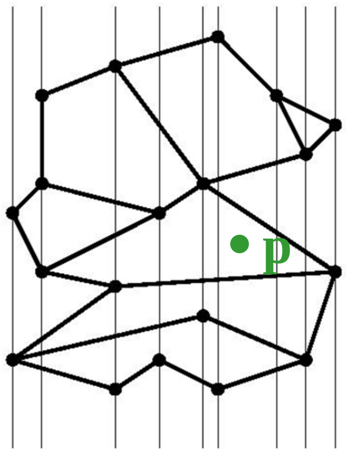
CMPS 3130/6130 Computational Geometry
Spring 2015



Planar Subdivisions and Point Location

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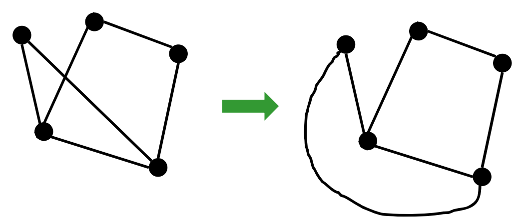
Based on:

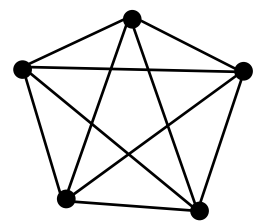
Computational Geometry: Algorithms and Applications
and David Mount's lecture notes

Planar Subdivision

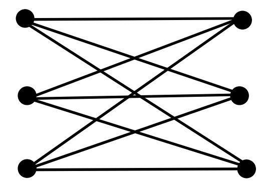
- Let $G=(V,E)$ be an undirected graph.
- G is planar if it can be embedded in the plane without edge crossings.



planar

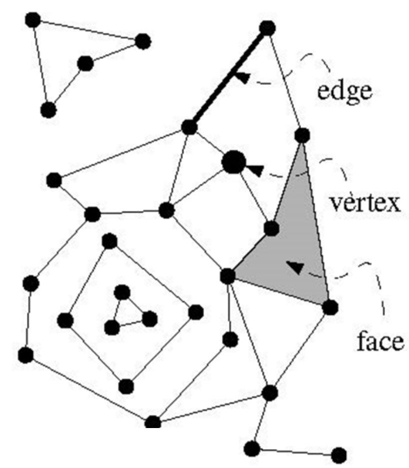


K_5, not planar



$K_{3,3}$, not planar

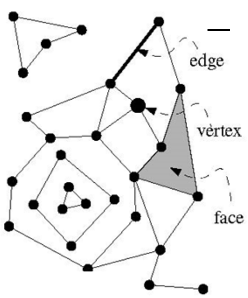
- A planar embedding (=drawing) of a planar graph G induces a **planar subdivision** consisting of vertices, edges, and faces.

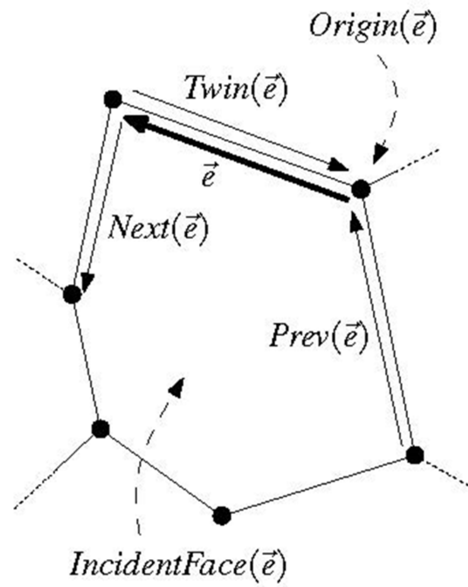


Doubly-Connected Edge List

- The **doubly-connected edge list (DCEL)** is a popular data structure to store the geometric and topological information of a planar subdivision.
  - It contains records for each face, edge, vertex
  - (Each record might also store additional application-dependent attribute information.)
  - It should enable us to perform basic operations needed in algorithms, such as walk around a face, or walk from one face to a neighboring face

- The DCEL consists of:
  - For each vertex *v*, its coordinates are stored in **Coordinates(*v*)** and a pointer **IncidentEdge(*v*)** to a half-edge that has *v* as it origin.
  - Two oriented **half-edges** per edge, one in each direction. These are called **twins**. Each of them has an **origin** and a **destination**. Each half-edge *e* stores a pointer **Origin(*e*)**, a pointer **Twin(*e*)**, a pointer **IncidentFace(e)** to the face that it bounds, and pointers **Next (e)** and **Prev(e)** to the next and previous half-edge on the boundary of **IncidentFace(e)**.
  - For each face *f*, **OuterComponent(*f*)** is a pointer to some half-edge on its outer boundary (null for unbounded faces). It also stores a list **InnerComponents(*f*)** which contains for each hole in the face a pointer to some half-edge on the boundary of the hole.





Complexity of a Planar Subdivision

- The complexity of a planar subdivision is:
  #vertices + #edges + #faces = $n_v + n_e + n_f$
- Euler's formula for planar graphs:
  1) $n_v - n_e + n_f \geq 2$
  2) $n_e \leq 3n_v - 6$

  2) follows from 1):

  Count edges. Every face is bounded by ≥ 3 edges.

  Every edge bounds ≤ 2 faces.

  $\Rightarrow 3n_f \leq 2n_e \Rightarrow n_f \leq 2/3 n_e$

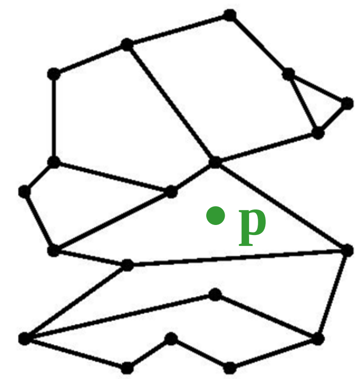
  $\Rightarrow 2 \leq n_v - n_e + n_f \leq n_v - n_e + 2/3\ n_e = n_v - 1/3\ n_e$

  $\Rightarrow 2 \leq n_v - 1/3\ n_e$

- Hence, the complexity of a planar subdivision is $O(n_v)$, i.e., linear in the number of vertices.

Point Location

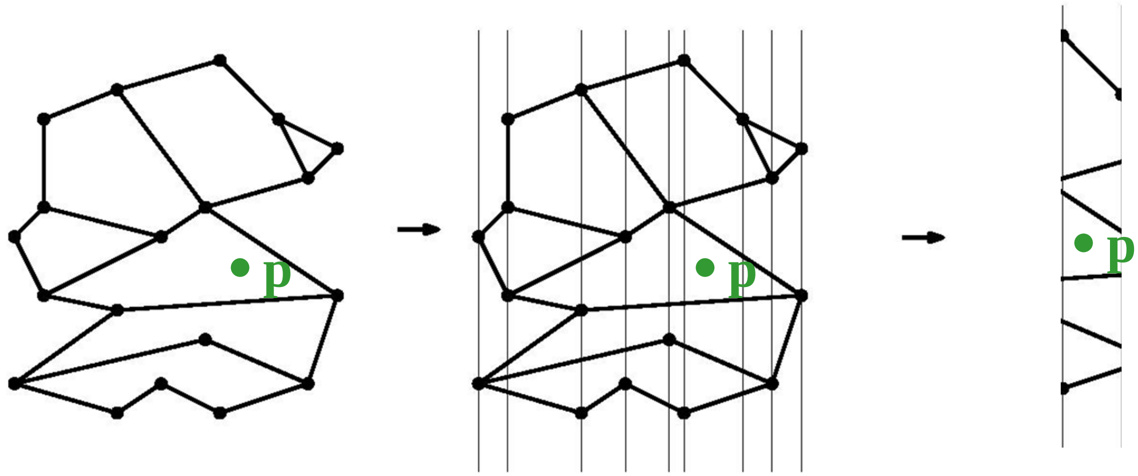
- **Point location task:**
  Preprocess a planar subdivision to efficiently answer **point-location queries** of the type: Given a point ***p*=(*p*$_x$,*p*$_y$)**, find the face it lies in.



- **Important metrics:**
  – Time complexity for preprocessing = time to construct the data structure
  – Space needed to store the data structure
  – Time complexity for querying the data structure

Slab Method

- **Slab method:**
  Draw a vertical line through each vertex. This decomposes the plane into slabs.

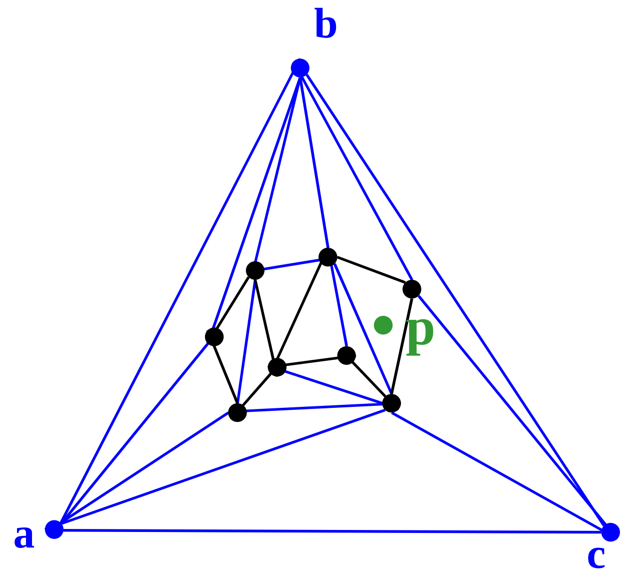


- In each slab, the vertical order of the line segments remains constant.
- If we know in which slab ***p*** lies, we can perform binary search, using the sorted order of the segments in the slab.
- Find slab that contains ***p*** by binary search on x among slab boundaries.
- A second binary search in slab determines the face containing ***p***.
- Search complexity $O(\log n)$, but space complexity $\Theta(n^2)$.

lower bound?

Kirkpatrick's Algorithm

- Needs a triangulation as input.
- Can convert a planar subdivision with n vertices into a triangulation:
  - Triangulate each face, keep same label as original face.
  - If the outer face is not a triangle:
    - Compute the convex hull of the subdivision.
    - Triangulate pockets between the subdivision and the convex hull.
    - Add a large triangle (new vertices **a**, **b**, **c**) around the convex hull, and triangulate the space in-between.

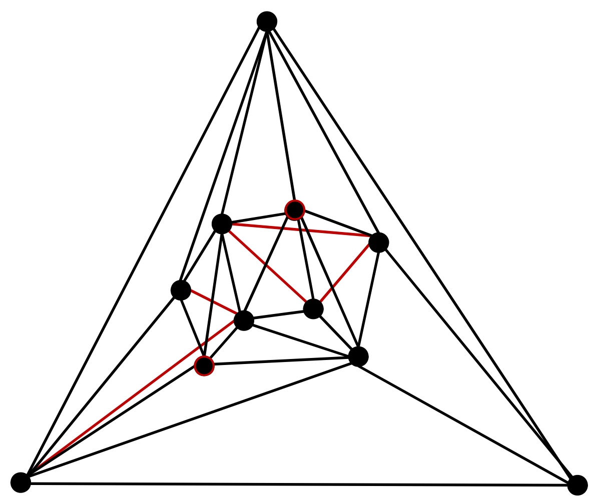


- The size of the triangulated planar subdivision is still $O(n)$, by Euler's formula.
- The conversion can be done in $O(n \log n)$ time.
- Given ***p***, if we find a triangle containing ***p*** we also know the (label of) the original subdivision face containing ***p***.

Kirkpatrick's Hierarchy

- Compute a sequence T_0, T_1, ..., T_k of increasingly coarser triangulations such that the last one has constant complexity.
- The sequence T_0, T_1, ..., T_k should have the following properties:
  - T_0 is the input triangulation, T_k is the outer triangle
  - $k \in O(\log n)$
  - Each triangle in T_{i+1} overlaps O(1) triangles in T_i

- How to build such a sequence?
  - Need to delete vertices from T_i .
  - Vertex deletion creates holes, which need to be re-triangulated.
- How do we go from T_0 of size $O(n)$ to T_k of size O(1) in k=O(log n) steps?
  - In each step, delete a constant fraction of vertices from T_i .

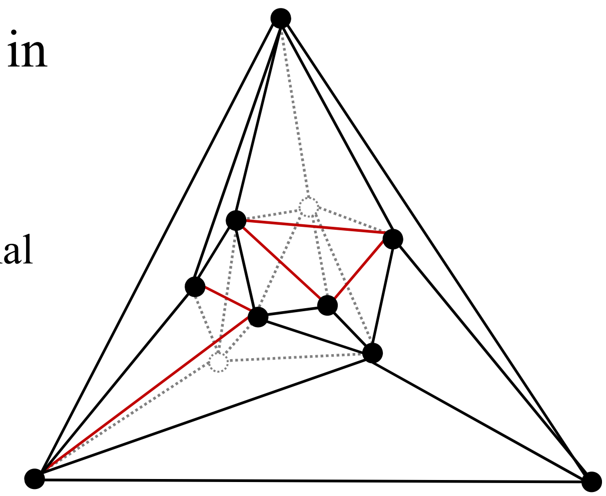


- We also need to ensure that each new triangle in T_{i+1} overlaps with only O(1) triangles in T_i .

Vertex Deletion and Independent Sets

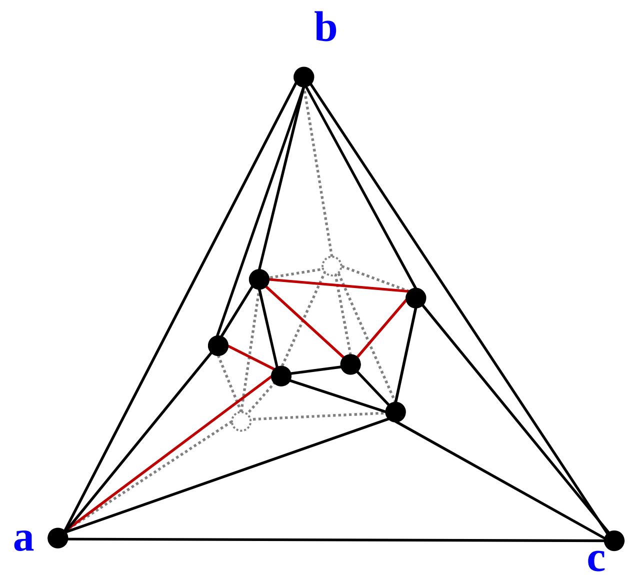
When creating T_{i+1} from T_i, delete vertices from T_i that have the following properties:

- **Constant degree:**
  Each vertex **v** to be deleted has O(1) degree in the graph T_i.
  - If **v** has degree d, the resulting hole can be re-triangulated with d-2 triangles
  - Each new triangle in T_{i+1} overlaps at most d original triangles in T_i
- **Independent sets:**
  No two deleted vertices are adjacent.
  - Each hole can be re-triangulated independently.



Independent Set Lemma

Lemma: Every planar graph on n vertices contains an independent vertex set of size $n/18$ in which each vertex has degree at most 8. Such a set can be computed in $O(n)$ time.



Use this lemma to construct Kirkpatrick's hierarchy:

- Start with T_0, and select an independent set S of size $n/18$ in which each vertex has maximum degree 8. [Never pick the outer triangle vertices **a**, **b**, **c**.]
- Remove vertices of S, and re-triangulate holes.
- The resulting triangulation, T_1, has at most $17/18n$ vertices.
- Repeat the process to build the hierarchy, until T_k equals the outer triangle with vertices **a**, **b**, **c**.
- The depth of the hierarchy is $k = \log_{18/17} n$

Hierarchy Example

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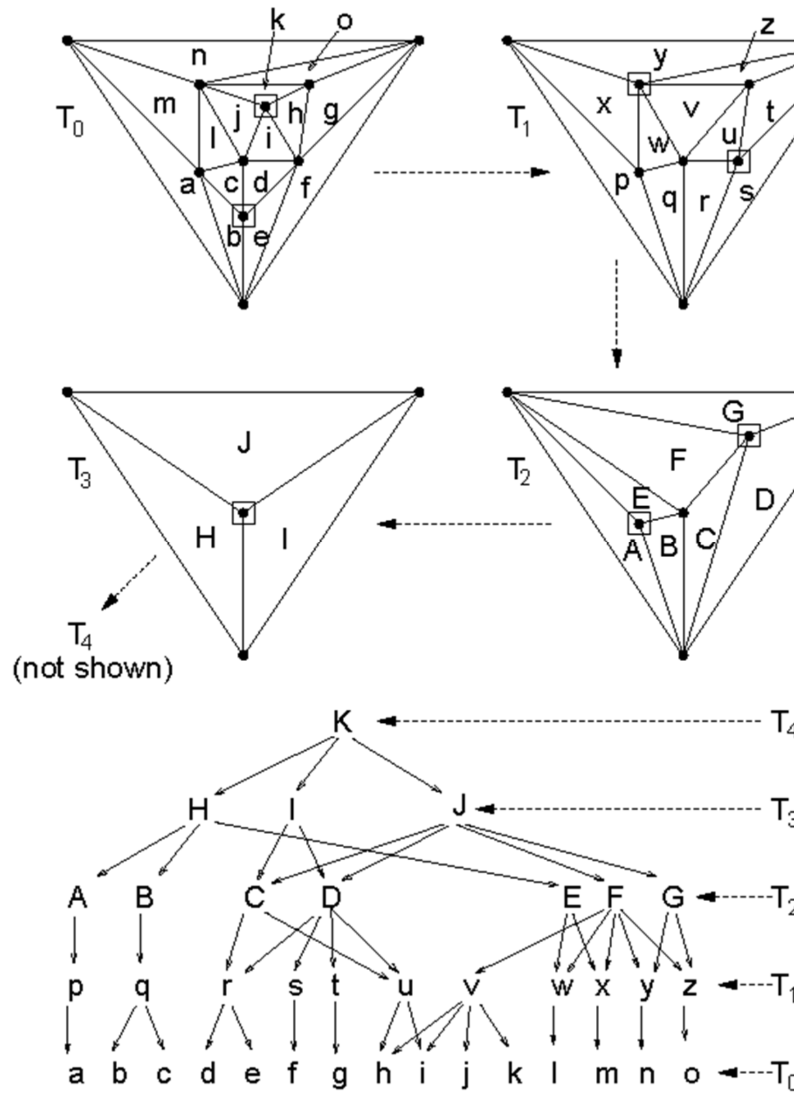


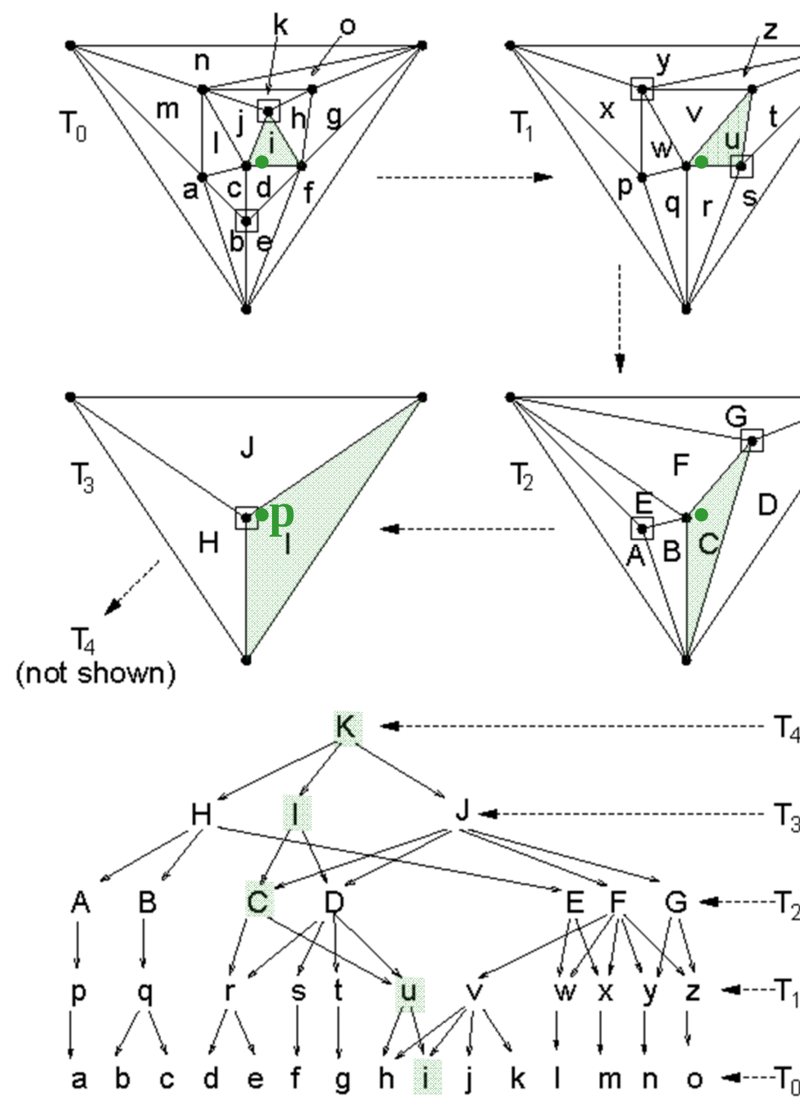
Hierarchy Data Structure

Store the hierarchy as a DAG:

- The root is T_k .
- Nodes in each level correspond to triangles T_i .
- Each node for a triangle in T_{i+1} stores pointers to all triangles of T_i that it overlaps.

How to locate point ***p*** in the DAG:

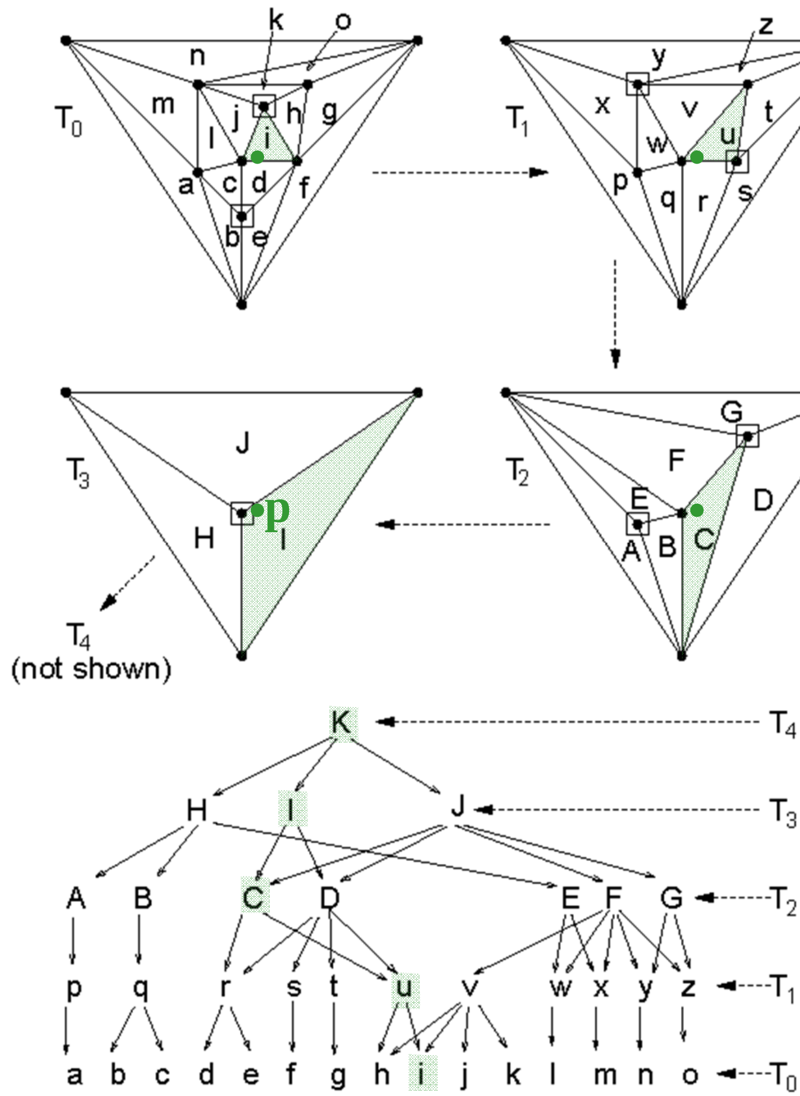
- Start at the root. If ***p*** is outside of T_k then ***p*** is in exterior face; done.
- Else, set Δ to be the triangle at the current level that contains ***p***.
- Check each of the at most 6 triangles of T_{k-1} that overlap with Δ, whether they contain ***p***. Update Δ and descend in the hierarchy until reaching T_0 .
- Output Δ .



Analysis

- **Query time** is $O(\log n)$: There are $O(\log n)$ levels and it takes constant time to move between levels.
- **Space complexity** is $O(n)$:
  - Sum up sizes of all triangulations in hierarchy.
  - Because of Euler's formula, it suffices to sum up the number of vertices.
  - Total number of vertices:

    $n + 17/18\, n + (17/18)^2\, n + (17/18)^3\, n + \ldots$
    $\leq 1/(1\text{-}17/18)\, n = 18\, n$
- **Preprocessing time** is $O(n \log n)$:
  - Triangulating the subdivision takes $O(n \log n)$ time.
  - The time to build the DAG is proportional to its size.



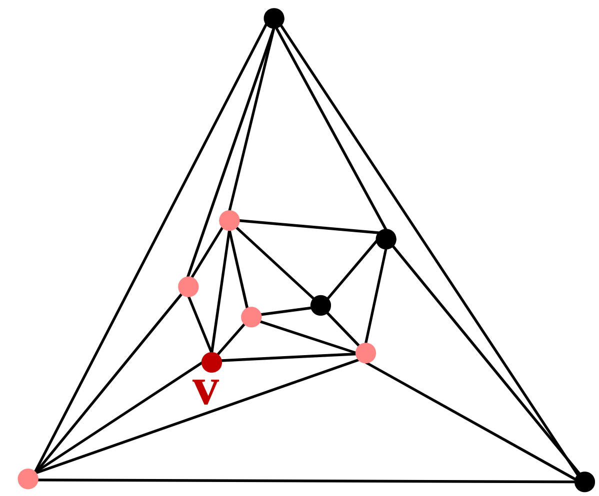
Independent Set Lemma

Lemma: Every planar graph on n vertices contains an independent vertex set of size $n/18$ in which each vertex has degree at most 8. Such a set can be computed in O(n) time.

Proof:

Algorithm to construct independent set:

- Mark all vertices of degree ≥ 9
- While there is an unmarked vertex
  - Let **v** be an unmarked vertex
  - Add **v** to the independent set
  - Mark **v** and all its neighbors
- Can be implemented in O(n) time: Keep list of unmarked vertices, and store the triangulation in a data structure that allows finding neighbors in O(1) time.



Independent Set Lemma

Still need to prove existence of large independent set.

- Euler's formula for a triangulated planar graph on n vertices:
  $\#\text{edges} = 3n - 6$
- Sum over vertex degrees:
  $\sum_v \deg(v) = 2\ \#\text{edges} = 6n - 12 < 6n$
- **Claim:** At least $n/2$ vertices have degree ≤ 8.
  Proof: By contradiction. So, suppose otherwise.
  $\rightarrow$ $n/2$ vertices have degree ≥ 9. The remaining have degree ≥ 3.
  $\rightarrow$ The sum of the degrees is $\geq 9\ n/2 + 3\ n/2 = 6n$. Contradiction.

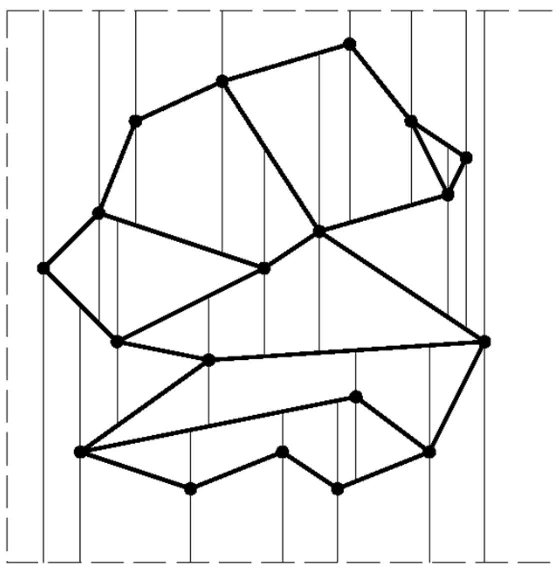
- In the beginning of the algorithm, at least $n/2$ nodes are unmarked. Each picked vertex **v** marks ≤ 8 other vertices, so including itself 9.
- Therefore, the while loop can be repeated at least $n/18$ times.
- This shows that there is an independent set of size at least $n/18$ in which each node has degree ≤ 8.

$\square$

Summing Up

- Kirkpatrick's point location data structure needs $O(n \log n)$ preprocessing time, $O(n)$ space, and has $O(\log n)$ query time.
- It involves high constant factors though.

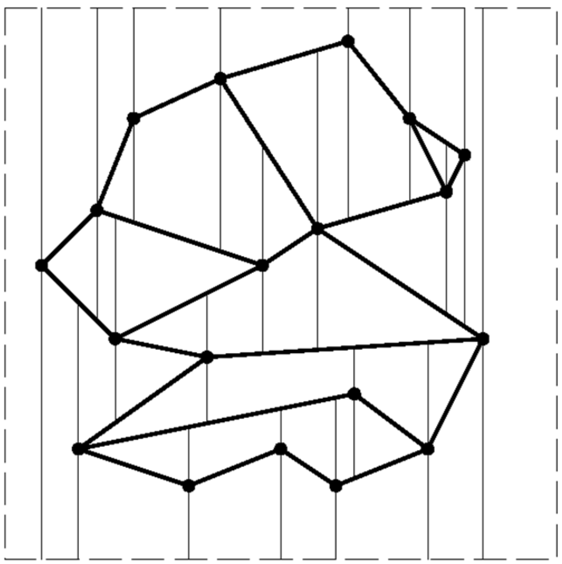
- Next we will discuss a randomized point location scheme (based on **trapezoidal maps**) which is more efficient in practice.



Trapezoidal map

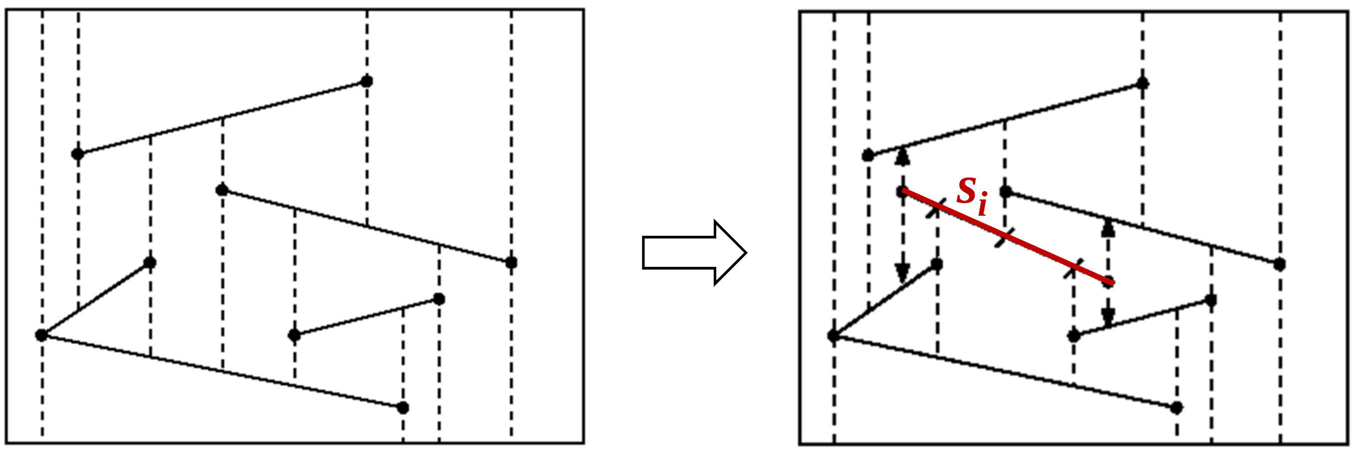
- **Input:** Set $S=\{s_1,\ldots,s_n\}$ of non-intersecting line segments.
- **Query:** Given point $\boldsymbol{p}$, report the segment directly above $\boldsymbol{p}$.

- Create trapezoidal map by shooting two rays vertically (up and down) from each vertex until a segment is hit. [Assume no segment is vertical.]
- **Trapezoidal map** = rays + segments
- Enclose S into bounding box to avoid infinite rays.
- All faces in subdivision are trapezoids, with vertical sides.
- The trapezoidal map has at most $6n+4$ vertices and $3n+1$ trapezoids:
  - Each vertex shoots two rays, so, $2n(1+2)$ vertices, plus 4 for the bounding box.
  - Count trapezoids by vertex that creates its left boundary segment: Corner of box for one trapezoid, right segment endpoint for one trapezoid, left segment endpoint for at most two trapezoids. $\rightarrow 3n+1$



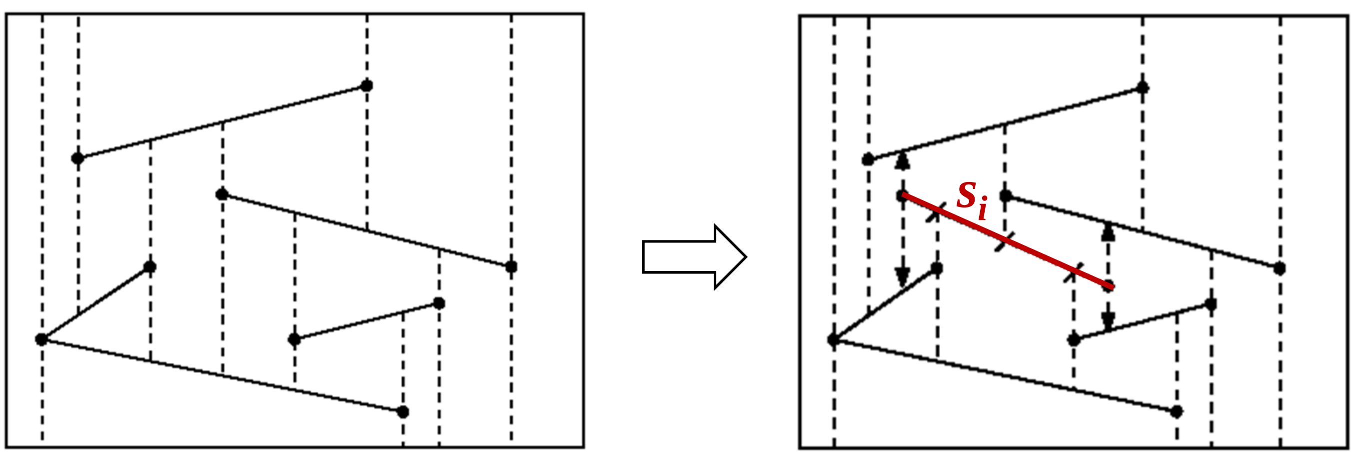
Construction

- **Randomized incremental construction**
- Start with outer box which is a single trapezoid. Then add one segment s_i at a time, in random order.



Construction

- Let $S_i=\{s_1,\ldots, s_i\}$, and let T_i be the trapezoidal map for S_i.
- Add s_i to T_{i-1}.
- Find trapezoid containing left endpoint of s_i. [Point location; details later]
- Thread s_i through T_{i-1}, by walking through it and identifying trapezoids that are cut.
- "Fix trapezoids up" by shooting rays from left and right endpoint of s_i and trim earlier rays that are cut by s_i.



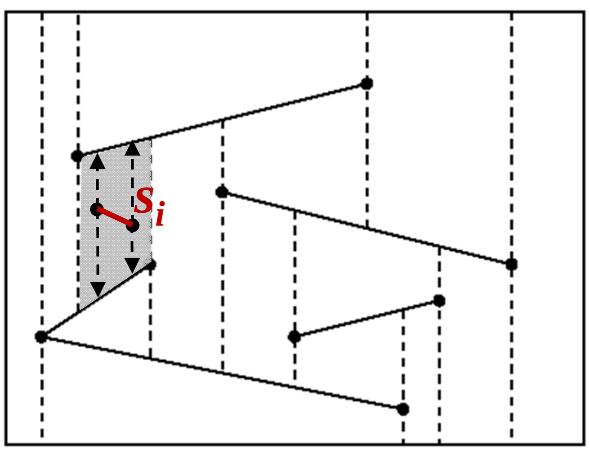
Analysis

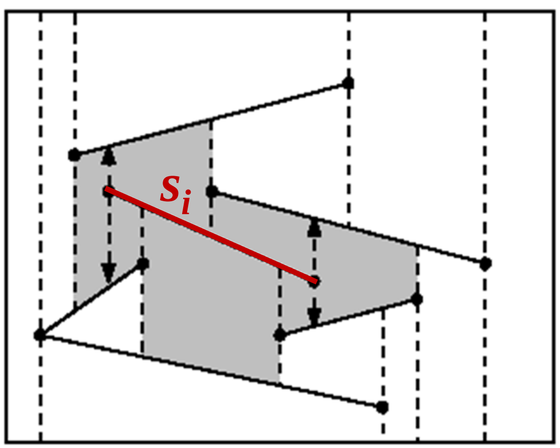
Observation: The final trapezoidal map T_i does not depend on the order in which the segments were inserted.

Lemma: Ignoring the time spent for point location, the insertion of s_i takes $O(k_i)$ time, where k_i is the number of newly created trapezoids.

Proof:

- Let k be the number of ray shots interrupted by s_i.
- Each endpoint of s_i shoots two rays
  $\rightarrow k_i = k+4$ rays need to be processed
- If $k=0$, we get 4 new trapezoids.
- Create a new trapezoid for each interrupted ray shot; takes $O(1)$ time with DCEL



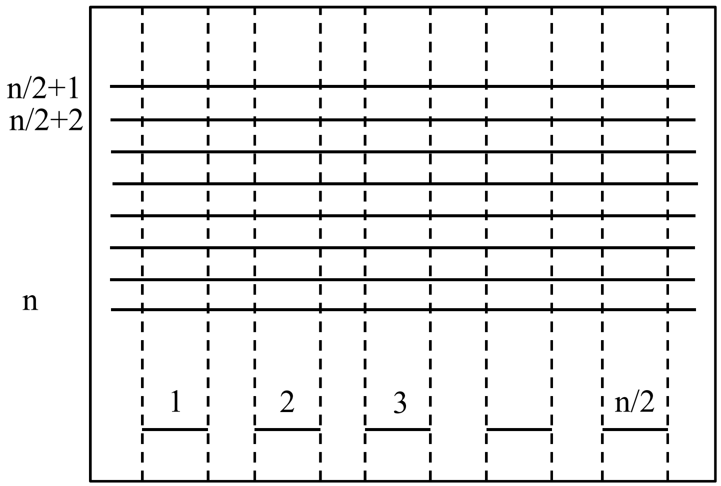


□

Analysis

Total runtime (without point location): $\sum_{i=1}^{n} k_i$

- Best case: $k_i = O(1)$, so $\sum_{i=1}^{n} k_i = O(n)$.
- Worst case: $k_i = O(i)$, so $\sum_{i=1}^{n} k_i = O(n^2)$.



- Insert segments in *random* order:
  - Π = {all possible permutations/orders of segments}; $|\Pi| = n!$ for n segments
  - $k_i = k_i(\pi)$ for some random order $\pi \in \Pi$
  - We will show that $E(k_i) = O(1)$
  - $\Rightarrow$ Expected runtime $E(T) = E(\sum_{i=1}^{n} k_i) = \sum_{i=1}^{n} E(k_i) = O(\sum_{i=1}^{n} 1) = O(n)$

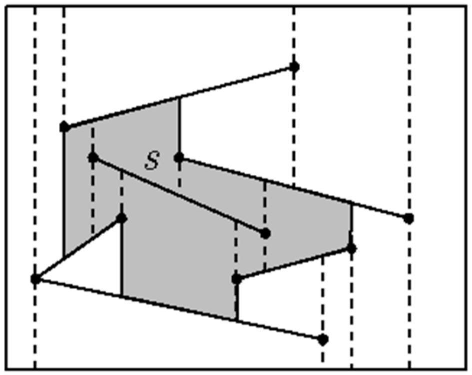
linearity of expectation

Analysis

Theorem: $E(k_i) = O(1)$, where k_i is the number of newly created trapezoids created upon insertion of $\boldsymbol{s_i}$, and the expectation is taken over all segment permutations of $S_i=\{s_1, \ldots, s_i\}$.

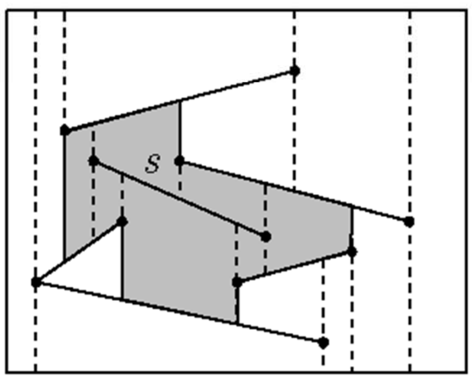
Proof:

- T_i does not depend on the order in which segments $s_1, \ldots, s_i$ were added.
- Of $s_1, \ldots, s_i$, what is the probability that a particular segment s was added last?
- $1/i$
- We want to compute the number of trapezoids that would have been created if s was added last.

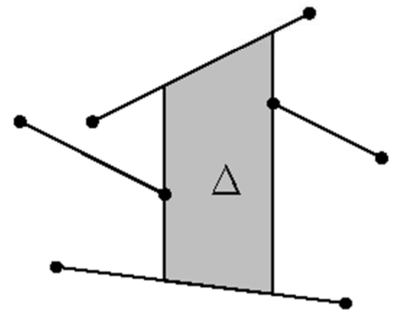


Analysis

- A trapezoid Δ **depends on** s if Δ would be created by s if s was added last.
- We want to count trapezoids that depend on s, and then compute the expectation over all choices of s.
- Let $\delta(\Delta,s)=1$, if Δ depends on s. And $\delta(\Delta,s)=0$, otherwise.



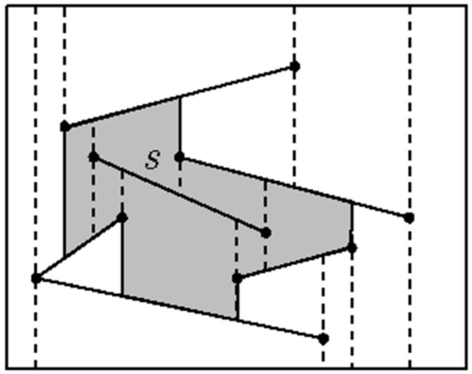
The trapezoids that depend on s



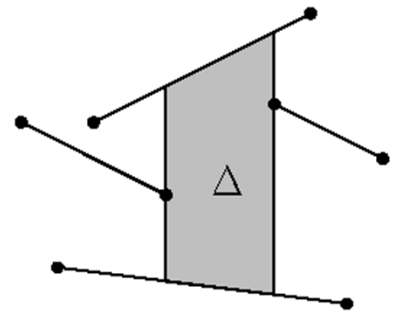
Segments that Δ depends on.

- Random variable $k_i(s)$= #trapezoids added when s was inserted last in S_i.
- $k_i(s)=\sum_{\Delta\in T_i}\delta(\Delta,s)$
- $E(k_i)=\sum_{s\in S_i}k_i(s)\,P(s) = \frac{1}{i}\sum_{s\in S_i}k_i(s) = \frac{1}{i}\sum_{s\in S_i}\sum_{\Delta\in T_i}\delta(\Delta,s)$

Analysis



The trapezoids that depend on s

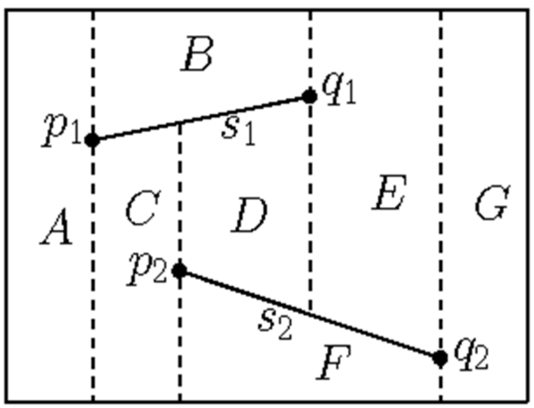


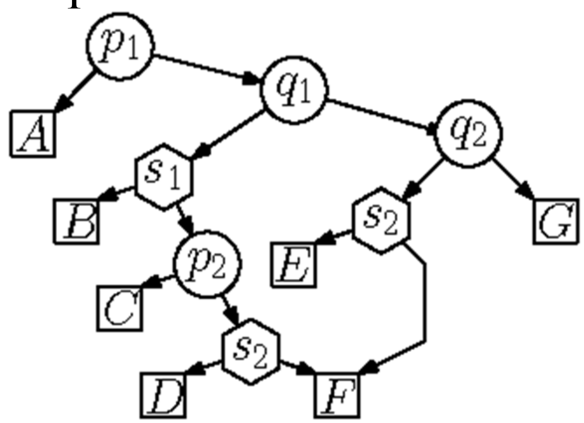
Segments that Δ depends on.

- Random variable $k_i(s)$= #trapezoids added when s was inserted last in S_i.
- $k_i(s) = \sum_{\Delta \in T_i} \delta(\Delta, s)$
- $E(k_i) = \sum_{s \in S_i} k_i(s)\, P(s) = \frac{1}{i} \sum_{s \in S_i} k_i(s) = \frac{1}{i} \sum_{s \in S_i} \sum_{\Delta \in T_i} \delta(\Delta, s)$
- $\qquad = \frac{1}{i} \sum_{\Delta \in T_i} \sum_{s \in S_i} \delta(\Delta, s)$
- How many segments does Δ depend on? At most 4.
- Also, T_i has $O(i)$ trapezoids (by Euler's formula).
- $E(k_i) = \frac{1}{i} \sum_{\Delta \in T_i} \sum_{s \in S_i} \delta(\Delta, s) = \frac{1}{i} \sum_{\Delta \in T_i} 4 = \frac{1}{i} 4|T_i| = \frac{1}{i} O(i) = O(1)$

Point Location

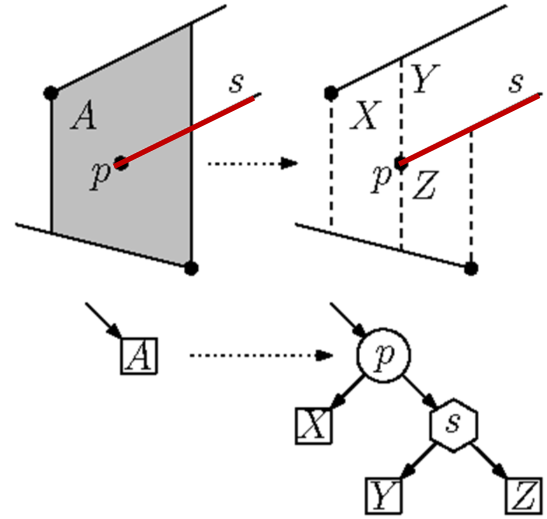
- Build a point location data structure; a DAG, similar to Kirkpatrick's
- DAG has two types of internal nodes:
  - x-node (circle): contains the x-coordinate of a segment endpoint.
  - y-node (hexagon): pointer to a segment
- The DAG has one leaf for each trapezoid.



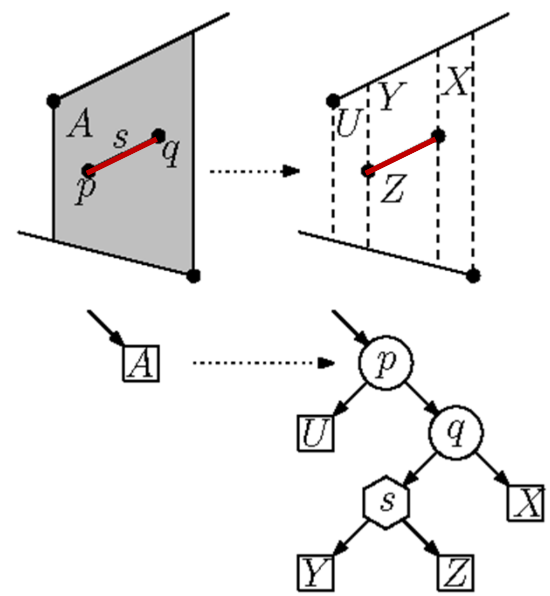


- Children of x-node: Space to the left and right of x-coordinate
- Children of y-node: Space above and below the segment
- y-node is only searched when the query's x-coordinate is within the segment's span.
- $\Rightarrow$ Encodes trapezoidal decomposition and enables point location during construction.

Construction



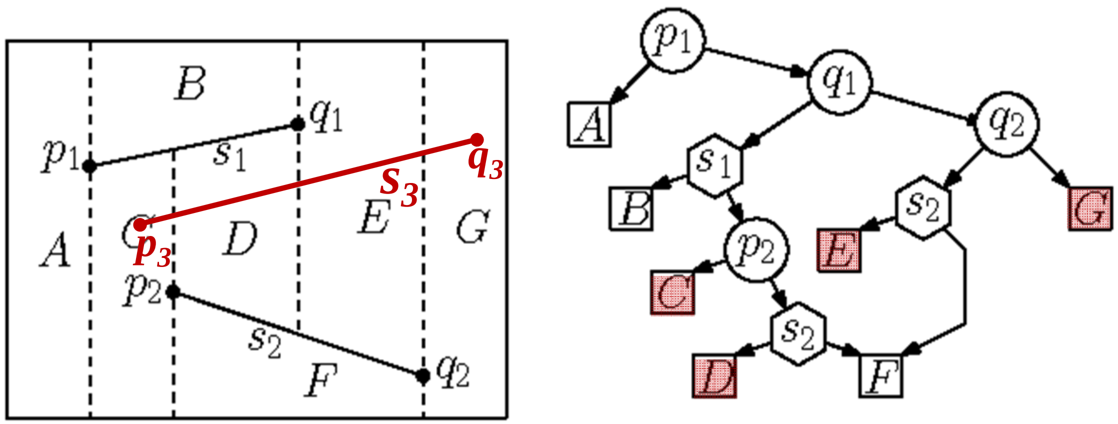
- Incremental construction during trapezoidal map construction.
- When a segment ***s*** is added, modify the DAG.
  - Some leaves will be replaced by new subtrees.
- Each old trapezoid will overlap $O(1)$ new trapezoids.
- Each trapezoid appears exactly once as a leaf.

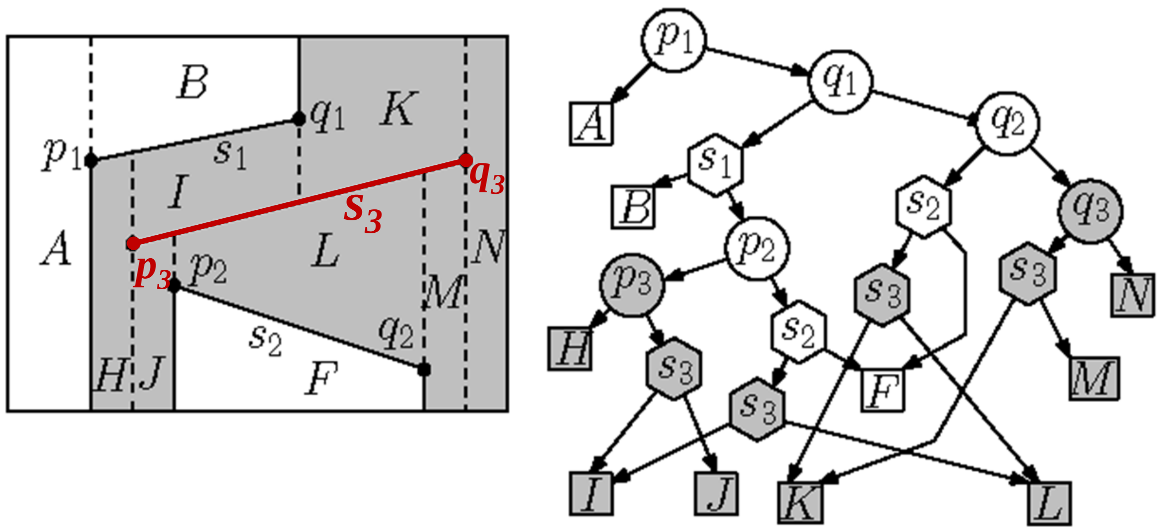


- Changes are highly local.
- If ***s*** passes entirely through trapezoid t, then t is replaced with two new trapezoids t' and t''.
  - Add new y-node as parent of t' and t'', in order to facilitate search later.
- If an endpoint of ***s*** lies in trapezoid t, then add an x-node to decide left/right and a y-node for the segment.

Inserting a Segment

- Insert segment s_3.

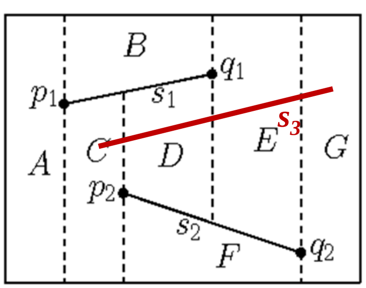


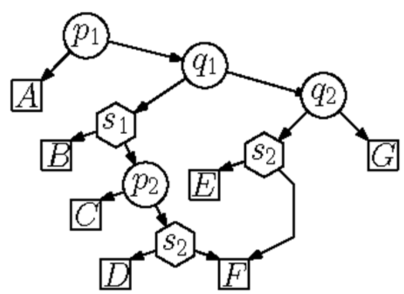


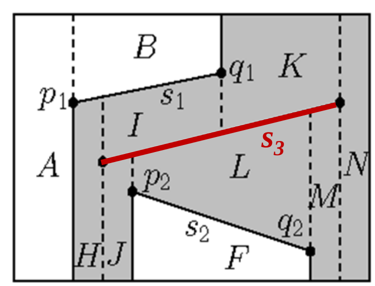
Analysis

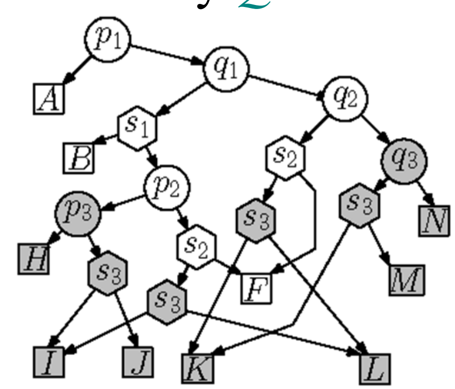
- **Space:** Expected $O(n)$
  - Size of data structure = number of trapezoids = $O(n)$ in expectation, since an expected $O(1)$ trapezoids are created during segment insertion
- **Query time:** Expected $O(\log n)$
- **Construction time:** Expected $O(n \log n)$ follows from query time

- **Proof** that the query time is expected $O(\log n)$:
  - Fix a query point Q.
  - Consider how Q moves through the trapezoidal map as it is being constructed as new segments are inserted.
  - Search complexity = number of trapezoids encountered by Q



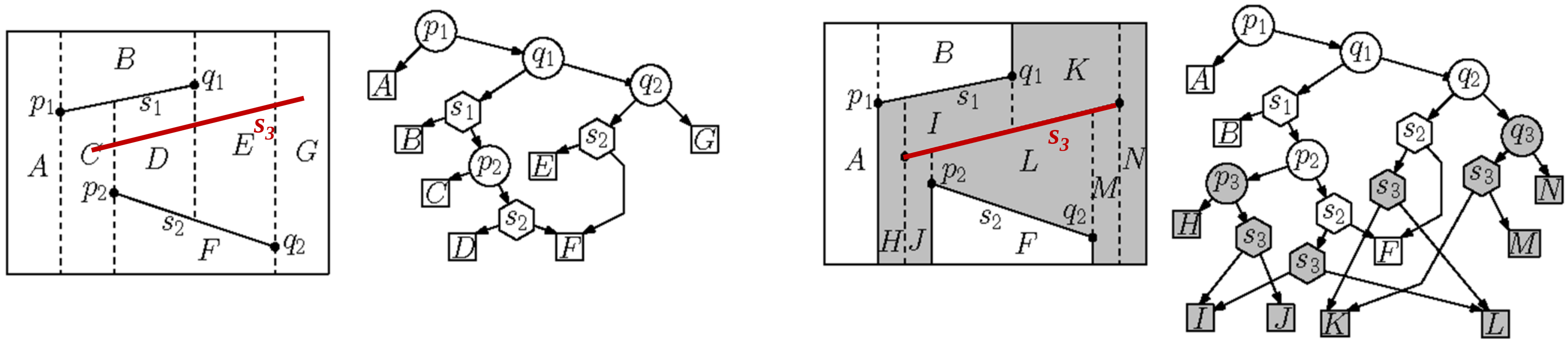




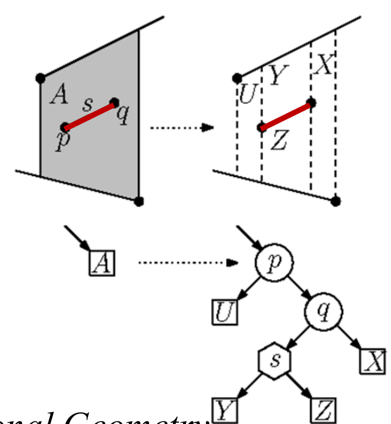


Query Time

- Let Δ_i be the trapezoid containing Q after the insertion of ith segment.
- If $\Delta_i = \Delta_{i-1}$ then the insertion does not affect Q's trapezoid (E.g., $Q \in B$).
- If $\Delta_i \neq \Delta_{i-1}$ then the insertion deleted Q's trapezoid, and Q needs to be located among the at most 4 new trapezoids.



- Q could fall 3 levels in the DAG.



Query Time

- Let X_i be the # nodes on path created in iteration i, and let P_i be the probability that there exists a node in iteration i, i.e., $\Delta_i \neq \Delta_{i\text{-}1}$
- The expected search path length is $\mathrm{E}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) \leq \sum_{i=1}^{n} 3\,P_i$ by lin. of expectation and since Q can drop at most 3 levels.
- **Claim:** $P_i \leq 4/i$.
  - Backwards analysis: Consider deleting segments, instead of inserting.
  - Trapezoid Δ_i depends on ≤ 4 segments. The probability that the ith segment is one of these 4 is $\leq 4/i$.
- The expected search path length is at most

$$\sum_{i=1}^{n} 3\,P_i = \sum_{i=1}^{n} 3\,\frac{4}{i} = 12\sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$$

Harmonic number

□