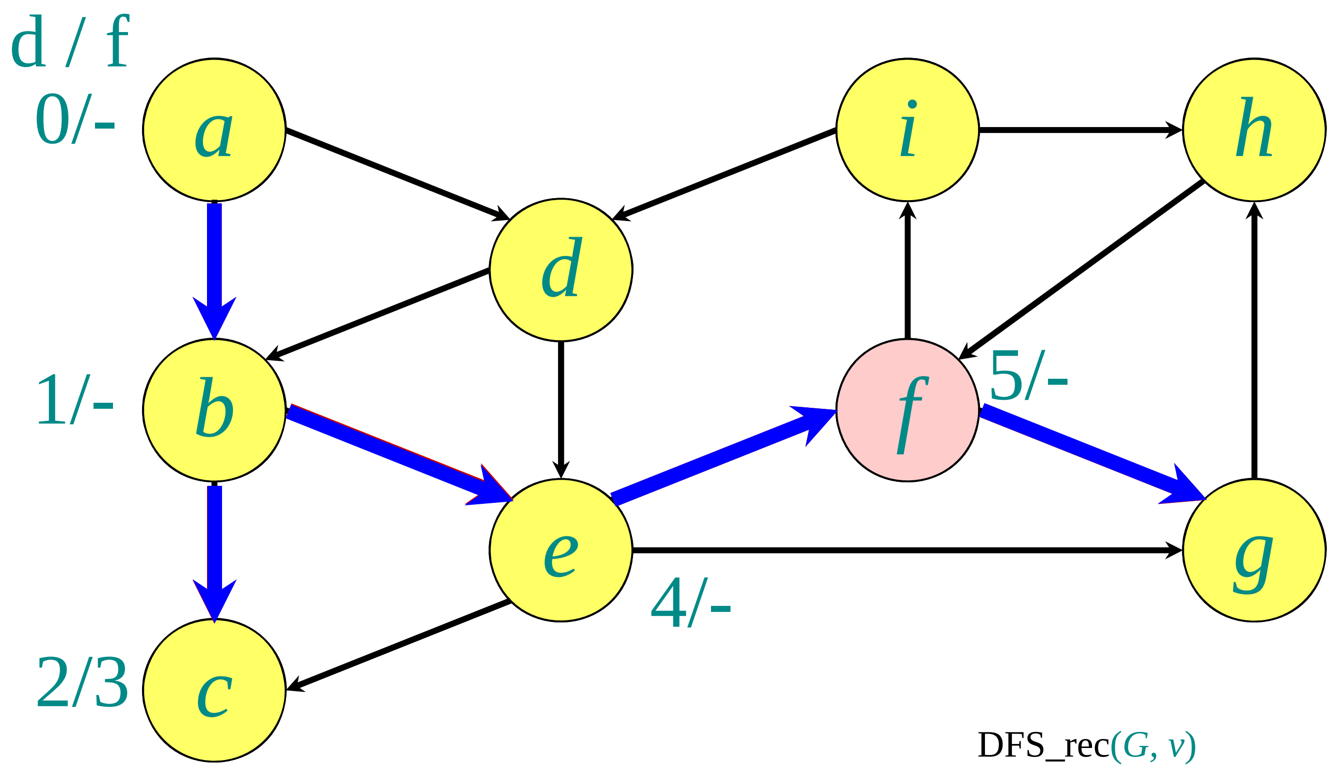

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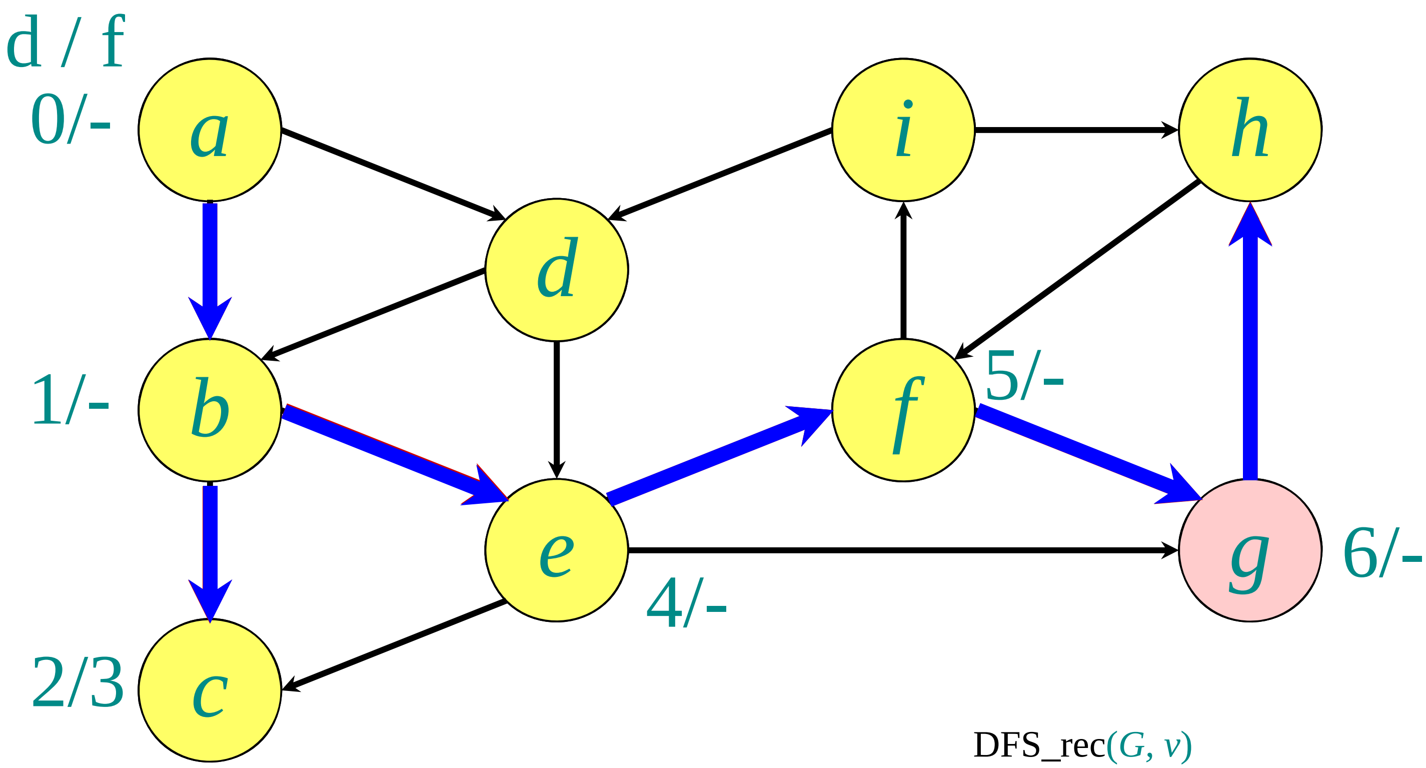

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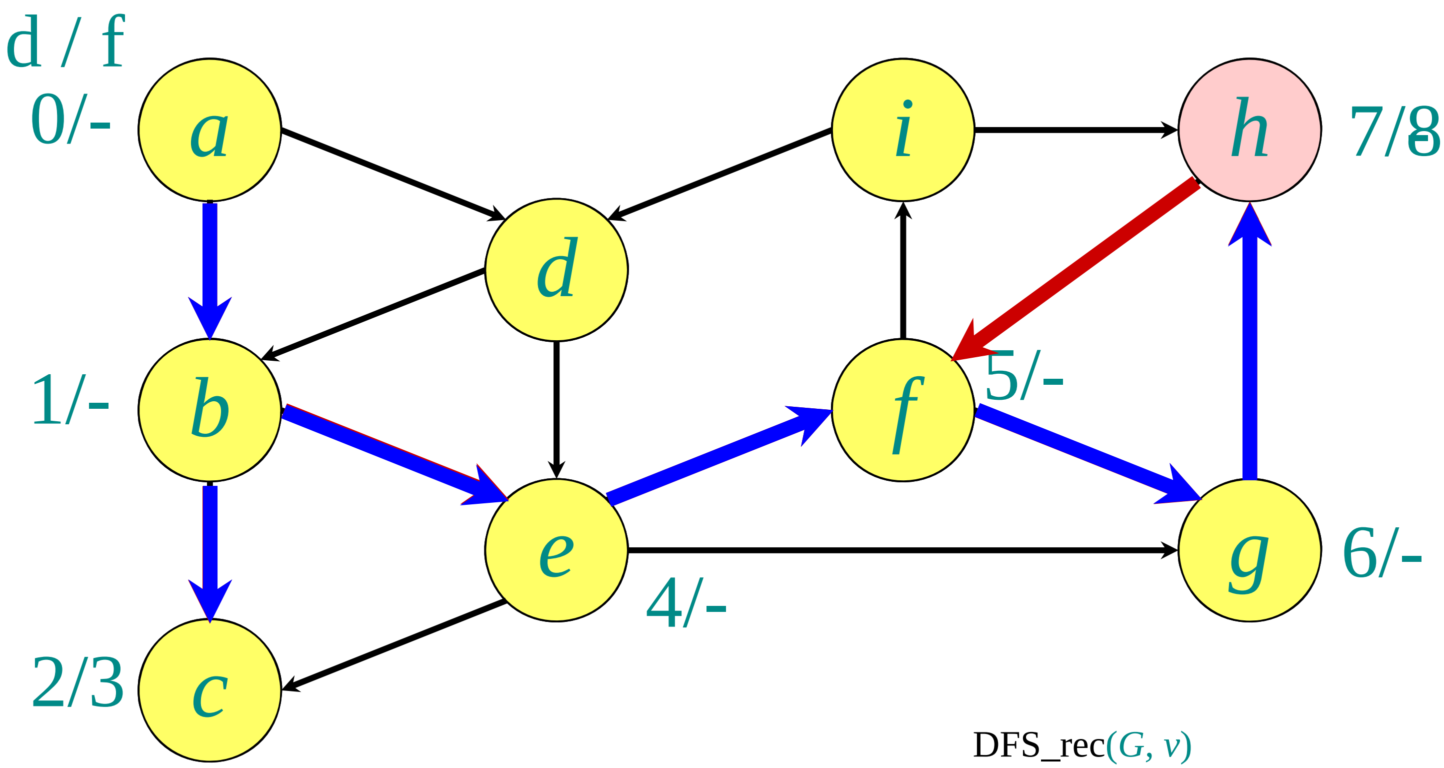

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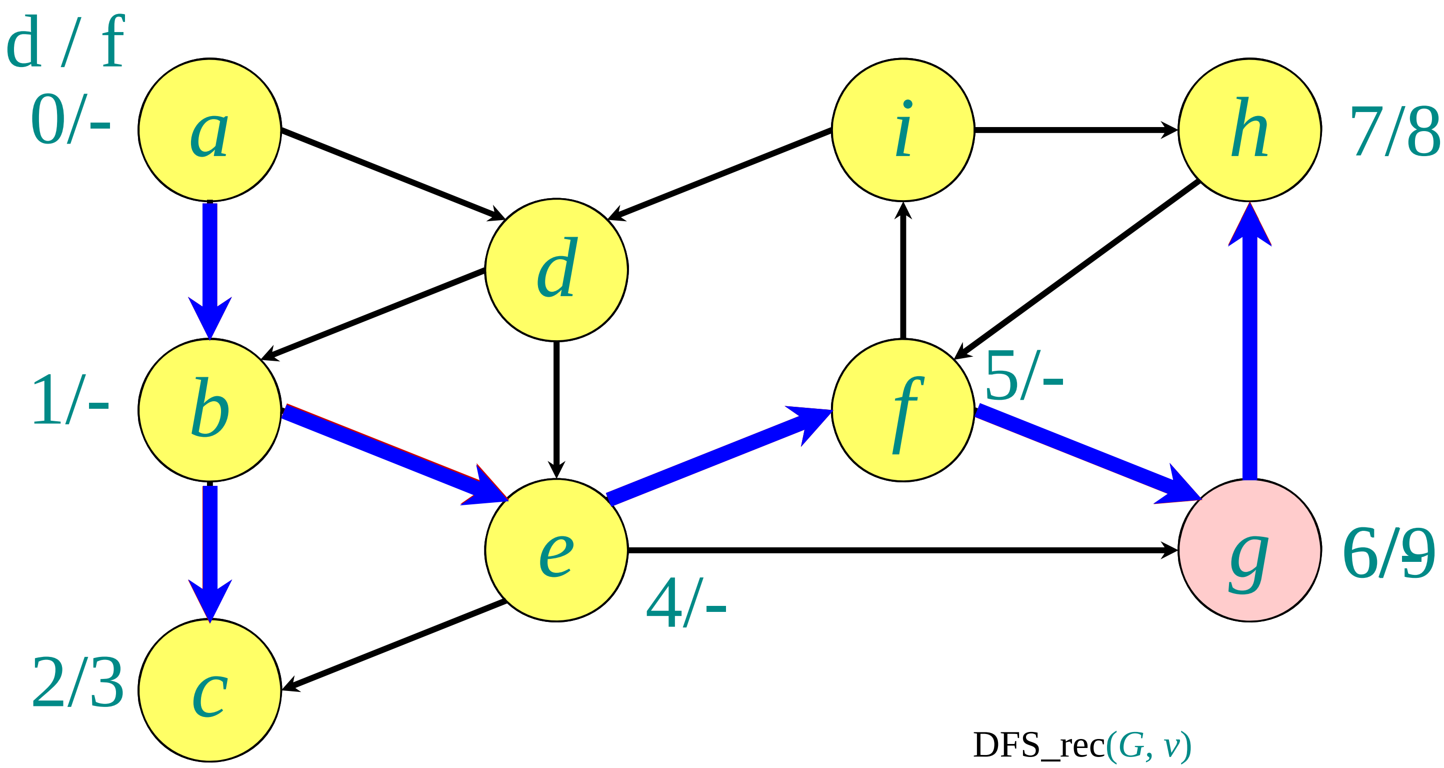

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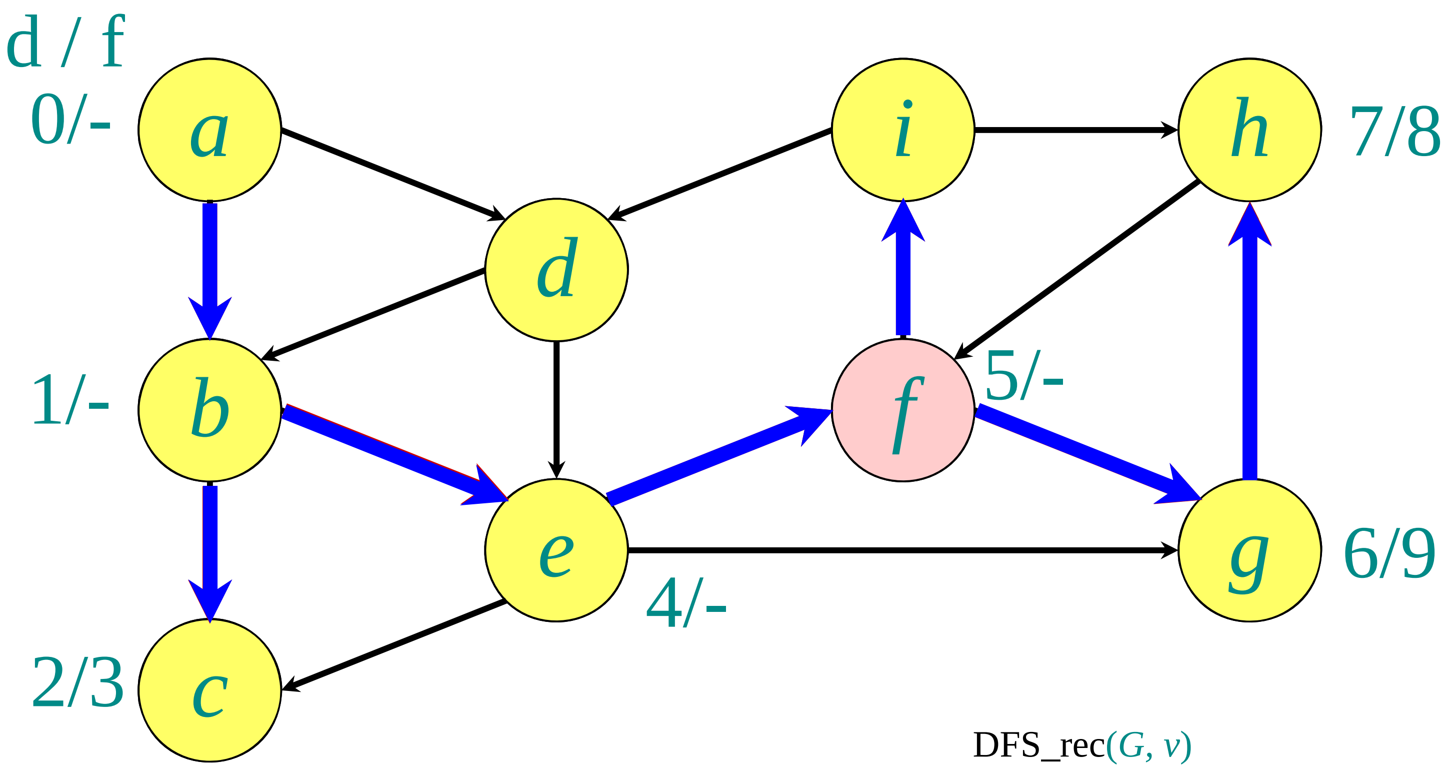

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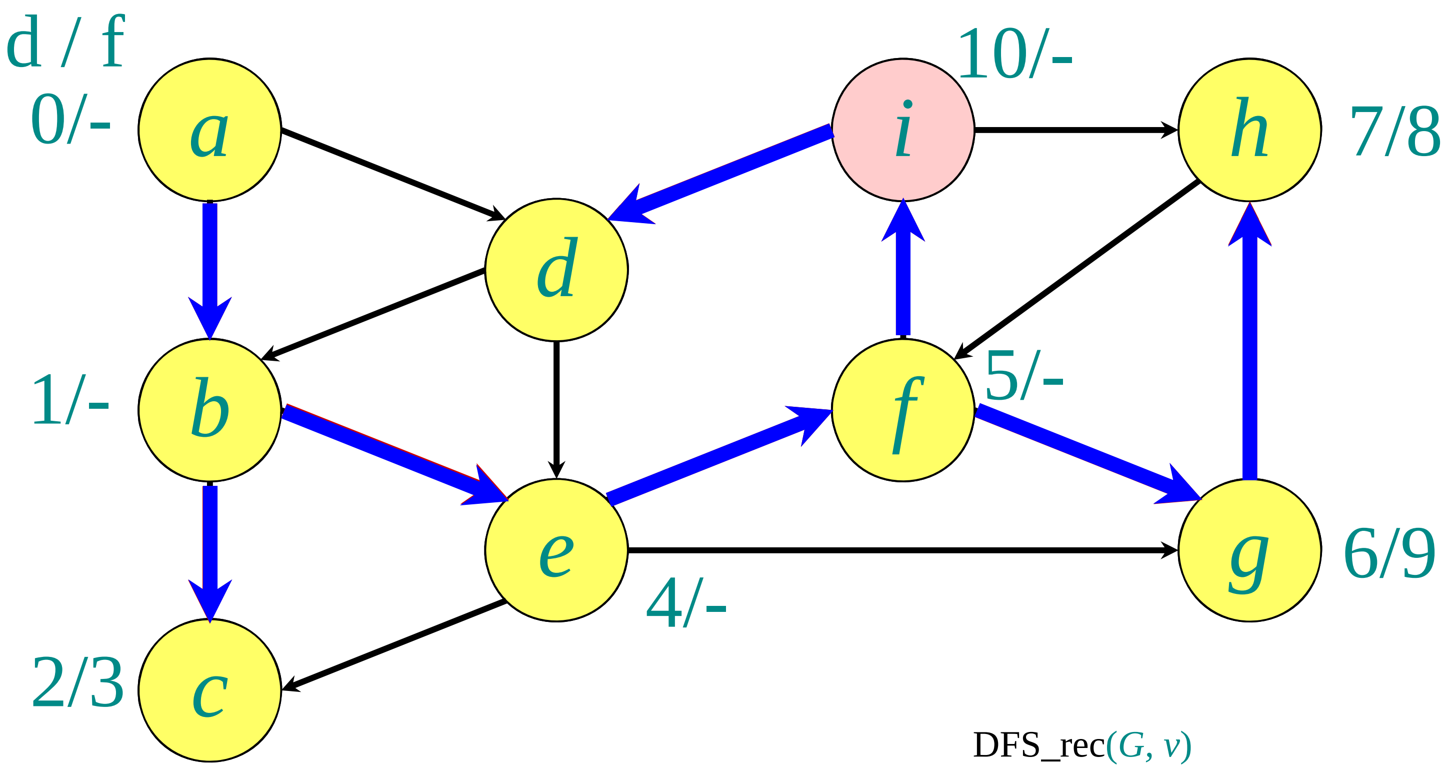

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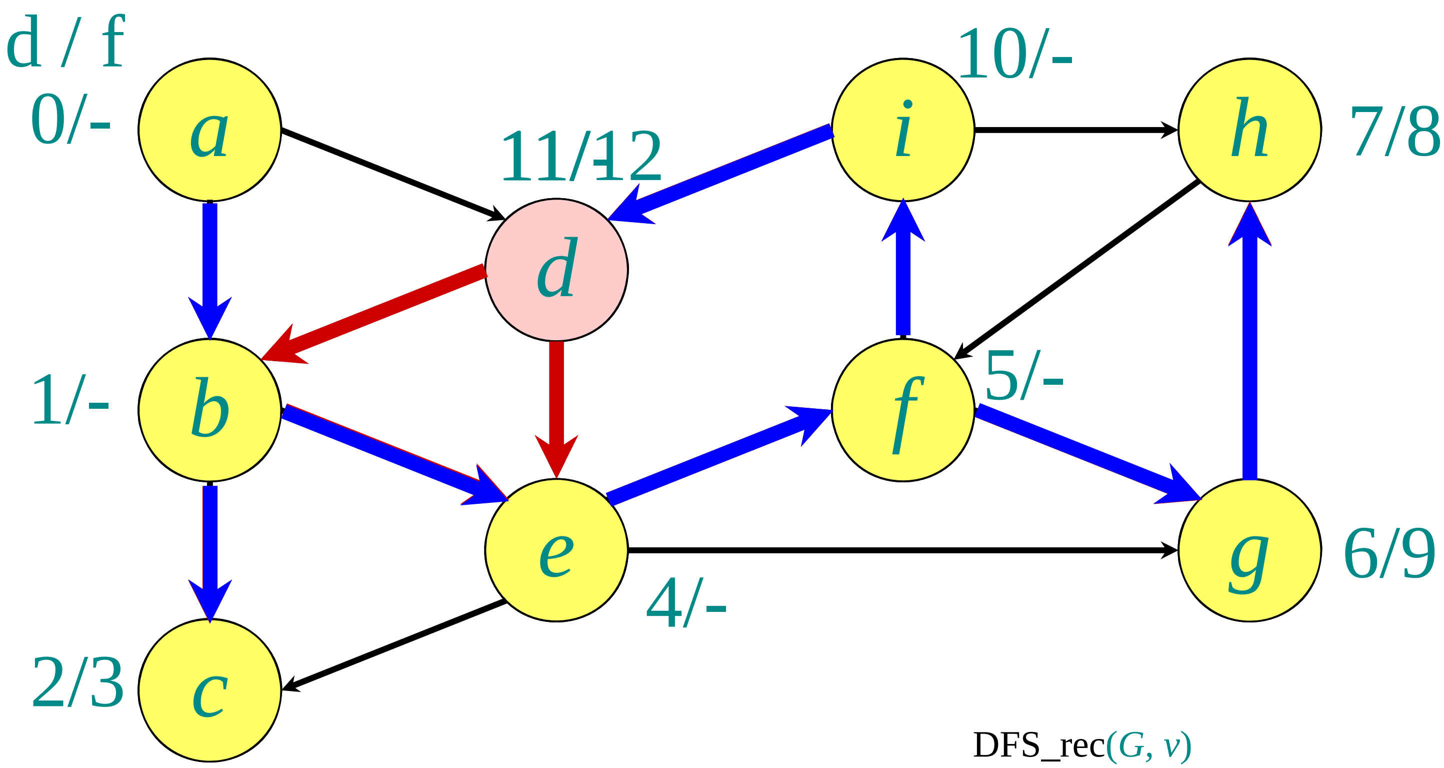

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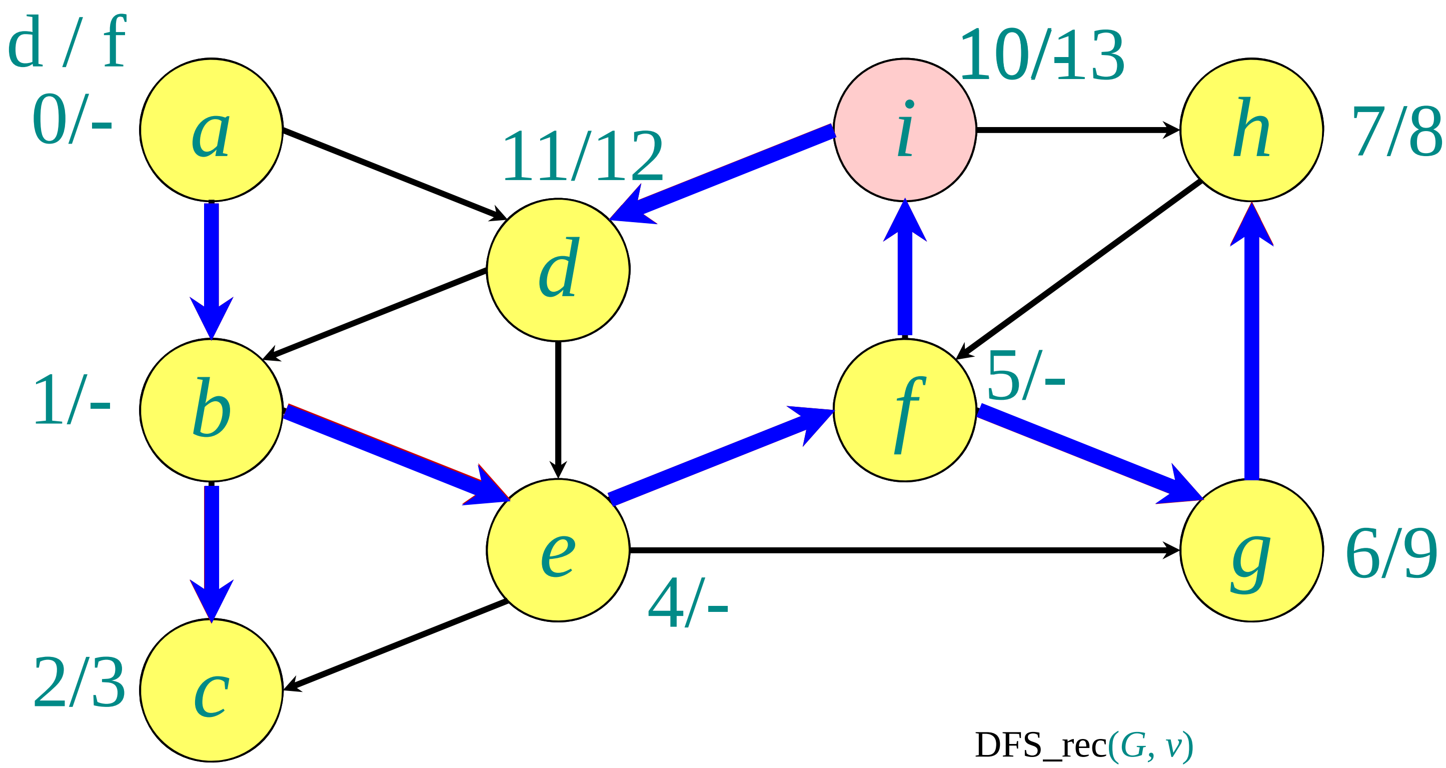

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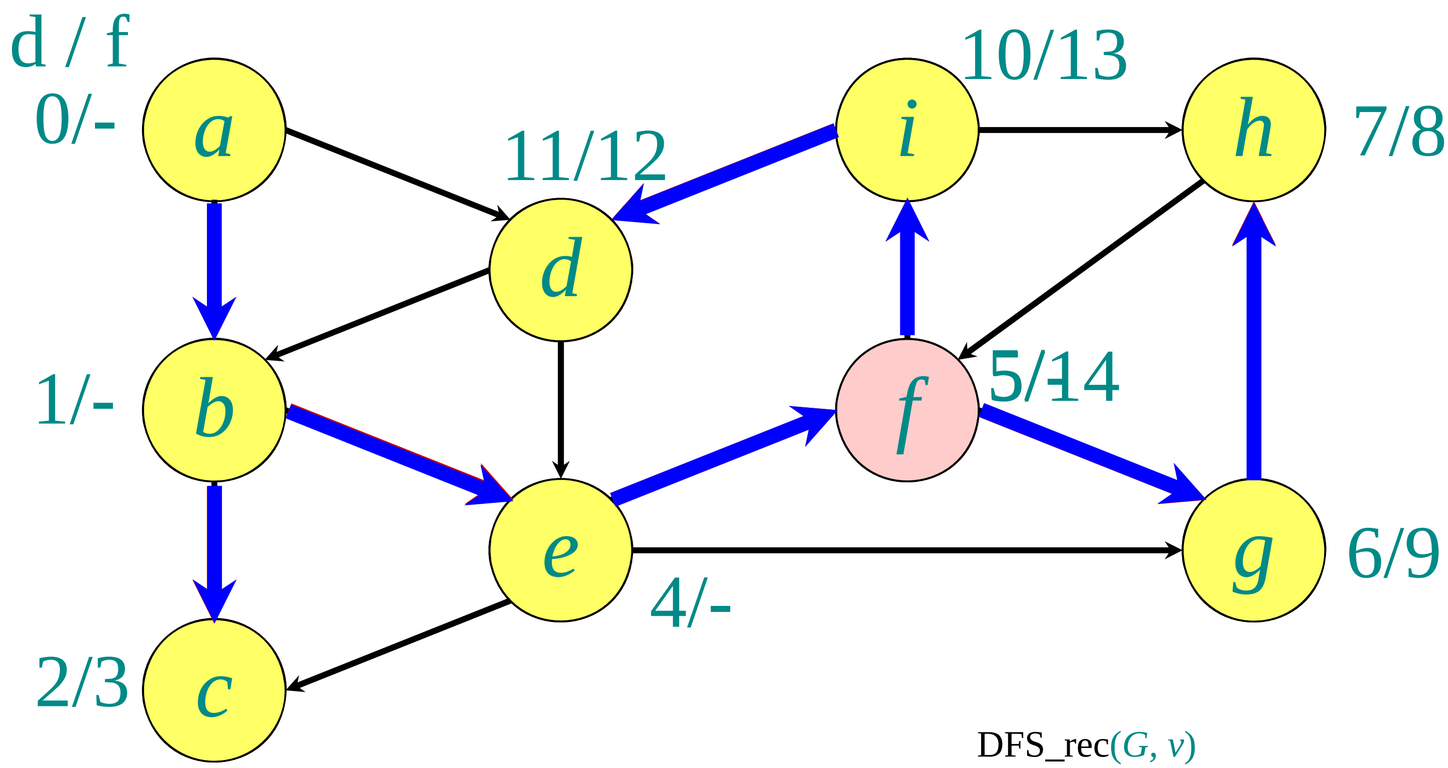

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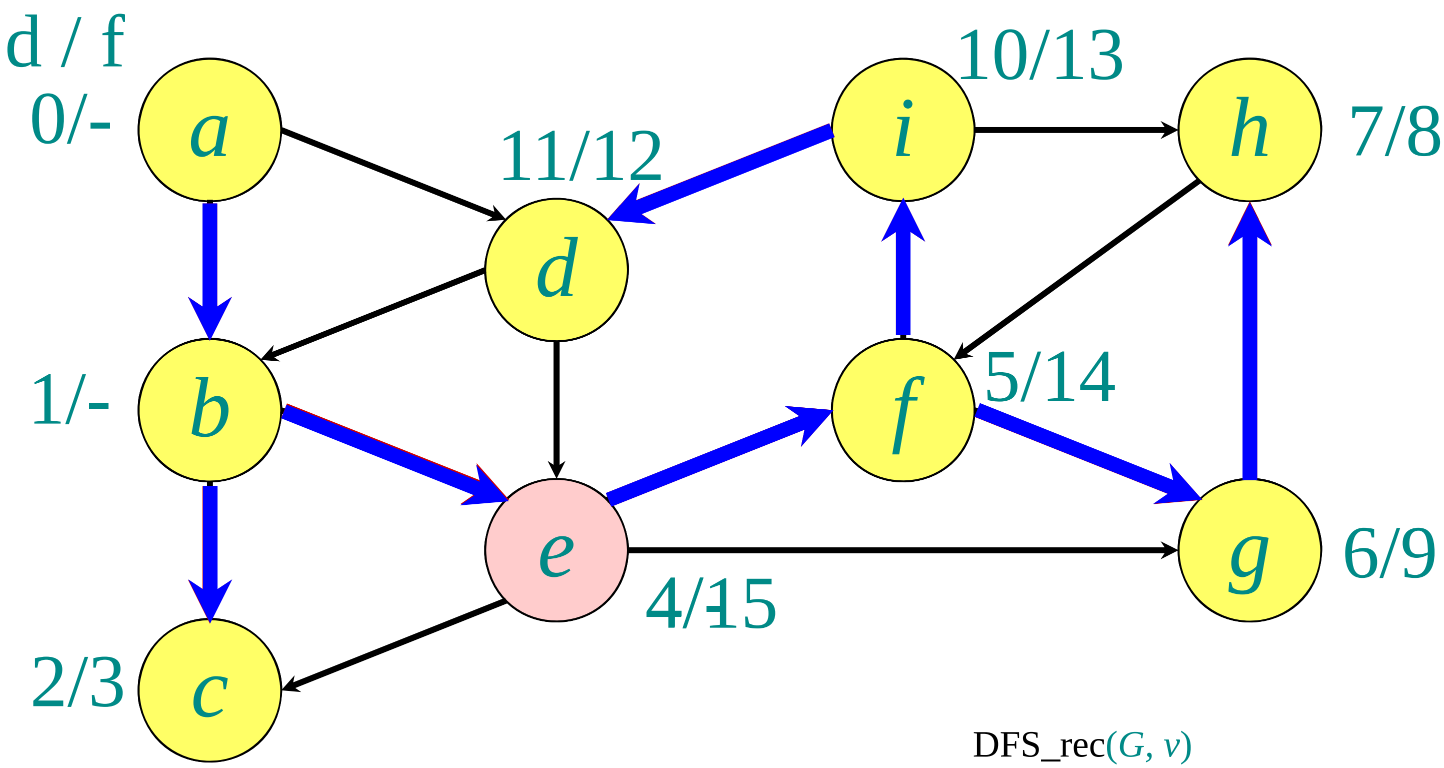

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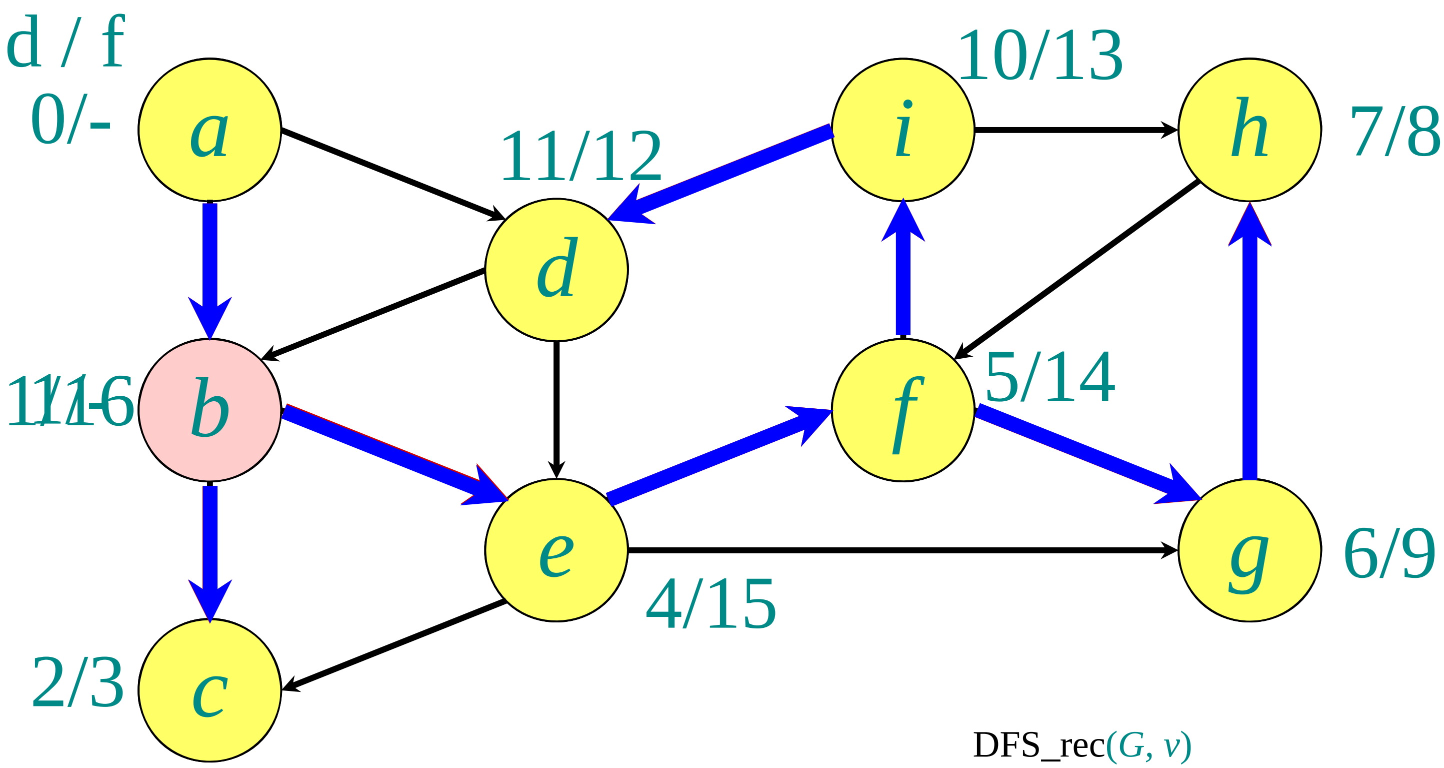
```

Example of depth-first search

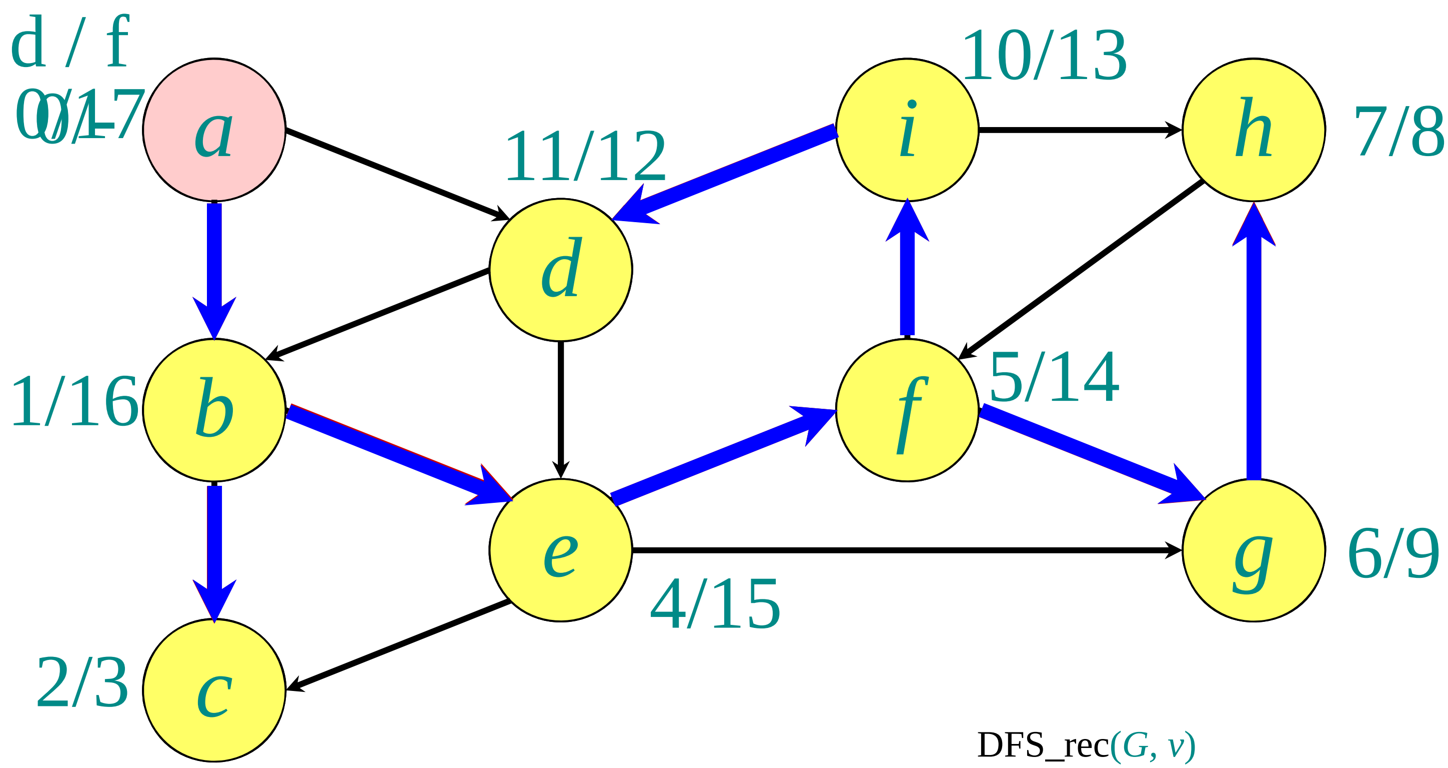


π:	a	b	c	d	e	f	g	h	i
	-	a	b	i	b	e	f	g	f

Store edges in predecessor array

```
DFS_rec(G, v)
    mark v as "visited" // d[v]=++time
    for each w adjacent to v do
        if w is unvisited
            Add edge (v,w) to tree T
            DFS_rec(G,w)
    mark v as "finished" // f[v]=++time
```

Example of depth-first search



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```

Depth-First Search (DFS)

O(n)

O(n) without DFS_rec

```
DFS(G=(V,E))
    Mark all vertices in G as "unvisited"  // time=0
    for each vertex v ∈ V do
        if v is unvisited
            DFS_rec(G,v)
```

O(1)

O($deg(v)$) without recursive call

```
DFS_rec(G, v)
    mark v as "visited" // d[v]=++time
    for each w adjacent to v do
        if w is unvisited
            Add edge (v,w) to tree T
            DFS_rec(G,w)
    mark v as "finished" // f[v]=++time
```

$\Rightarrow$ With Handshaking Lemma, all recursive calls are O(m), for a total of O($n + m$) runtime

DFS runtime

- Each vertex is visited at most once $\Rightarrow$ O(n) time
- The body of the **for** loops (except the recursive call) take constant time per graph edge
- All **for** loops take O(m) time
- Total runtime is O(n+m) = O(|V| + |E|)

Paths, Cycles, Connectivity

Let $G=(V,E)$ be a directed (or undirected) graph

- A **path** from v_1 to v_k in G is a sequence of vertices $v_1, v_2,\ldots,v_k$ such that $(v_i,v_{\{i+1\}})\in E$ (or $\{v_i,v_{\{i+1\}}\} \in E$ if G is undirected) for all $i\in\{1,\ldots,k\text{-}1\}$.
- A path is **simple** if all vertices in the path are distinct.
- A path $v_1, v_2,\ldots,v_k$ forms a **cycle** if $v_1=v_k$.
- A graph with no cycles is **acyclic**.
  - An undirected acyclic graph is called a **tree**. (Trees do not have to have a root vertex specified.)
  - A directed acyclic graph is a **DAG**. (A DAG can have undirected cycles if the direction of the edges is not considered.)
- An undirected graph is **connected** if every pair of vertices is connected by a path. A directed graph is **strongly connected** if for every pair $u,v\in V$ there is a path from u to v and there is a path from v to u.
- The **(strongly) connected components** of a graph are the equivalence classes of vertices under this reachability relation.